WHEN IS THE ALGEBRA OF REGULAR SETS FOR A FINITELY ADDITIVE BOREL MEASURE A $\sigma$-ALGEBRA?

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(Received 4 May 1981; revised 3 December 1981)

Communicated by J. B. Miller

Abstract

It is shown that the algebra of regular sets for a finitely additive Borel measure $\mu$ on a compact Hausdorff space is a $\sigma$-algebra only if it includes the Baire algebra and $\mu$ is countably additive on the $\sigma$-algebra of regular sets. Any infinite compact Hausdorff space admits a finitely additive Borel measure whose algebra of regular sets is not a $\sigma$-algebra. Although a finitely additive measure with a $\sigma$-algebra of regular sets is countably additive on the Baire $\sigma$-algebra there are examples of finitely additive extensions of countably additive Baire measures whose regular algebra is not a $\sigma$-algebra. We examine the particular case of extensions of Dirac measures. In this context it is shown that all extensions of a $\{0,1\}$-valued countably additive measure from a $\sigma$-algebra to a larger $\sigma$-algebra are countably additive if and only if the convex set of these extensions is a finite dimensional simplex.

Keywords: Borel measure, regularity, extensions of measures, completion regular compact space, Borel regular compact space.

Introduction and synopsis

In [16], Kupka noted that if a vector-valued Borel measure on a compact Hausdorff space $X$ is countably additive then its algebra of regular sets is in fact a $\sigma$-algebra. In Question 3.3.1 of [16], he asked whether countable additivity is necessary for this result. We essentially answer this question in the negative but do show that a good deal of countable additivity is implicit in the assumption that the algebra of regular sets of a finitely additive Borel measure is a $\sigma$-algebra. More specifically we show that, on any $\sigma$-algebra contained in the algebra of regular

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sets of a finitely additive Borel measure $\mu$, $\mu$ is countably additive (Lemma 1). If in fact the algebra of regular sets for $\mu$ is a $\sigma$-algebra then it includes the $\mu$-completion of the Baire algebra and $\mu$ agrees with the canonical regular extension of $\mu$ to the Borel algebra from the Baire algebra at least on the $\sigma$-algebra of regular sets (Propositions 3 and 5). In fact, the latter statement holds even if $\mu$ is only assumed to be countably additive on the Baire algebra but with the algebra of regular sets not necessarily a $\sigma$-algebra. Corollary 3.1 answers Kupka’s question affirmatively for completion regular compact Hausdorff spaces. Here a finitely additive Borel measure is countably additive if and only if its algebra of regular sets is $\sigma$-algebra. Corollary 3.2 shows that on any infinite compact Hausdorff space there is a finitely additive Borel measure which does not have a $\sigma$-algebra of regular sets. This follows from Proposition 4 which asserts that a Boolean algebra admits a non-countably additive measure if and only if it is not Cantor separable if and only if its Stone space is not an almost $P$-space, a result of independent interest.

The latter part of the paper examines the regular algebras of finitely additive Borel measures $\mu$ whose restriction to the Baire algebra is countably additive when $\mu$ is $\{0, 1\}$-valued on the Baire algebra. Proposition 6 deals with the convex compact set of all extensions of a countably additive $\{0, 1\}$-valued measure $\delta$ on a $\sigma$-algebra $\Sigma_1$ to a larger $\sigma$-algebra $\Sigma_2$. It is shown that this convex compact set is finite dimensional if and only if all extensions of $\delta$ are countably additive. Otherwise, there exist $2^c$ mutually singular non-atomic purely finitely additive extensions or $c \{0, 1\}$-valued extensions where $c = 2^{\aleph_0}$, (Corollary 6.1). This is applied to the case where $\Sigma_1$ is the Baire algebra, $\Sigma_2$ is the Borel algebra and $\delta$ is $\delta_x$ for some non-$G_\delta$-point $x \in X$. If the extensions of $\delta_x$ to the Borel algebra are all countably additive there is a countably additive extension $\mu$ whose regular algebra is just the $\delta_x$-completion of the Baire algebra. However, for this to be true $X$ must be topologically pathological near $x$.

We conclude with an example which yields finitely additive Borel measures whose regular algebras are not $\sigma$-algebras yet contain the Baire algebra. If real valued measurable cardinals exist an example is given of a countably additive Borel measure whose regular $\sigma$-algebra is properly contained in the Borel algebra and properly contains the completed Baire algebra.

1. When is the algebra of regular sets for a finitely additive Borel measure a $\sigma$-algebra?

$\mathfrak{B}_0$ and $\mathfrak{B}$ denote, respectively, the Baire and Borel $\sigma$-algebras on $X$. $C(X)$ denotes the real continuous functions on $X$ and $\mathfrak{M}(X)$ the dual of $C(X)$. $\mathfrak{M}(X)$ is identified, as usual, with both $CA(\mathfrak{B}_0)$ the countable additive Baire measures.
and with \( CA_\lambda(\mathcal{B}) \) the regular countably additive Borel measures. For any Boolean algebra \( \mathcal{A} \), \( BA(\mathcal{A}) \) denotes the finitely additive real measures of bounded variation on \( \mathcal{A} \) with \( CA(\mathcal{A}) \) the band of countably additive elements of \( BA(\mathcal{A}) \). If \( \mu \in BA^+(\mathcal{B}) \) we denote by \( \text{Reg}(\mu) \) all \( A \in \mathcal{B} \) so that \( \inf\{\mu(\theta \setminus K) : K \text{ compact } \subset A \subset \theta \text{ open}\} = 0 \). Note that \( \text{Reg}(\mu) \) is an algebra which is \( \mu \)-complete in \( \mathcal{B} \) in that whenever \( \{A_n\} \) is an increasing sequence and \( \{B_n\} \) is a decreasing sequence in \( \text{Reg}(\mu) \) with \( A_n \subset B_n \) for all \( n \) and with \( \lim_{n \to \infty} \mu(B_n \setminus A_n) = 0 \) then \( A \in \text{Reg}(\mu) \) provided \( A \in \mathcal{B} \) and \( A_n \subset A \subset B_n \) for all \( n \). For any algebra \( \mathcal{A} \subset \mathcal{B} \), \( \mathcal{A}^\lambda \) will denote its completion in \( \mathcal{B} \) with respect to the finitely additive Borel measure \( \mu \). Thus, \( \text{Reg}(\mu) = (\text{Reg}(\mu))^\lambda \). This lemma was pointed out by Douglas Dokken. It is a generalization of Problem 7 on page 11 of [6].

**Lemma 1.** If \( \Sigma \) is a \( \sigma \)-algebra contained in \( \text{Reg}(\mu) \) for \( \mu \in BA^+(\mathcal{B}) \) then \( \mu \) is countably additive on \( \Sigma \).

**Proof.** It must be shown that if \( \{D_n\} \subset \Sigma \) is a disjoint sequence with union \( D \) then \( \mu(D) = \sum_{n=1}^{\infty} \mu(D_n) \). That \( \mu(D) \geq \sum_{n=1}^{\infty} \mu(D_n) \) is immediate. If we show that \( \mu(D) \leq \sum_{n=1}^{\infty} \mu(D_n) + \varepsilon \) for any \( \varepsilon > 0 \) the assertion will be established. Pick \( K \) compact \( \subset D \) with \( \mu(D) \leq \mu(K) + \varepsilon/2 \). Pick \( \theta_n \) open with \( D_n \subset \theta_n \) and with \( \mu(\theta_n \setminus D_n) \leq \varepsilon 2^{-n-1} \). Since \( K \subset D \subset \bigcup_{n=1}^{\infty} \theta_n \) there is an integer \( m \) so that \( K \subset \theta_1 \cup \cdots \cup \theta_m \). For this \( m \) it is true that \( \mu(K) \leq \sum_{n=1}^{\infty} \mu(\theta_m) \leq \sum_{n=1}^{\infty} \mu(\theta_n) < \sum_{n=1}^{\infty} \mu(D_n) + \varepsilon/2 \). Thus, \( \mu(D) < \sum_{n=1}^{\infty} \mu(D_n) + \varepsilon \).

**Remark.** Lemma 1 is a consequence of Proposition 1.6 in Chapter V of [4] and of Lemma 1 of [25].

**Corollary 1.1.** a) If \( \mu \in BA^+(\mathcal{B}) \) and \( \text{Reg}(\mu) \) is a \( \sigma \)-algebra then \( \mu \) is countably additive on \( \text{Reg}(\mu) \).

b) \( \text{Reg}(\mu) \) is a \( \sigma \)-algebra if and only if \( \mu \) is countably additive on the \( \sigma \)-algebra generated by \( \text{Reg}(\mu) \).

**Proof.** Only b) needs to be established. This is done in the standard fashion. Let \( \{D_n\} \) be a disjoint sequence in \( \text{Reg}(\mu) \) with union \( D \). Let \( \theta_n \) be open with \( D_n \subset \theta_n \) and \( \mu(\theta_n \setminus D_n) \leq 2^{-n-1} \varepsilon \) for a given \( \varepsilon > 0 \). Let \( m \) be such that \( \mu(\bigcup_{n=m+1}^{\infty} D_n) \leq \varepsilon/4 \). Let \( K_n \subset D_n \) for \( n = 1, \ldots, m \) be compacts with \( \mu(D_n \setminus K_n) < \frac{1}{4} \varepsilon m^{-1} \). We have \( \mu([\bigcup_{n=1}^{\infty} \theta_n] \setminus \bigcup_{n=1}^{m} K_n) \leq \varepsilon \) with \( \bigcup_{n=1}^{m} K_n \subset D \subset \bigcup_{n=1}^{\infty} \theta_n \). Thus, \( D \in \text{Reg}(\mu) \). Thus, \( \text{Reg}(\mu) \) is a \( \sigma \)-algebra if \( \mu \) is countably additive on the \( \sigma \)-algebra generated by \( \text{Reg}(\mu) \). The converse follows from a).
LEMMA 2. Let \( A \in \text{Reg}(\mu) \).

i) There exists a \( G_\delta A_\delta \in \text{Reg}(\mu) \) and an \( F_\sigma A_\sigma \in \text{Reg}(\mu) \) with \( A_\sigma \subset A \subset A_\delta \) and 
\[ \mu(A_\delta \backslash A_\sigma) = 0. \]

ii) There exists a \( G_\delta A_\delta \in \mathcal{B}_0 \cap \text{Reg}(\mu) \) and an \( F_\sigma A_\sigma \in \mathcal{B}_0 \cap \text{Reg}(\mu) \) with 
\[ A_\sigma \subset A_\sigma \subset A_\delta \subset A_\delta. \]

iii) \( \mu(A) = \mu(A_\sigma) = \mu(A_\delta) = \mu(A_\sigma) = \mu(A_\delta) = \sup\{\mu(K) : K \text{ compact Baire} \subset A_\sigma\} = \inf\{\mu(G) : G \text{ open Baire} \supset A_\delta\}. \)

iv) There is an \( A_0 \in \mathcal{B}_0 \cap \text{Reg}(\mu) \) with \( \mu(A \Delta A_0) = 0. \)

PROOF.

i) Immediate from the definition of regularity.

ii) Let \( A_\sigma = \bigcup_{n=1}^{\infty} K_n \) and \( A_\sigma = \bigcap_n G_n \) where \( K_n \) is compact and \( G_n \) is open for all \( n \). By Urysohn's Theorem there is a compact \( G_\delta, K_\delta, m \) satisfying \( K_\delta \subset K_\delta, m \subset G_\delta \) for all \( n, m \). Set \( K_n = \bigcap_{m=1}^{\infty} K_\delta, m \). \( K_\delta \) is a compact \( G_\delta \) and \( K_\delta \subset K_\delta \subset A_\delta \) for all \( n \). Set \( A_\sigma \) equal to the \( F_\sigma \), \( \bigcup_{n=1}^{\infty} K_\delta \). \( A_\sigma \) is obtained analogously as a countable intersection of open \( F_\sigma \) sets.

iii) From the definition of regularity the \( K_n \) in ii) may be chosen with 
\[ \mu(A) = \sup\mu(K_n) \leq \sup\mu(K_n) \leq \sup\{\mu(K) : K \text{ compact Baire} \subset A_\sigma\} = \mu(A). \]
Thus, \( \mu(A) = \sup\{\mu(K) : K \text{ compact Baire} \subset A_\sigma\} \). Similarly, \( \mu(A) = \inf\{\mu(G) : G \text{ open Baire} \supset A_\delta\} \).

iv) Set \( A_0 = A_\delta \) or \( A_\sigma \).

Plachky, [20], shows that if \( \nu \) is a finitely additive probability on a Boolean algebra \( \mathcal{A} \) and \( BA_+ (\mathcal{A}, \nu, \mathcal{B}_2) \) denotes the convex compact set of extensions of \( \nu \) to a probability measure on a larger algebra \( \mathcal{B}_2 \) then \( \mu \in BA_+ (\mathcal{A}, \nu, \mathcal{B}_2) \) is extreme if and only if for all \( A_2 \in \mathcal{B}_2 \) and \( \epsilon > 0 \) there is an \( A_1 \in \mathcal{A} \) with 
\[ \mu(A_1 \Delta A_2) < \epsilon. \]
Thus, in Lemma 2, \( \mu \), on \( \text{Reg}(\mu) \), is an extreme extension of its restriction to \( \mathcal{B}_0 \cap \text{Reg}(\mu) \).

PROPOSITION 3. If \( \mu \in BA_+ (\mathcal{B}) \) is such that \( \text{Reg}(\mu) \) is a \( \sigma \)-algebra then 
\( \mathcal{B}_0 \subset \text{Reg}(\mu) \).

To establish this we first consider the case \( X = [0,1] \). Let \( Y \) denote those \( x \in (0,1) \) so that \( \inf\{\mu(\theta) : x \in \theta \text{ open}\} = 0. \) The complement of \( Y \) is at most countably hence \( Y \) is dense. Each \( \{x\} \) with \( x \in Y \) is in \( \text{Reg}(\mu) \) with \( \mu(\{x\}) = 0. \)

For \( \epsilon > 0 \) let \( \theta \) be an open set containing \( x \in Y \) with \( \mu(\theta_x) < \epsilon, K_\epsilon^- = [0, x) \setminus \theta_x \) and 
\( K_\epsilon^+ = (x, 1] \setminus \theta_x. \) Both \( K_\epsilon^- \) and \( K_\epsilon^+ \) are compact. It is easily verified that 
\[ \lim_{\epsilon \to 0} \mu(K_\epsilon^-) = \mu(\{0, x\}) \) and \( \lim_{\epsilon \to 0} \mu(K_\epsilon^+) = \mu((x, 1]). \)
Thus, \( \{0, x\}, (x, 1]) \subset \text{Reg}(\mu) \). It follows that all intervals, open, closed, or half open, whose endpoints
are chosen from \( Y \) belong to \( \text{Reg}(\mu) \). The \( \sigma \)-algebra generated by these intervals is \( \mathcal{B}_0 = \mathfrak{B} \). Since \( \text{Reg}(\mu) \) is a \( \sigma \)-algebra \( \mathcal{B}_0 = \text{Reg}(\mu) \). This establishes this case.

Let \( X \) be arbitrary and let \( f: X \to [0, 1] \) be continuous. Let \( \nu \) be the finitely additive Borel measure on \([0, 1]\) which is the image of \( \mu \) under \( f \). Thus, for Borel \( A \subset [0, 1] \), \( \nu(A) = \mu(f^{-1}(A)) \). Just as in the countably additive case \( A \in \text{Reg}(\nu) \) if and only if \( f^{-1}(A) \in \text{Reg}(\mu) \). Consequently, \( \text{Reg}(\nu) \) is a \( \sigma \)-algebra hence is equal to the Borel algebra of \([0, 1]\) by the special case just established. Thus, \( f \) is measurable for the \( \sigma \)-algebra \( \text{Reg}(\mu) \). Since \( f \) is arbitrary it follows that all \( f \in \mathcal{C}(X) \) are \( \text{Reg}(\mu) \)-measurable. Thus, since \( \mathcal{B}_0 \) is the smallest \( \sigma \)-algebra so that all \( f \in \mathcal{C}(X) \) are \( \mathcal{B}_0 \)-measurable, \( \mathcal{B}_0 \subset \text{Reg}(\mu) \). This establishes the proposition.

In [4], Babiker and Knowles define a space \( X \) to be completion regular if and only if every \( \mu \in \mathcal{C}(\mathcal{B}_0) + \mathcal{C}(\mathfrak{B}) \) is completion regular in the sense of Berberian [5]. That is, each \( \mu \in \mathcal{C}(\mathcal{B}_0) + \mathcal{C}(\mathfrak{B}) \) has a unique extension in \( \mathcal{B} \mathcal{A}(\mathfrak{B}) \). Alternatively \( X \) is completion regular if and only if \( \mathfrak{B} \) is the \( \mu \)-completion of \( \mathcal{B}_0 \) for all \( \mu \in \mathcal{C}(\mathcal{B}_0) + \mathcal{C}(\mathfrak{B}) \). Examples of completion regular spaces include all perfectly normal compact Hausdorff spaces \( X \). In [5] Berberian notes that if \( X \) is completion regular all points must be \( G_\delta \)'s. Under the assumption that the continuum is real valued measurable an example may be constructed of a non-completion regular \( X \) each of whose points is a \( G_\delta \). In order that \( X \) be completion regular it is necessary and sufficient that every Borel set be regular with respect to the paving of compact \( G_\delta \)'s for all countably additive Borel measures. This corollary is easily deduced from the definition of completion regularity.

**Corollary 3.1.** Let \( X \) be completion regular. The following are equivalent for \( \mu \in \mathcal{B} \mathcal{A}(\mathfrak{B}) \)

a) \( \text{Reg}(\mu) \) is a \( \sigma \)-algebra

b) \( \text{Reg}(\mu) = \mathfrak{B} \)

c) \( \mu \in \mathcal{C}(\mathcal{B}_0) + \mathcal{C}(\mathfrak{B}) = \mathcal{C}(\mathcal{B}_0) + \mathcal{C}(\mathfrak{B}) \).

**Corollary 3.2.** If \( X \) is an infinite compact Hausdorff space there is a \( \mu \in \mathcal{B} \mathcal{A}(\mathfrak{B}) \) so that \( \text{Reg}(\mu) \) is not a \( \sigma \)-algebra.

**Proof.** Any extension \( \mu \) to \( \mathfrak{B} \) of a member of \( \mathcal{B} \mathcal{A}(\mathcal{B}_0) \setminus \mathcal{C}(\mathcal{B}_0) \) will do. The non-emptiness of \( \mathcal{B} \mathcal{A}(\mathcal{B}_0) \setminus \mathcal{C}(\mathcal{B}_0) \) is a special case of Proposition 4.

We are interested in determining for which infinite Boolean algebras \( \mathcal{A} \) every element of \( \mathcal{B} \mathcal{A}(\mathcal{A}) \) is countably additive. If no infinite strictly decreasing sequence in \( \mathcal{A} \) has a lower bound then, automatically, \( \mathcal{B} \mathcal{A}(\mathcal{A}) = \mathcal{C}(\mathcal{A}) \). Such Boolean algebras are termed Cantor separable in [28]. Cantor separable Boolean algebras \( \mathcal{A} \) are characterized in terms of their Stone space \( X_\mathfrak{B} \) by the fact that each
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non-empty zero set has a non-empty interior. Completely regular spaces $X$ with the aforementioned property are called *almost P-spaces* in [17] and have been studied in [7], [10] and [27]. Thus, $\mathcal{G}$ is Cantor separable if $X_\mathcal{G}$ is an almost $P$-space. Notice that if $\mathcal{G}$ is $\sigma$-complete it is not Cantor separable if it is infinite. $\beta N \setminus N$ is the most familiar example of an almost $P$-space [9, 65.8]. Graves and Wheeler in [10] give a method for producing a large class of almost $P$-spaces. The following proposition was pointed out by R. F. Wheeler.

**Proposition 4.** *The following are equivalent for an infinite Boolean Algebra $\mathcal{G}$.*

a) $\mathcal{G}$ is Cantor separable

b) $X_\mathcal{G}$ is an almost $P$-space

c) $BA^+(\mathcal{G}) = CA^+(\mathcal{G})$.

**Proof.** We already have a) $\Leftrightarrow$ b) $\Rightarrow$ c). Let us assume c) and see that this implies b). Notice that all $\{0,1\}$-valued elements of $BA^+(\mathcal{G})$ are countably additive. Phrased in terms of the corresponding ultrafilters on $\mathcal{G}$, this says that if $\{A_n; n \in \mathbb{N}\}$ is a decreasing sequence in an ultrafilter then $\emptyset \neq \inf_n A_n$. That is, there is an $A_\infty \in \mathcal{G}$ with $\emptyset \neq A_\infty \subset A_n$ for all $n$. Since every decreasing sequence of non-empty elements of $\mathcal{G}$ lies in an ultrafilter this says that no decreasing sequence of non-empty elements of $\mathcal{G}$ has $\emptyset$ as infimum. In particular, regarding $\mathcal{G}$ as the clopen algebra of $X_\mathcal{G}$, the intersection of a decreasing sequence of non-empty clopen sets (that is, a zero set) has non-empty interior. Thus, c) implies both a) and b).

**Remark.** We use the term $\delta$-ultrafilter for an arbitrary Boolean algebra to denote any ultrafilter whose corresponding $\{0,1\}$-valued measure is countably additive.

A compact Hausdorff space $X$ is called *Borel regular* [19], or *Radon*, [21], if and only if $CA^+(\mathcal{B}) = CA_1^+(\mathcal{B})$ if and only if every $\mu \in CA^+(\mathcal{B}_0)$ has a unique extension, the regular extension, to $\mathcal{B}$ belonging to $CA^+(\mathcal{B})$. If $\mu \in CA_1^+(\mathcal{B}) \setminus CA_1^+(\mathcal{B}_0)$ then $\text{Reg}(\mu)$ is a super-$\sigma$-algebra of $\mathcal{B}_0$ properly contained in $\mathcal{B}$. The canonical example of a non-Borel regular space is the compact ordinal space $[0, \omega_1]$ where $\omega_1$ is the first uncountable ordinal. There are countably additive $\{0,1\}$-valued extensions of the Dirac measure $\delta_{\omega_1}$ from $\mathcal{B}_0$ to $\mathcal{B}$ other than the regular extension [9, ex. 53.10a]. An example of a Borel regular space $X$ which is not completion regular is the one point compactification $D \cup \{\infty\}$ of a discrete space $D$ with uncountable non-real-valued measurable cardinal, [8, ex. 6.2]. The Dirac measure $\delta_{\infty}$ has extensions from $\mathcal{B}_0$ to $\mathcal{B}$ other than the regular one but all must be purely finitely additive [2], [13], since they induce on $D$ finitely additive,
diffuse [2], probability measures. We shall be primarily concerned with $\text{Reg}(\mu)$ for $\mu$ non-countably additive yet with $\mu$ countably additive on $\mathcal{B}_0$ but occasionally with $\mu$ countably additive and non-regular on $\mathcal{B}$. In any case, $\mu_{\text{reg}}$ will denote the unique element of $CA^+_1(\mathcal{B})$ agreeing with $\mu$ on $\mathcal{B}_0$.

**Proposition 5.** Let $\mu \in BA^+(\mathcal{B})$ be countably additive on $\mathcal{B}_0$. On $\text{Reg}(\mu)$, $\mu$ and $\mu_{\text{reg}}$ coincide.

**Proof.** Let $A \in \text{Reg}(\mu)$. One can, in the proof of Lemma 2, find $A_\sigma$ an $F_\sigma$ in $\text{Reg}(\mu)$ and $A_\delta$ a $G_\delta$ in $\text{Reg}(\mu)$, so that $A_\sigma \subset A \subset A_\delta$ and so that $\mu(A_\delta \setminus A_\sigma) = 0$. Let $(A^\sigma, A^\delta) \subset \mathcal{B}_0 \cap \text{Reg}(\mu)$ with $A_\sigma \subset A^\sigma \subset A^\delta \subset A_\delta$. Then, $\mu(A) = \mu(A^\sigma) = \mu_{\text{reg}}(A^\sigma) = \mu_{\text{reg}}(A)$.

In the remainder of the paper we will be dealing fairly exclusively with extensions $\mu$ of Dirac measures $\delta_x$ for $x \in X$ from $\mathcal{B}_0$ to $\mathcal{B}$. All such extensions must be $\{0,1\}$-valued on $\text{Reg}(\mu)$. If $A \in \text{Reg}(\mu)$ then $\mu(A) = 0$ if and only if $x \notin A$.

**Proposition 6.** Let $\Sigma_1 \subset \Sigma_2$ be $\sigma$-algebras of subsets of a set $\Omega$. Let $\delta \in CA^+_1(\Sigma_1)$ be $\{0,1\}$-valued. Let $\eta$ be the $\sigma$-ideal in $\Sigma_2$ of sets of outer measure 0 under $\delta$.

i) If the quotient algebra $\Sigma_2/\eta$ is finite then $BA^+_1(\Sigma_1, \delta, \Sigma_2)$ is a finite dimensional subset of $CA^+_1(\Sigma_2)$.

ii) If $\Sigma_2/\eta$ is infinite there is a family $\{\mu^1_i\} \subset BA^+_1(\Sigma_1, \delta, \Sigma_2)$ of mutually singular, non-atomic, purely finitely additive measures whose cardinality is $2^c$ where $c$ is the continuum.

**Proof.** There is an affine bijection from $BA^+_1(\Sigma_1, \delta, \Sigma_2)$ to $BA^+_1(\Sigma_2/\eta)$. If $\mu \in BA^+_1(\Sigma_1, \delta, \Sigma_2)$ then $\mu(A) = 0$ for all $A \in \eta$ hence $\mu$ induces on $\Sigma_2/\eta$ an element, also denoted by $\mu$, in the usual fashion. This gives the affine bijection.

ii) If $\Sigma_2/\eta$ is infinite it is an infinite $F$-algebra as in [3]. By Corollary 3.2.3 of [3] there is a family $\{\mu_i\}$, of cardinality $2^c$, of mutually singular non-atomic probability measures on $\Sigma_2/\eta$ all with the same negligible sets. Pulling back under the affine bijection from $BA^+_1(\Sigma_1, \delta, \Sigma_2)$ to $BA^+_1(\Sigma_2/\eta)$ one obtains the same sort of family in $BA^+_1(\Sigma_1, \delta, \Sigma_2)$. If $\mu_s \in \{\mu_i\}$ is countably additive there can be no other countably additive $\mu_r \in \{\mu_i\}$ for $\mu_r \perp \mu_s$ and both have the same nullsets. Delete $\mu_s$ if necessary so that no element of $\{\mu_i\}$ is countably additive. Each $\mu_i$ has a non-trivial purely finitely additive part which is a multiple of a purely finitely additive $\mu_i'$ which is easily verified to belong to $BA^+_1(\Sigma_1, \delta, \Sigma_2)$. Furthermore, $\mu_i'$ must be non-atomic for each $i$. This establishes ii).
Suppose that $\Sigma_2/\eta$ is finite and has $n$ atoms \{a_1, \ldots, a_n\}. Corresponding to each $a_i$ is an $A_i \in \Sigma_2$ which is such that if $A \in \Sigma_2$ then $A_i \setminus A \in \eta$ or $A \cap A_i \in \eta$. The \{(0,1)\}-valued measure $\delta_i$ on $\Sigma_2/\eta$ or in $BA_1^+(\Sigma_2, \delta, \Sigma_2)$ corresponding to $a_i$ is an extreme point of $BA_1^+(\Sigma_2/\eta)$ and $BA_1^+(\Sigma_2/\eta) = \text{conv}(\delta_1, \ldots, \delta_n)$. To show that $BA_1^+(\Sigma_2, \delta, \Sigma_2) \subseteq CA^+(\Sigma_2)$ it suffices to show that each $\delta_i$, considered as an element of $BA_1^+(\Sigma_1, \delta, \Sigma_2)$, is in $CA^+(\Sigma_2)$. To this end let $\{E_n\}$ be an increasing sequence in $\Sigma_2$ with $\delta_i(E_n) = 0$ for all $n$. We have $E_n \cap A_i \in \eta$ for all $n$ hence, by the $\sigma$-completeness of $\eta$, we have $(\bigcup_n E_n) \cap A_i \in \eta$. Thus, $\delta_i(\bigcup_n E_n) = 0$. This establishes countable additivity of $\delta_i$ hence establishes i).

**Remarks.** Recall from [2] that a measure $\mu$ is strongly finitely additive if and only if there is a partition \{A_n: n \in N\} with $\mu(A_n) = 0$ for all $n$. Any purely finitely additive probability measure is the sum of countably many strongly finitely additive measures, [2]. In ii) purely finitely additive measures may be replaced by strongly finitely additive measures.

Actually ii) asserts only that such a family of probabilities exists in $BA(\Sigma_2/\eta)$. This is true if $\eta$ is replaced by the ideal generated by the null sets of a non \{(0,1)\}-valued measure or $\Sigma_2/\eta$ by an arbitrary $F$-algebra.

**Corollary 6.1.** If $\Sigma_2/\eta$ is infinite there exist $c$ purely finitely additive \{(0,1)\}-valued elements of $BA_1^+(\Sigma_1, \delta, \Sigma_2)$.

**Proof.** There is a strongly finitely additive non-atomic $\mu \in BA_1^+(\Sigma_1, \delta, \Sigma_2)$. Let $\{A_n\} \subset \Sigma_2$ be an increasing sequence with $\mu(A_n) = 0$ for all $n$ and with $\bigcup_n A_n = \Omega$. Let $\mathcal{B}$ denote the algebra $\Sigma_2/\eta$ and let $X_\mathcal{B}$ be its Stone space. $BA_1^+(\Sigma_1, \delta, \Sigma_2)$ is affinely homeomorphic to the Bauer simplex of Radon probability measures on $X_\mathcal{B}$. Let $\tilde{\mu}$ be the Radon measure on $X_\mathcal{B}$ corresponding to $\mu$ so that if $A \in \Sigma_2/\eta$ or if $A \in \Sigma_2$ then $\mu(A) = \tilde{\mu}(A]$ where $[A]$ is the clopen set in $X_\mathcal{B}$ corresponding to $A$. We have $\mu(A) = \int 1_A(x)\tilde{\mu}(dx) = \int \chi_{[A]}(x)\tilde{\mu}(dx)$ (where $x \in X_\mathcal{B}$ are considered as ultrafilters on $\mathcal{B}$). If there were a set $Z$ with outer measure $\tilde{\mu}^*(Z) > 0$ of $\delta$-ultrafilters $x \in X_\mathcal{B}$ (so that each $\chi_x$ is countably additive on $\mathcal{B}$), it would follow that $0 = \lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \int 1_{[A_n]}(x)\tilde{\mu}(dx) \geq \tilde{\mu}^*(Z) > 0$. Since this is impossible $\tilde{\mu}$-almost all $x \in X_\mathcal{B}$ have $\chi_x$ purely finitely additive. Since $\tilde{\mu}$ is non-atomic there is a compact perfect set $Y \subset \text{supp}(\tilde{\mu}) \subset X_\mathcal{B}$ so that if $x \in Y$ then $\chi_x$ is purely finitely additive. $Y$ contains at least $c$ elements.

**Corollary 6.2.** If $\mathcal{B}$ is a Boolean algebra then $\mu \in BA_1^+(\mathcal{B})$ is purely finitely additive with corresponding measure $\tilde{\mu}$ on the Stone space $X_\mathcal{B}$ only if $\mu$-almost all $x \in X_\mathcal{B}$ are not $\delta$-ultrafilters.
We may apply the preceding results to the case where $\mathcal{B}_0 = \Sigma_1$ and $\mathcal{B} = \Sigma_2$. A \{0, 1\}-valued measure $\delta$ on $\mathcal{B}_0$ is a Dirac measure $\delta_x$. $\eta$ will be denoted by $\eta_x$. $\eta_x$ consists of those Borel sets in $X$ contained in a $\sigma$-compact subset of $X' = X \setminus \{x\}$. We are only interested in the case where $\mathcal{B}/\eta_x = \mathcal{B}_x$ has cardinality larger than 2 so that $\{x\}$ is not a $G_\delta$.

**Proposition 7.** Let $x$ be a non-$G_\delta$ point in $X$.

i) If $\mathcal{B}_x$ is finite the elements of $BA^+_1(\mathcal{B}_0, \delta_x, \mathcal{B})$ form a finite dimensional simplex in $CA^+_1(\mathcal{B})$. In this case there is a $\mu \in BA^+_1(\mathcal{B}_0, \delta_x, \mathcal{B})$ with $\text{Reg}(\mu) = \mathcal{B}_0^\delta = \mathcal{B}_0^\mu$.

ii) If $\mathcal{B}_x$ is infinite there is a family of cardinality $2^\omega$ of singular non-atomic purely finitely additive elements of $BA^+_1(\mathcal{B}_0, \delta_x, \mathcal{B})$ and a family of cardinality $c$ of \{0, 1\}-valued purely finitely additive elements.

**Proof.** We need only find in case i) a $\mu \in BA^+_1(\mathcal{B}_0, \delta_x, \mathcal{B})$ with $\text{Reg}(\mu) = \mathcal{B}_0^\mu$. Let $\{\delta_x, \delta_1, \ldots, \delta_n\}$ denote the extreme points of $BA^+_1(\mathcal{B}_0, \delta_x, \mathcal{B})$ where $\delta_x$ is the usual Dirac measure on $\mathcal{B}$. We assert that $\mu = \frac{1}{n}(\delta_1 + \cdots + \delta_n)$ has $\text{Reg}(\mu) = \mathcal{B}_0^\mu$. Suppose not. Note that $\mathcal{B}_0^\mu = \mathcal{B}_0^\delta$ is the largest subalgebra of $\mathcal{B}$ to which $\delta_x$ has a unique extension. Note also that $\delta_x$ agrees with $\mu$ on $\text{Reg}(\mu)$ by Proposition 5. There is an extreme extension $\delta$ of $\delta_x$ from $\mathcal{B}_0^\mu$ to $\text{Reg}(\mu)$ other than $\delta_x$ hence other than $\mu$. This extreme extension $\delta$ is the restriction of one of $\{\delta_1, \ldots, \delta_n\}$ to $\text{Reg}(\mu)$, say $\delta_1$. Since all extreme extensions of $\delta_x$ to $\text{Reg}(\mu)$ are $\{0, 1\}$-valued there is an $A \in \text{Reg}(\mu)$ with $0 = \delta_x(A) = \mu(A)$ and $\delta_1(A) = 1$. But $\mu(A) = \frac{1}{n}(\delta_1(A) + \cdots + \delta_n(A)) \geq \frac{1}{n}$ which is impossible. Thus, $\text{Reg}(\mu) = \mathcal{B}_0^\mu$.

**Corollary 7.1.** If $\mathcal{B}_x$ is infinite and $\mu \in BA^+_1(\mathcal{B}_0, \delta_x, \mathcal{B})$ has $\text{Reg}(\mu) \neq \mathcal{B}_0^\mu$ there is a $\nu \in BA^+_1(\mathcal{B}_0, \delta_x, \mathcal{B})$ with $\text{Reg}(\nu)$ a proper subset of $\text{Reg}(\mu)$.

**Remark.** We know of no case in which $x$ is a non-$G_\delta$ point for which i) holds in Proposition 7. For the case $X = [0, \omega_1]$ and $x = \omega_1$ one may set $A_0$ equal to the relatively closed set in $[0, \omega_1]$ consisting of limit ordinals, and set $A_n = \{\alpha + 1: \alpha \in A_{n-1}\}$ for $n \in \omega$. Then $[0, \omega_1) = \cup_n A_n$. Each $A_n$ is in $\mathcal{B} \setminus \eta_x$ hence $\mathcal{B}_x$ is infinite. A similar argument shows that if $D$ is an infinite discrete set with uncountable cardinality then $X = D \cup \{\infty\}$ has $\mathcal{B}_x$ infinite then $x = \infty$.

**Corollary 7.2.** If $\mathcal{B}_x$ is finite there is a closed set $E \subset X'$ whose complement is $\sigma$-compact and is such that $E$ has a partition $\{E_1, \ldots, E_n\}$ with each $E_i$ closed. Within each $E_i$ the set $\mathcal{B}_i$ of non-$\sigma$-compact closed sets forms a $\delta$-ultrafilter of closed sets. If $E_i \cup \{x\} = X_i$ is considered as the one point compactification of $E_i$ then $\delta_x$
has a one dimensional simplex of extensions to the Borel sets of \( X_i \). The extreme extension \( \delta_i \) is defined by \( \delta_i(A) = 1 \) if and only if \( A \) contains an element of \( \mathcal{F}_i \) for \( i = 1, \ldots, n \).

**Proof.** Let \( \{ \delta_0, \delta_1, \ldots, \delta_n \} \) be the extreme elements of \( BA^+_i(\mathcal{B}_0, \delta_x, \mathcal{B}) \) with \( \delta_0 \) the regular extension. For each \( i = 1, \ldots, n \) there is a \( \delta \)-ultrafilter \( \mathcal{F}_i \) of closed subsets of \( X' \) so that \( \delta_i(A) = 1 \) if and only if \( A \) meets each element of \( \mathcal{F}_i \). One may find \( \{ F_1, \ldots, F_n \} \) so that \( F_i \in \mathcal{F}_i \) for \( i = 1, \ldots, n \) and so that \( F_i \cap F_j \in \eta_x \) for all \( i \neq j \). One may find an open \( \sigma \)-compact \( \theta \subset X' \) with \( F_i \cap F_j \subset \theta \) for all \( i, j \). Let \( E_i = F_i \setminus \theta \) for all \( i \) and let \( E = \bigcup_{i=1}^n E_i = X' \setminus \theta \). Any extension \( \delta \) of \( \delta_x \) to the Borel sets of \( X_i \) with \( \delta(x) = 0 \) may be extended to an element of \( BA^+_i(\mathcal{B}_0, \delta_x, \mathcal{B}) \) with \( \delta(E_i) = 1 \). We must have \( \delta = \delta_i \) which establishes the corollary.

**Corollary 7.3.** If \( \mathcal{B}_x \) is finite every closed set in \( X' \) contains a dense \( \sigma \)-compact subset.

**Proof.** We may, by Corollary 7.2, assume that \( BA^+_i(\mathcal{B}_0, \delta_x, \mathcal{B}) = \{ \delta_x, \delta \} \) so that \( \mathcal{F} = \{ F \text{ closed in } X': \delta(F) = 1 \} \) is the set of non-\( \sigma \)-compact closed sets in \( X' \).

Assume that \( X' \neq \mathcal{F} \) for any \( E \in \eta_x \). If this is the case then \( E \in \eta_x \) implies that \( \mathcal{F} \in \eta_x \). To see this note that if \( \mathcal{F} \notin \eta_x \), then \( \mathcal{F} \in \mathcal{F} \) and \( \mathcal{F} \notin \eta_x \), Since \( \mathcal{F} \) is the closure of \( E \cup \mathcal{F} \in \eta_x \) one has a contradiction.

Let \( \{ \theta_\alpha \} \subset \eta_x \) be a sequence indexed by ordinals \( \alpha \) defined by transfinite induction so that \( \theta_\alpha \) is a proper subset of \( \theta_{\alpha+1} \) and so that \( \theta_\alpha = \bigcup_{\beta < \alpha} \theta_\beta \) if \( \alpha \) is a limit ordinal. The last element \( \theta_\lambda \) of this sequence occurs for a limit ordinal \( \lambda \) so that \( \mathcal{F} \in \mathcal{F} \) hence so that \( \theta_\lambda \notin \eta_x \). Since \( \eta_x \) is \( \sigma \)-complete \( \lambda \) is of uncountable cofinality. Let \( \psi_\alpha = \theta_{\alpha+1} \setminus \theta_\alpha \) for \( \alpha < \lambda \) and let \( \psi_\lambda = X' \setminus \theta_\lambda \). We have \( X' = [\bigcup \{ \psi_\alpha: \alpha < \lambda \}] \cup [\bigcup \{ \theta_\alpha: \alpha < \lambda \}] \). The open set \( [\bigcup \{ \psi_\alpha: \alpha < \lambda \}] \) is dense in \( X' \) hence is not in \( \eta_x \). The closed set \( \bigcup \{ \theta_\alpha: \alpha < \lambda \} \) is \( \sigma \)-compact hence is in an open \( \theta_\infty \in \eta_x \). Let \( D = \{ \alpha < \lambda: \psi_\alpha \setminus \theta_\infty \neq \emptyset \} \). The open sets \( \{ \psi_\alpha: \alpha \in D \} \) together with \( \theta_\infty \) cover \( X' \). Thus, \( \text{card}(D) \geq \aleph_1 \). If \( K \) is a compact set in \( X' \) it is covered by \( \theta_\infty \) together with finitely many \( \psi_\alpha \) with \( \alpha \in D \) hence a \( \sigma \)-compact set is covered by \( \theta_\infty \) together with countably many \( \psi_\alpha \) with \( \alpha \in D \). Let \( \{ D_n: n \in N \} \) be a countable partition of \( D \) into uncountable sets. For each \( n \) let \( U_n = [\bigcup \{ \psi_\alpha: \alpha \in D_n \}] \). The family \( \{ U_n: n \in N \} \) is a disjoint family of open sets with \( \bigcup \{ U_n: n \in N \} = \bigcup \{ \psi_\alpha: \alpha \in D \} \). Since \( \sigma \)-compact \( F \) meets only countably many \( \psi_\alpha \), no \( U_n \) is in \( \eta_x \). Thus, \( \mathcal{B}_x \) is infinite which is impossible. Thus, \( X' = \mathcal{F} \) for some \( E \in \eta_x \). This demonstration also establishes, if \( F \in \mathcal{F} \) replaces \( X' \), that \( F = \mathcal{F} \) for some \( E \in \eta_x \), which establishes the corollary.
In the unlikely event that $\mathcal{B}_x$ be finite for some non-$G_\delta$-point $x$, Proposition 7 gives a countably additive $\mu \in BA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$ with $\text{Reg}(\mu) = \mathcal{B}_0^\mu$. We conclude by giving an example where $\text{Reg}(\mu)$ is always larger than $\mathcal{B}_0^\mu$.

**Example 8.** Let $X$ be the one point compactification $D \cup \{x\}$ of an uncountable discrete space. $\mathcal{B}_0$ consists of countable sets in $D$ and their complements in $X$, $\mathcal{B} = 2^X$ and $\eta_x$ consists of countable sets in $D$ hence is a maximal ideal in $\mathcal{B}_0$ and $\mathcal{B}_0$ is $\mu$-complete for any $\mu \in BA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$. The $\mu \in BA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$ with $\mu(\{x\}) = 0$ are identified with elements of $BA_1^+(2^D/\eta_x)$ or with elements of $BA_1^+(2^D)$ which annihilate $\eta_x$ hence are those $\mu \in BA_1^+(2^X)$ with $\mu(A) = 0$ if $A$ is countable in $X$. If $\mu \in BA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$ then $\mu$ agrees with $\delta_x$ on $\text{Reg}(\mu)$. If $A \subset D$ has $\mu(A) = 0$ then $A \in \text{Reg}(\mu)$ since $A$ is open whereas $A \cup \{x\} \not\in \text{Reg}(\mu)$. Thus, $\text{Reg}(\mu)$ consists of $A \subset D$ with $\mu(A) = 0$ and the complements in $X$ of these $A$. Let $\eta_\mu$ denote the ideal in $2^D$ of $\mu$-negligible sets. $\eta_\mu$ is a maximal ideal in $\text{Reg}(\mu)$ and $2^D/\eta_\mu$ satisfies the countable chain condition. On the other hand $2^D/\eta_x$ does not satisfy the countable chain condition since $D$ has an uncountable partition into uncountable sets. Thus, $\eta_x \not\subseteq \eta_\mu$ and $\mathcal{B}_0^\mu \not\subseteq \text{Reg}(\mu)$.

Note that if the cardinality of $D$ is not real-valued measurable, [1], [2], then all elements $\mu$ of $BA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$ with $\mu(\{x\}) = 0$ must be purely finitely additive. If the cardinality of $D$ is real-valued measurable any countably additive diffuse measure $m$ on $2^D$ gives an element of $CA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$ singular to $\delta_x$ and $\text{Reg}(\mu)$ is guaranteed to be strictly between $\mathcal{B}_0$ and $\mathcal{B}$. If $\mu \in BA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$ is purely finitely additive it is a countable convex combination $\sum \{\lambda_n \mu_n: n \in \mathbb{N}\}$ of strongly finitely additive $\{\mu_n\} \subset BA_1^+(\mathcal{B})$. Each $\mu_n$ must be in $BA_1^+(\mathcal{B}_0, \delta_x, \mathcal{B})$. From the definition of strong finite additivity there exist $\{A_m^n: m \in \mathbb{N}\} \subset \eta_{\mu_n}$ which partition $D$. We have $\{A_m^n: m \in \mathbb{N}\} \subset \text{Reg}(\mu_n)$. Since $D \not\subseteq \text{Reg}(\mu_n)$ it is impossible for $\text{Reg}(\mu_n)$ to be $\sigma$-algebra even though $\mathcal{B}_0 \subset \text{Reg}(\mu_n)$.

**Remark.** Karel Prikry and Richard Gardner pointed out Example 8.

**References**


The algebra of regular sets for a Borel measure


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