On Mathieu Functions of Higher Order.

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The definition and properties of Mathieu (or elliptic-cylinder) functions are well known to the members of this Society, owing to the appearance in its *Proceedings* from time to time of various papers by different authors, wherein these functions are discussed. The object of the present paper is to introduce a new kind of function which can be considered as a generalisation of Mathieu functions, and for which we propose the name of "Mathieu Functions of Higher Order."

1. *Definition of the functions by a differential equation.* Let us consider the differential equation of the second order.

$$\frac{d^2 y}{dz^2} + 2\nu \cot z \frac{dy}{dz} + (a + k^2 \cos^2 z)\, y = 0 \qquad (1)$$

where k and ν are given constants, and a a constant of which the value is still unfixed. It is obvious, from the general theory of differential equations with periodic coefficients, that, when special values (depending on k and ν) are given to a, equation (1) can be satisfied by periodic functions of z, with period 2π. These periodic solutions we propose to study.

When k is zero, the differential equation becomes

$$\frac{d^2 y}{dz^2} + 2\nu \cot z \frac{dy}{dz} + ay = 0 \qquad (2)$$

and it is readily seen that it admits of periodic solutions when a is equal to $n(n+2\nu)$, where n is any positive integer. The differential equation is then identical to Gegenbauer's equation, and its periodic solutions are the polynomial $C_n^\nu(\cos z)$ of Gegenbauer, and its associated function of the second kind (which shall be denoted by $H_n^\nu(\cos z)$). The periodic solutions of equation (2) are therefore, according to the value of a,

$$C_0^\nu(\cos z),\ C_1^\nu(\cos z),\ C_2^\nu(\cos z), \ldots\ldots$$

$$H_0^\nu(\cos z),\ H_1^\nu(\cos z),\ H_2^\nu(\cos z), \ldots\ldots$$

Hence the following result: *The differential equation* (1) *is satisfied, when suitable values are given to* a, *by a set of periodic functions, reducing, when k is zero, to a function C or a function H.* We shall denote any of these functions by the symbol $M_n^\nu(z;k)$, or, more briefly, $M_n^\nu(z)$; but they can be divided into two sets—*even* functions of z, which shall be denoted by $E_n^\nu(z)$, and which reduce to $C_n^\nu(\cos z)$ when k is zero; and *odd* functions, $O_n^\nu(z)$, reducing to $H_n^\nu(\cos z)$. Periodic solutions of equation (1), *i.e.* Mathieu functions of higher order, are therefore, according to the value of a,

$$E_0^\nu(z),\ E_1^\nu(z),\ E_2^\nu(z),\ \ldots\ldots,$$

$$O_0^\nu(z),\ O_1^\nu(z),\ O_2^\nu(z),\ \ldots\ldots,$$

When ν is zero, equation (1) reduces to Mathieu's equation, so that its solutions reduce to Mathieu functions, $E_n^\nu(z)$ to $ce_n(z)$, and $O_n^\nu(z)$ to $se_n(z)$.

2. *The integral-equation.*—As may be expected, these functions satisfy an integral equation analogous to Whittaker's integral-equation for Mathieu functions. When ν *is an integer*, one readily obtains the equation

$$E_n^\nu(z) = \lambda \int_{-\pi}^{+\pi} e^{k\cos z\cos u} \sin^{2\nu} u\, E_n^\nu(u)\, du \quad\ldots\ldots\ldots\ldots\ldots(3)$$

If ν is not an integer, this integral equation must be written

$$E_n^\nu(z) = \lambda \int_{-\pi}^{\pi} e^{k\cos z\cos u} \sin^{2\nu} u\, M_n^\nu(u)\, du,$$

the function M which occurs there having the property that the

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product $\sin^{2\nu} u\, M_n^\nu(u)$ must not be an odd function of u. For instance, if $\nu = \frac{1}{2}$, we have

$$E_n^{\frac{1}{2}}(z) = \lambda \int_{-\pi}^{\pi} e^{k\cos z \cos u} \sin u\; O_n^{\frac{1}{2}}(u)\, du$$

which is an integral equation of the first kind.

3. *The functions* $\mathcal{M}$, $\mathcal{E}$ *and* $\mathcal{O}$.—Let us make a change of function by putting

$$\mathcal{M}_n^\nu(z) = \sin^\nu z\, M_n^\nu(z).$$

Then this new function is a periodic solution of the differential equation

$$\frac{d^2 y}{dz^2} + \left[a + \nu^2 - \frac{\nu(\nu-1)}{\sin^2 z} + k^2\cos^2 z\right] y = 0 \quad \text{............}(4)$$

If we denote by $\mathcal{E}_n^\nu$ and $\mathcal{O}_n^\nu$ even and odd solutions of this equation, we have

$$\mathcal{E}_n^\nu = \sin^\nu z\, E_n^\nu, \text{ if } \nu \text{ is even};$$

$$\mathcal{E}_n^\nu = \sin^\nu z\, O_n^\nu, \text{ if } \nu \text{ is odd, and so on,}$$

and we can write the homogeneous integral equation of the second kind, *with a symmetrical nucleus*,

$$\mathcal{E}_n^\nu(z) = \lambda \int_{-\pi}^{\pi} e^{k\cos z\cos u} \sin^\nu z \sin^\nu u\; \mathcal{E}_n^\nu(u)\, du,$$

where ν is supposed to be an even integer, the modifications when ν is odd or not integral being obvious.

When $k = 0$, $\mathcal{E}_n^\nu$ reduces (ν being even) to $\sin^\nu z\, C_n^\nu(\cos z)$, and when ν is zero, to $ce_n(z)$.

Now, when $\nu = 1$, equation (4) reduces to

$$\frac{d^2 y}{dz^2} + (a + 1 + k^2\cos^2 z)\, y = 0 \quad \text{....................}(5)$$

which is of Mathieu's type. But, when k is zero, we know that the only valid values of a are $n(n+2\nu)$, *i.e.* $n(n+2)$; then $a + 1 = (n+1)^2$, and the solutions of (5) are $ce_{n+1}(z)$ and $se_{n+1}(z)$; therefore

$$\mathcal{E}_n^1(z) = ce_{n+1}(z), \quad \mathcal{O}_n^1(z) = se_{n+1}(z),$$

and

$$E_n^1(z) = \frac{se_{n+1}(z)}{\sin z}, \quad O_n^1(z) = \frac{ce_{n+1}(z)}{\sin z}.$$

If we make $k=0$ se_{n+1} reduces to $\sin(n+1)z$, so that $E_n^1(z)$ becomes $\frac{\sin(n+1)z}{\sin z}$, which is, as is well known, equal to $C_n^1(\cos z)$, thus confirming our new result.

The $\mathcal{M}$-functions being solutions of a homogeneous integral equation with a symmetrical nucleus, there exist between two of them, with different lower indices, such relations as, if ν is even,

$$\int_{-\pi}^{\pi} \mathcal{E}_n^\nu(z)\,\mathcal{E}_m^\nu(z)\,dz=0,$$

which, when $k=0$, becomes

$$\int_{-\pi}^{\pi} C_n^\nu(\cos z)\,C_m^\nu(\cos z)\sin^{2\nu} z\,dz=0,$$

or

$$\int_{-1}^{+1} C_n^\nu(x)\,C_m^\nu(x)\,(1-x^2)^{\nu-\frac{1}{2}}\,dx=0,$$

a well-known formula for Gegenbauer's polynomials.

Various Properties of the M-functions.—When 2ν is integral if we make the change of function

$$M_n^\nu(z)=\sin^{1-2\nu}z\;F(z),$$

we obtain for $F(z)$ the differential equation

$$\frac{d^2F}{dz^2}+2(1-\nu)\cot z\frac{dF}{dz}+F(a+2\nu-1+k^2\cos^2 z)=0,$$

which can be written

$$\frac{d^2F}{dz^2}+2\nu'\cot z\frac{dF}{dz}+(a'+k^2\cos^2 z)\;F=0,$$

where $\nu'=1-\nu$, and where a', when $k=0$, becomes equal to $(n+2\nu-1)\;(n+2\nu-1+2\nu')$. This equation for F, therefore, is a Mathieu equation of higher order, and its periodic solutions are of the form $M^{n-\nu}_{n+2\nu-1}$. Hence the remarkable formula

$$M_n^\nu(z)=\sin^{1-2\nu}z\,M^{1-\nu}_{n+2\nu-1}(z),$$

where M, in each member, must be replaced by E or O according to the value of ν. For instance, if $\nu = 0$, we obtain

$$E_n^0 = \sin z \; O_{n-1}^1 ,$$

and, if $\nu = \frac{1}{2}$, the identity,

$$E_n^{\frac{1}{2}} = E_n^{\frac{1}{2}} .$$

It is readily seen that, when k tends to zero, the function $M_n^\nu\left(\cos^{-1}\dfrac{x}{k}\right)$ becomes equal (a constant factor being omitted) to the product $x^{-\nu} J_{n+\nu}(ix)$, where J is a Bessel function. When $k = 0$, the integral equation (3) reduces to the known form

$$C_n^\nu(\cos z) = \lambda_1 \int_0^\infty e^{it\cos z}\, t^{\nu-1} J_{n+\nu}(t)\, dt.$$

5. *The functions of the elliptic-hypercylinder.*—Let us consider a four-dimensional space, where the Cartesian coordinates are x, y, z, t, and make the change of variables

$$\begin{aligned} x &= \sin\rho \;\sin\sigma \;\cos\phi \\ y &= \sin\rho \;\sin\sigma \;\sin\phi \\ z &= i\cos\rho \;\cos\sigma \\ t &= t. \end{aligned}$$

The hypersurfaces $t = \text{const.}$ and $\phi = \text{const.}$ are hyperplanes, and the hypersurfaces $\rho = \text{const.}$ (or $\sigma = \text{const.}$) are hypercylinders parallel to the t-axis. In three-dimensional space these are the hyperboloids of revolution

$$\frac{x^2 + y^2}{\sin^2\rho} - \frac{z^2}{\cos^2\rho} = 1.$$

We shall term these hypersurfaces, hyperbolic-hypercylinders, which by a slight change of notations become elliptic-hypercylinders.

Laplace's equation $\Delta U = 0$ with four variables is readily found to be in this new system

$$0 = \frac{\partial}{\partial\rho}\left(\sin\rho\;\sin\sigma\frac{\partial U}{\partial\rho}\right) - \frac{\partial}{\partial\sigma}\left(\sin\rho\;\sin\sigma\frac{\partial U}{\partial\sigma}\right) + \frac{\cos^2\rho - \cos^2\sigma}{\sin\rho\,\sin\sigma}\frac{\partial^2 U}{\partial\phi^2}$$

$$+ \sin\rho\,\sin\sigma\,(\cos^2\rho - \cos^2\sigma)\frac{\partial^2 U}{\partial t^2}.$$

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We can try to solve it by taking

$$U(\rho, \sigma, \phi, t) = \cos m\phi\, e^{kt} \sin^m \rho \sin^m \sigma\; V_{m,h}(\rho, \sigma)$$

where the function $V_{m,h}(\rho, \sigma)$, which can be called a function of the hyperbolic (or elliptic) hypercylinder, satisfies the partial differential equation

$$\frac{\partial^2 V}{\partial \rho^2} - \frac{\partial^2 V}{\partial \sigma^2} + (2m+1)\cot\rho\,\frac{\partial V}{\partial \rho} - (2m+1)\cot\sigma\,\frac{\partial V}{\partial \rho} + h^2(\cos^2\rho - \cos^2\sigma)\,V = 0 \quad\ldots\ldots\ldots\ldots(6)$$

It is obvious that we can take, as a solution of this equation (of which I made a general study in *Comptes Rendus Académie des Sciences*, January 1922), a product of a function of ρ alone, $y_1(\rho)$, and of a function of σ alone, $y_2(\sigma)$, which functions shall satisfy ordinary differential equations. That for $y_1(\rho)$ is

$$\frac{d^2 y_1}{d\rho^2} + (2m+1)\cot\rho\,\frac{dy_1}{d\rho} + (h^2\cos^2\rho + \lambda)\,y_1 = 0,$$

whose λ is an arbitrary constant, the equation for $y_2(\sigma)$ being exactly similar. Now this is exactly the differential equation for Mathieu functions of higher order, so that we obtain the following interesting result: a solution of equation (6) is

$$M_n^{m+\frac{1}{2}}(\rho\,;\,h)\;M_n^{m+\frac{1}{2}}(\sigma\,;\,h),$$

or, in other words, *the product of two Mathieu functions of higher order is an elliptic-hypercylinder function.* It is analogous to the fact that the product of two Gegenbauer's polynomials is a harmonic hyperspherical function.

If m is zero, the corresponding function will be a zonal one; we have then to consider a function $M_n^{\frac{1}{2}}$, which reduces, when k is zero, to the Legendre polynomial $P_n(\cos z)$, itself a zonal spherical function.

6. *The general integral-equation.*—It is easy to verify that our functions are solutions of other integral equations, analogous to (3), but with new nuclei. For instance, generalising a result of the theory of Mathieu functions, we can write

$$E_n^\nu(z) = \lambda \int_{-\pi}^{\pi} (\cos z + \cos u)^{-\nu} J_\nu\left[ik(\cos z + \cos u)\right]\sin^{2\nu} u\, E_n^\nu(u)\,du.$$

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But it is a matter of greater interest to extend this result, and to consider the following problem: to find the general function $G(z, u)$ such that

$$E_n^\nu(z) = \lambda \int_{-\pi}^{\pi} G(z, u) \sin^{2\nu} u \, E_n^\nu(u) \, du.$$

It is obvious, first of all, that G must be periodic and even in z and u; afterwards, by a very simple method, using the differential equation for E and integrating by parts, we can show that G must satisfy the partial differential equation

$$\frac{\partial^2 G}{\partial z^2} - \frac{\partial^2 G}{\partial u^2} + 2\nu \cot z \frac{\partial G}{\partial z} - 2\nu \cot u \frac{\partial G}{\partial u} + k^2 (\cos^2 z - \cos^2 u) G = 0.$$

But this is equation (6) for elliptic-hypercylinder functions, where $2m + 1 = 2\nu$, or $m = \nu - \frac{1}{2}$, and $h = k$. Hence the very curious result: even Mathieu functions of higher order are solutions of the homogeneous integral equation

$$y(z) = \lambda \int_{-\pi}^{\pi} V_{\nu - \frac{1}{2}, k}(z, u) \sin^{2\nu} u \, y(u) \, du,$$

where V is an elliptic-hypercylinder function, even and periodic in z and u.