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LOCALLY NILPOTENT SKEW LINEAR GROUPS II

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Our paper [**6**] studied in some depth certain locally nilpotent skew linear groups, but our conclusions there left some obvious gaps. By means of a trick, which now seems obvious, but then did not, we are able to tidy up the situation very satisfactorily. This present paper should be viewed as a follow up to [**6**]. In particular we do not repeat the motivation, basic definitions and references to related work given here.

The following was conjectured in [**6**], where substantial steps were taken towards its solution.

1. Theorem. *Let H be a locally nilpotent normal subgroup of the absolutely irreducible skew linear group G. Then H is centre by locally-finite and $G/C_G(H)$ is periodic.*

As pointed out in [**6**] this reduces the study of such groups H to considering unipotent-free locally nilpotent skew linear groups over locally finite-dimensional division algebras, about which much is known, see for example Chapter 3 of the forthcoming book [**4**]. It follows immediately from 1 and [**6**] 1.4 that if H is a radical (in the sense of Plotkin, i.e. $H \in \acute{P}L\mathfrak{N}$) normal subgroup of the absolutely irreducible skew linear group G, then H and $G/C_G(H)$ are both abelian-by-periodic. More generally these two results with Theorem A of [**7**] yield the following.

2. Corollary. *Let H be a normal subgroup of the absolutely irreducible skew linear group G, where $H \in \acute{P}L(\mathfrak{N} \cup \mathfrak{F})$. Then there is an abelian normal subgroup $A \leqq H$ of G with H/A locally finite and $G/C_G(H)$ is abelian by periodic.*

The symbolism here and above is part of P. Hall's calculus of group classes, see the opening pages of [**3**] for an account of this. Results 1 and 2 above focus attention on the class of locally soluble groups and our techniques make a small dent into the corresponding problem for this class. For any group X let $T(X)$ denote the maximal periodic normal subgroup of X and let $B(X)/T(X)$ be the Hirsch–Plotkin radical of $X/T(X)$. We are able to prove the following.

3. *Let G be an absolutely irreducible skew linear group of degree* 1.

(a) *If G is locally soluble then G is abelian by locally-finite.*

(b) *Let H be a locally soluble normal subgroup of G and set $B=B(H)$, $K=C_H(B)$ and $A=B \cap K$. Suppose that K/A can be made into an ordered group. Then H and $G/C_G(H)$ are both abelian by periodic.*

The orderable condition here seems quite out of place. It does not show up explicitly in the locally nilpotent case since every torsion-free locally nilpotent group is orderable (e.g. [**2**] 13.1.6 and 13.2.2).

We start our proofs with the trick we missed in [**6**]. Let K be a normal subgroup of the group H with H/K an ordered group, and suppose that $R=E[H]$ is a crossed product of the division ring E by H/K, so $K=E\cap H$. Pick a transversal T_0 of K to H and let D be the set of all formal sums

$$x=\sum_{t\in T} t\xi_t,$$

where $T\subseteq T_0$, the $\xi_t\in E^*$, the set of non-zero elements of E, and $\{tK: t\in T\}$ is a well-ordered subset of the ordered group H/K. (If $T=\varnothing$ then $x=0$.) Then the obvious addition and multiplication on D is well-defined and makes D into a division ring containing R as a subring. This may be proved in a similar way to 13.2.11 of [**2**]; in particular the crucial lemmas 13.2.9 and 13.2.10 apply directly to H/K.

4. *With the notation above assume also that H centralizes E. Then*

$$N_{D^*}(H)=HC_{D^*}(H).$$

Proof. Trivially $N_{D^*}(H)\supseteq HC_{D^*}(H)$. Let $x=\sum_T t\xi_t\in N_{D^*}(H)$ and $h\in H$. Then $k=h^x\in H$ and $hx=xk$. Thus

$$\sum_T ht\xi_t=\sum_T tk\xi_t.$$

Now left and right multiplication in H/K preserves the order and of course the supports of these two sums are equal subsets of H/K. Consequently $htK=tkK$ for all $t\in T$. But then $h^tK=kK=h^{t'}K$ and $t't^{-1}\in C_H(hK/K)$ for all t, $t'\in T$. This is for all $h\in H$ and so $t't^{-1}\in C_H(H/K)$ for all such t and t'. Thus choosing a fixed $t\in T$ we have $x=(\sum_{c\in C} c\xi_{ct})t$ where $C=Tt^{-1}\subseteq C_H(H/K)$.

Trivially t normalizes H and hence so does $y=xt^{-1}$. But then for $h\in H$ and $l=h^y\in H$ we have $hy=yl$ and

$$\sum_C ch[h,c]\xi_{ct}=\sum_C cl\xi_{ct}.$$

Consequently $chK=clK$ for any $c\in C$ and $h^{-1}l\in K$. Comparing coefficients we obtain $[h,c]\xi_{ct}=h^{-1}l\xi_{ct}$ and so $h^c=l$ for all $c\in C$. Hence $c'c^{-1}$ centralizes h for all $c,c'\in C$, and this is for all h in H. Therefore if we pick any one $c\in C$, then $yc^{-1}\in C_{D^*}(H)$ and so

$$x\in C_{D^*}(H)ct\subseteq HC_{D^*}(H).$$

The proof of the lemma is complete.

The theorem follows easily from 4 and Section 5 of [**6**]. It follows even quicker using the following result, which we need for the locally soluble case.

5. *Let F be a field, $R=F[G]$ an F-algebra, generated as such by the subgroup G of its group of units, K a normal subgroup of G and L a division F-subalgebra of R generated as such by K. Let C denote the centre of L and set $A=K\cap C$. Assume that K/A is orderable and that $C[K]\leqq L$ is both an Ore domain and a crossed product of C by K/A. Then $K=A$.*

Proof. Set $X=G\cap L^*C_{L^*G}(K)$. Then R is a crossed product of $L[C_R(K)]$ by G/X by [**6**] 2.6. But $R=F[G]$, so $L[C_R(K)]=F[X]$. Trivially $X=G\cap N_{L^*}(K)C_{L^*G}(K)$ in fact, so $L[C_R(K)]=C[N_{L^*}(K),C_R(K)]$. Now L/C is central simple. Hence $L\otimes_C C_R(K)\cong L[C_R(K)]\leqq R$ by [**1**] p. 363, Theorem 2. Therefore $L=C[N_{L^*}(K)]$.

By hypothesis $C[K]$ is a crossed product of C by the orderable group K/A. By 4 there is a division ring D containing $C[K]$ as a subring such that $N_{D^*}(K)=KC_{D^*}(K)$. But $C[K]$ is also Ore. Consequently L is embedded naturally in D and $N_{L^*}(K)=KC^*$. Then $L=C[K]$ is a crossed product of C by K/A and the orderable group K/A is periodic ([**5**] 2.2). This implies that $K=A$, as required.

6. *The Proof of the Theorem.*

Assume the notation of [**6**] Section 5, which is consistent with that of 5 above. Then K/A is torsion-free, locally nilpotent and hence orderable. L exists by [**6**] 5.5 and 4.4 and $C[K]$ is Ore by Goldie's Theorem. Further $C[K]$ is a crossed product of C by K/A by [**6**] 2.5. Therefore $K=A$ by 5 and the theorem is a trivial consequence of [**6**] 5.3 and 1.1.

7. *Let $R=F[G]$ be an F-algebra, where F is a field and G is a locally soluble subgroup of the group of units of R such that for every infinite subgroup X of G the left annihilator of $X-1$ in R is $\{0\}$. Then R is a crossed product of $F[B(G)]$ by $G/B(G)$.*

Proof. Let H, K and L be finitely generated subgroups of G. Set $\bar{B}(H)=\bigcap_{L\geqq H}B(L)$. Then $\bar{B}(H)\subseteq\bar{B}(K)$ whenever $H\leqq K$ and so $B=\bigcup_H\bar{B}(H)$ is a normal subgroup of G. Also as $B(H)/T(H)$ is locally nilpotent, $T(H)$ is the set of elements of $B(H)$ of finite order. Consequently $T(\bar{B}(H))=\bar{B}(H)\cap T(\bar{B}(K))$ whenever $H\leqq K$, and so B is periodic by locally nilpotent. That is $B\leqq B(G)$. Trivially $L\cap B(G)\leqq B(L)$, so $H\cap B(G)\leqq\bar{B}(H)\leqq B$ and $B=B(G)$.

Suppose $\sum_{i=1}^r t_i\alpha_i=0$ where the t_i are distinct elements of a transversal of B to G and the α_i are non-zero elements of $F[B]$. Then there is a finitely generated subgroup H of G such that $t_1,\dots,t_r\in H$ and $\alpha_1,\dots,\alpha_r\in F[\bar{B}(H)]$. By a theorem of Zalesskiĭ, $F[K]$ is a crossed product of $F[B(K)]$ by $K/B(K)$ for all finitely generated subgroups K of G containing H, see [**2**] 11.4.10. Then for each K there exists $i\neq j$ with $t_i^{-1}t_j\in B(K)$. Since there are only a finite number of i and j there exists a pair i,j with $i\neq j$ such that the set of all finitely generated subgroups K of G containing H with $t_i^{-1}t_j\in B(K)$ is a local system for G. Then $\bar{B}(H)=H\cap\bigcap_{\text{such } K}B(K)$ and so $t_i^{-1}t_j\in B$. This contradiction completes the proof.

8. *The Proof of 3.* By hypothesis there is a division ring D, with centre F say, containing G such that $D=F[G]$.

(a) Let $B=B(G)$. Then D is a crossed product of $F[B]$ by G/B by 7 and therefore G/B

is periodic by [5] 2.2. But then G is locally-finite by locally-nilpotent by locally-finite and the desired conclusion follows from 2 for example.

(b) Trivially $F[K]\subseteq D$ is a domain and K is locally soluble, so $F[K]$ is an Ore domain. Let L be its division ring of quotients in D and let C be the centre of L. Note that

$$B(K)\leqq B(H)\cap K=A\leqq C_H(K)\cap K\leqq B(K),$$

so $A=B(K)$ is the centre of K. Thus $C[K]$ is a crossed product of C by K/A by 7. Clearly $C[K]$ is also an Ore domain. Hence $K=A$ by 5. But $G/C_G(B)$ is abelian-by-periodic by 2 and so H is metabelian by locally-finite. The conclusion now follows from 2 again.

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