

A GENERALIZED TAUBERIAN THEOREM

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Let $\{s(n)\}$ be a real sequence and let x be any number in the interval $0 < x \leq 1$. Representing x by a non-terminating binary decimal expansion we shall denote by $\{s(n, x)\}$ the subsequence of $\{s(n)\}$ obtained by omitting $s(k)$ if and only if there is a 0 in the k th decimal place in the expansion of x . With this correspondence it is then possible to speak of "a set of subsequences of the first category," "an everywhere dense set of subsequences," and so on.

Suppose that T is a regular summability transform given by the matrix (a_{mn}) and let $t(m, x) = \sum a_{mn}s(n, x)$. In a previous note (3), extending a theorem of Buck (2), we proved that a real sequence $\{s(n)\}$ is convergent if there exists a T which sums a set of subsequences of the second category. Our object now is to generalize this Tauberian theorem to the following:

THEOREM. *Suppose that $\{s(n)\}$ is a real sequence and there is a T such that*

$$\limsup t(m, x) - \liminf t(m, x) < \epsilon$$

in a set of the second category. Then

$$\limsup s(n) - \liminf s(n) < \epsilon.$$

The possibility of such a generalization of a Tauberian theorem has been pointed out by Bowen and Macintyre (1).

We first show that, under the hypothesis of the theorem, $\{s(n)\}$ is bounded. Suppose, on the contrary, that $\{s(n)\}$ is unbounded. In (3) we proved that when $\{s(n)\}$ is unbounded then, on the one hand, if (a_{mn}) has infinitely many rows of finite length, $\limsup t(m, x) - \liminf t(m, x)$ is finite only in a set of the first category and, on the other hand, if (a_{mn}) has only a finite number of rows of finite length, $\{s(n, x)\}$ is in the domain of T only in a set of the first category. In either case we have a contradiction and it follows that $\{s(n)\}$ is bounded. We may now prove the conclusion of the theorem with the added hypothesis that $\{s(n)\}$ is bounded. Under this hypothesis, by the following lemma, we may further assume that (a_{mn}) is row finite.

LEMMA 1. *Given a regular transform T with matrix (a_{mn}) we can find a transform with a row finite matrix (a'_{mn}) such that, for every bounded sequence $\{s(n)\}$,*

$$\sum a_{mn}s(n) - \sum a'_{mn}s(n) \rightarrow 0.$$

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Suppose that $|s(n)| < K$. For each m choose k_m so that

$$\sum_{n=k_m}^{\infty} |a_{mn}| < \frac{1}{m}$$

and define $a_{mn}' = a_{mn}$ for $n < k_m$, $a_{mn}' = 0$ for $n \geq k_m$. Then

$$\left| \sum a_{mn}s(n) - \sum a_{mn}'s(n) \right| = \left| \sum_{n=k_m}^{\infty} a_{mn}s(n) \right| < \frac{K}{m} \rightarrow 0.$$

We next prove

LEMMA 2. *Let $\{v(n)\}$ be any sequence of 0's and 1's containing an infinity of both 0's and 1's. Then for any integers p, N and any regular row finite matrix (a_{mn}) , there is a subsequence $\{v(j_n)\}$ such that*

- (i) $\limsup \sum a_{mn}v(j_n) - \liminf \sum a_{mn}v(j_n) \geq 1$,
- (ii) $j_p > N$.

The subsequence described in the lemma is obtained in the following way. Writing $a_{mn} = a(m, n)$, let $a(m, N(m))$ be the last non-zero number in the m th row of (a_{mn}) . We first choose m_1 so that

$$\left| \sum_{n=1}^{N(m_1)} a(m_1, n) - 1 \right| < 1$$

and start the subsequence with $N(m_1)$ 1's. We then choose m_2 , with $N(m_2) > N(m_1)$, so that

$$\left| \sum_{n=1}^{N(m_1)} a(m_2, n) \right| < \frac{1}{2},$$

and continue the subsequence with $N(m_2) - N(m_1)$ 0's. At the k th stage, if k is odd, we choose m_k so that $N(m_k) > N(m_{k-1})$ and

$$\left| \left(\sum_{n=1}^{N(m_1)} + \sum_{N(m_2)+1}^{N(m_3)} + \sum_{N(m_4)+1}^{N(m_5)} + \dots + \sum_{N(m_{k-1})+1}^{N(m_k)} \right) a(m_k, n) - 1 \right| < \frac{1}{k}.$$

We then continue the subsequence with $N(m_k) - N(m_{k-1})$ 1's. If k is even we choose m_k so that $N(m_k) > N(m_{k-1})$ and

$$\left| \left(\sum_{n=1}^{N(m_1)} + \sum_{N(m_2)+1}^{N(m_3)} + \dots + \sum_{N(m_{k-2})+1}^{N(m_{k-1})} \right) a(m_k, n) \right| < \frac{1}{k}.$$

We then continue the subsequence with $N(m_k) - N(m_{k-1})$ 0's. The possibility of this construction is ensured by the facts that, (a_{mn}) being regular,

$$\lim_{m \rightarrow \infty} \sum a_{mn} = 1, \quad \lim_{m \rightarrow \infty} a_{mn} = 0.$$

Plainly the subsequence $\{v(j_n)\}$ so constructed satisfies the inequality (i). It is obvious, moreover, since $\{v(n)\}$ contains an infinity of both 0's and 1's, that given any integers p, N , we may choose j_p so that $j_p > N$, the inequality (ii).

We now proceed to the proof of the equivalent of the theorem: if

$$\limsup s(n) - \liminf s(n) \geq \epsilon,$$

then $\limsup t(m, x) - \liminf t(m, x) < \epsilon$ only in a set of the first category. We prove first that the set D of x such that $\limsup t(m, x) - \liminf t(m, x) \geq \epsilon$ is everywhere dense.

Let $\liminf s(n) = L$, $\limsup s(n) - \liminf s(n) = H$ and define

$$(1) \quad u(n) = \frac{1}{H} (s(n) - L).$$

Then $\limsup u(n) = 1$, $\liminf u(n) = 0$ and we can choose two subsequences $\{u(k_n)\}$, $\{u(p_n)\}$, such that $\lim u(k_n) = 1$, $\lim u(p_n) = 0$, and $k_i \neq p_j$ for all i, j . Let $\{u(i_n)\}$ be the subsequence of $\{u(n)\}$ obtained by combining these two subsequences, arranging them so that the suffixes are in ascending order. Now let $\{v(i_n)\}$ be defined by $v(i_n) = 1$ if $i_n = k_j$ for some j , $v(i_n) = 0$ if $i_n = p_j$ for some j . Then

$$(2) \quad \lim (v(i_n) - u(i_n)) = 0.$$

By Lemma 2 (i), since it is a sequence of 0's and 1's and contains an infinity of both 0's and 1's, $\{v(i_n)\}$ has a subsequence $\{v(j_n)\}$, say, such that

$$(3) \quad \limsup \sum a_{mn} v(j_n) - \liminf \sum a_{mn} v(j_n) \geq 1.$$

By Lemma 2 (ii), moreover, given p and any subsequence $\{s(q_n)\}$ of $\{s(n)\}$ we may choose j_p so that $j_p > q_{p-1}$, and then $\{s(r_n)\} \equiv s(q_1), s(q_2), \dots, s(q_{p-1}), s(j_p), s(j_{p+1}), \dots$ is a subsequence of $\{s(n)\}$. By varying $\{s(q_n)\}$ and p , we obtain an everywhere dense set of subsequences, whose representative points will be shown to lie in D . In fact, since

$$\lim_{m \rightarrow \infty} a_{mn} = 0,$$

we have by (1) and (2)

$$\begin{aligned} \limsup \sum a_{mn} s(r_n) &= \limsup \sum a_{mn} s(j_n) = \limsup \sum a_{mn} (Hu(j_n) + L) \\ &= \limsup \sum a_{mn} (Hv(j_n) + L) \end{aligned}$$

and similar equalities with \limsup replaced by \liminf . Thus, by (3),

$$\begin{aligned} \limsup \sum a_{mn} s(r_n) - \liminf \sum a_{mn} s(r_n) \\ = H(\limsup \sum a_{mn} v(j_n) - \liminf \sum a_{mn} v(j_n)) \geq H \geq \epsilon. \end{aligned}$$

Finally, let S_n^k , ($k = 1, 2, \dots$; $n = 1, 2, \dots$) denote the set of x such that there exist $\mu, \nu > n$ for which

$$|t_\mu(x) - t_\nu(x)| > \epsilon - \frac{1}{k}.$$

Since (a_{mn}) is row finite, S_n^k is obviously open and, since it contains D , it is everywhere dense. If

$$x \in \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} S_n^k$$

then

$$\limsup t(m, x) - \liminf t(m, x) \geq \epsilon - \frac{1}{k}$$

for all k and so $\limsup t(m, x) - \liminf t(m, x) \geq \epsilon$. The set of x for which $\limsup t(m, x) - \liminf t(m, x) < \epsilon$ therefore belongs to

$$\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{C} S_n^k$$

and so is of the first category.

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REFERENCES

1. N. A. Bowen and A. J. Macintyre, *An oscillation theorem of Tauberian type*, Quart. J. Math., 2 (1950), 243–247.
2. R. C. Buck, *A note on subsequences* Bull. Amer. Math. Soc., 49 (1943), 898–899.
3. F. R. Keogh and G. M. Petersen, *A universal Tauberian theorem*, J. Lond. Math. Soc. (to appear).

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