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MULTIPLICITY AND ŁOJASIEWICZ EXPONENT OF GENERIC LINEAR SECTIONS OF MONOMIAL IDEALS

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Abstract

We obtain a characterisation of the monomial ideals $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ of finite colength that satisfy the condition $e(I) = \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I)$, where $\mathcal{L}_0^{(1)}(I), \ldots, \mathcal{L}_0^{(n)}(I)$ is the sequence of mixed Łojasiewicz exponents of I and $e(I)$ is the Samuel multiplicity of I. These are the monomial ideals whose integral closure admits a reduction generated by homogeneous polynomials.

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1. Introduction

Let $(R, \mathbf{m})$ denote a local ring of dimension n. Let I be an $\mathbf{m}$-primary ideal of R. There are two important numbers attached to I: the multiplicity of I, denoted by $e(I)$ (see for instance [9, 14] or [24]), and the Łojasiewicz exponent of I, that is usually denoted by $\mathcal{L}_0(I)$ (see [15, 22, 23]). We shall also refer to $\mathbf{m}$-primary ideals as ideals of finite colength. We recall that $\mathcal{L}_0(I)$ was originally defined for ideals of the ring $\mathcal{O}_n$ of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$ around the origin. That is, if I is generated by $g_1, \ldots, g_r \in \mathcal{O}_n$, then $\mathcal{L}_0(I)$ is defined as the infimum of all positive real numbers α such that

$$\|x\|^{\alpha} \leq C \sup_i |g_i(x)|$$

for some constant $C > 0$ and all x that belong to some open neighbourhood of the origin in $\mathbb{C}^n$. Lejeune and Teissier showed in [15] a relation between $\mathcal{L}_0(I)$ and the asymptotic Samuel function of I and, consequently, with the integral closure of I. This relation is the motivation of the definition of $\mathcal{L}_0(I)$ for an arbitrary ideal I of finite colength in a local ring $(R, \mathbf{m})$. We shall now explain this more precisely.

Fix a local ring $(R, \mathbf{m})$. Let I be an ideal of R and let $h \in R$. Then the *order of h with respect to I* is defined as $\mathrm{ord}_I(h) = \sup\{r \geq 1 : h \in I^r\}$. By convention, $\mathrm{ord}_I(0) = +\infty$.

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It is proven in [15, Section 0.2] and [17] that the sequence $\{r^{-1}\operatorname{ord}_I(h^r)\}_{r\geq 1}$ has a limit in $\mathbb{R}_{\geq 0}\cup\{+\infty\}$. The *asymptotic Samuel function* of I is defined as the function $\overline{\nu}_I : R \to \mathbb{R}_{\geq 0}\cup\{+\infty\}$ given by

$$\overline{\nu}_I(h) = \lim_{r\to\infty}\frac{\operatorname{ord}_I(h^r)}{r}$$

for all $h \in R$, where we set $\overline{\nu}_I(0) = +\infty$. We remark that $\overline{\nu}_I(h) = 0$ for all $h \notin \sqrt{I}$. The number $\overline{\nu}_I(h)$ is also known as the *reduced order of h with respect to I*. It is known that the range of $\overline{\nu}_I$ is $\mathbb{Q}_{\geq 0}\cup\{+\infty\}$ (see for instance [14, Section 10]).

If I is an ideal of $\mathcal{O}_n$ such that $\overline{I}$ is a monomial ideal (that is, $\overline{I}$ is generated by monomials), then $\overline{\nu}_I$ can be expressed in terms of the Newton polyhedron of I (see [2]). If I and J are ideals of R, then we define

$$\overline{\nu}_I(J) = \min\{\overline{\nu}_I(h) : h \in J\}.$$

The result of Lejeune–Teissier to which we referred before states: if I is an ideal of $\mathcal{O}_n$ of finite colength and $\mathbf{m}_n$ denotes the maximal ideal of $\mathcal{O}_n$, then

$$\mathcal{L}_0(I) = \frac{1}{\overline{\nu}_I(\mathbf{m}_n)}. \tag{1.1}$$

The above equality is used as the definition of the Łojasiewicz exponent of an arbitrary ideal I of finite colength in a local ring $(R, \mathbf{m})$. We also remark that the equality (1.1) is equivalent to $\mathcal{L}_0(I) = \inf\{r/s : r, s \in \mathbb{Z}_{\geq 1}, \mathbf{m}_n^r \subseteq \overline{I^s}\}$ (see [15, Section 7]).

The notion of multiplicity of an ideal was extended by Risler and Teissier [22] to sequences of $\mathbf{m}$-primary ideals, thus leading to the notion of *mixed multiplicity* $e(I_1, \ldots, I_n)$ of n $\mathbf{m}$-primary ideals in R (see [14, Section 17.4]). The motivation for this generalisation has its origin in the study developed by Teissier of the Milnor number of the restriction of a given function germ $f \in \mathcal{O}_n$ to generic subspaces of $\mathbb{C}^n$ of different dimensions [22]. The study of mixed multiplicities of ideals was further developed by Rees [18]. Let (R, m) be a local ring and let $I_1, \ldots, I_n$ be ideals of R. Then we define

$$\sigma(I_1, \ldots, I_n) = \sup_{r\in\mathbb{Z}_{\geq 1}} e(I_1 + \mathbf{m}^r, \ldots, I_n + \mathbf{m}^r). \tag{1.2}$$

In general, $\sigma(I_1, \ldots, I_n)$ can be infinite. In [3, page 393], we characterised the finiteness of $\sigma(I_1, \ldots, I_n)$. From (1.2), it is clear that if each ideal has finite colength, then $\sigma(I_1, \ldots, I_n)$ exists and it is equal to the usual mixed multiplicity $e(I_1, \ldots, I_n)$. We remark that $\sigma(I_1, \ldots, I_n)$ coincides with the mixed multiplicity defined by Rees [19, page 181] and we also refer to $\sigma(I_1, \ldots, I_n)$ as the *Rees mixed multiplicity of* $I_1, \ldots, I_n$.

Analogous to the generalisation of the notion of multiplicity leading to mixed multiplicities, we introduced in [4] the notion of Łojasiewicz exponent of n ideals $I_1, \ldots, I_n$ in a local ring $(R, \mathbf{m})$ of dimension n (see Section 2 and [5] for details). We denote this number by $\mathcal{L}_0(I_1, \ldots, I_n)$. In order to define $\mathcal{L}_0(I_1, \ldots, I_n)$, the ideals $I_1, \ldots, I_n$ are not assumed to have finite colength but the condition $\sigma(I_1, \ldots, I_n) < \infty$ is needed. Therefore, if I denotes an ideal of R of finite colength and $i \in \{1, \ldots, n\}$, then we define the *ith relative Łojasiewicz exponent of I* as

$$\mathcal{L}_0^{(i)}(I) = \mathcal{L}_0(I, \ldots, I, \mathbf{m}, \ldots, \mathbf{m}),$$

where I is repeated i times and $\mathbf{m}$ is repeated $n - i$ times. In particular, we have $\mathcal{L}_0^{(n)}(I) = \mathcal{L}_0(I)$ and $\mathcal{L}_0^{(1)}(I) = \mathrm{ord}(I)$.

Let $(R, \mathbf{m})$ denote an equicharacteristic regular local ring of dimension $n \geq 2$ with residue field $\mathbf{k}$, $\mathrm{char}(\mathbf{k}) = 0$. Let I be an ideal of R of finite colength and let us fix an index $i \in \{1, \ldots, n\}$. Hickel proved in [12, Théorème 1.1] that there exists a Zariski open set $U^{(i)}$ of the Grassmannian $G_{\mathbf{k}}(i, n)$ of subspaces of dimension i of $\mathbf{k}^n$ such that $\overline{\nu}_{IR_H}(\mathbf{m}_H)$ does not depend on H for all $H \in U^{(i)}$. Here we assume that H is the zero set of the collection of $\mathbf{k}$-linear forms $h_1, \ldots, h_{n-i}$, the quotient ring $R/\langle h_1, \ldots, h_{n-i}\rangle$ is denoted by R_H and $\mathbf{m}_H$ is the maximal ideal of R_H. By [6, Lemma 4.9], we have $\mathcal{L}_0^{(i)}(I) = (\overline{\nu}_{IR_H}(\mathbf{m}_H))^{-1}$ for all $i = 1, \ldots, n$. We remark that $(\overline{\nu}_{IR_H}(\mathbf{m}_H))^{-1}$ is denoted by $\nu_I^{(i)}$ in [12] for all $i = 1, \ldots, n$.

Moreover, Hickel proved in [12] that

$$e(I) \leq \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I). \tag{1.3}$$

We remark that this inequality was generalised in [6, Theorem 4.7]. There now arises the problem of characterising when equality holds in (1.3) and understanding the structure of the ideals satisfying that equality. This was already done by Hickel in dimension $n = 2$ [12, Proposition 5.1].

In this article, we consider this problem in the case that I is a monomial ideal of $\mathcal{O}_n$ or $\mathbb{C}[[x_1, \ldots, x_n]]$ (see Theorem 3.5). We prove that the only monomial ideals that satisfy the equality $e(I) = \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I)$ are those such that $\overline{I}$ admits a reduction generated by homogeneous polynomials. This condition reduces considerably the possibilities for the shape of the Newton polyhedron of the ideal. As is seen in Section 3, we translate this problem into a combinatorial problem that, at first sight, is independent from Łojasiewicz exponents and captures a special class of monomial ideals.

2. Mixed Łojasiewicz exponents

In this section we recall briefly the notion of mixed Łojasiewicz exponent and some basic facts about this concept.

Let $(R, \mathbf{m})$ denote a Noetherian local ring of dimension $n \geq 1$ and let $I_1, \ldots, I_n$ be ideals of R such that $\sigma(I_1, \ldots, I_n) < \infty$. Let J be a proper ideal of R. Define

$$r_J(I_1, \ldots, I_n) = \min\{r \in \mathbb{Z}_{\geq 0} : \sigma(I_1, \ldots, I_n) = \sigma(I_1 + J^r, \ldots, I_n + J^r)\}.$$

We recall that $\sigma(I_1, \ldots, I_n)$ denotes the Rees mixed multiplicity of $I_1, \ldots, I_n$, defined in (1.2).

If we suppose that $I_1 = \cdots = I_n = I$ for some ideal I of R of finite colength and we assume that R is formally equidimensional, then we can apply Rees's multiplicity theorem (see [11, page 147] or [14, page 222]) to deduce that

$$r_J(I, \ldots, I) = \min\{r \in \mathbb{Z}_{\geq 0} : J^r \subseteq \overline{I}\}.$$

Definition 2.1 [5]. Under the above conditions, we define the *Łojasiewicz exponent of* $I_1, \dots, I_n$ *with respect to* J, denoted by $\mathcal{L}_J(I_1, \dots, I_n)$, as

$$\mathcal{L}_J(I_1, \dots, I_n) = \inf_{s \geq 1} \frac{r_J(I_1^s, \dots, I_n^s)}{s}.$$

We also refer to the number $\mathcal{L}_J(I_1, \dots, I_n)$ as the *mixed Łojasiewicz exponent of* $I_1, \dots, I_n$ *with respect to* J. When $J = \mathbf{m}$, we denote this number by $\mathcal{L}_0(I_1, \dots, I_n)$.

Let us observe that in order to define $\mathcal{L}_J(I_1, \dots, I_n)$, we only need the ring R to be local and Noetherian and no additional condition on R is assumed. As mentioned in the Introduction, if I is an ideal of finite colength of R, then we can associate to I the vector $\mathcal{L}_0^*(I) = (\mathcal{L}_0^{(n)}(I), \dots, \mathcal{L}_0^{(1)}(I))$, where $\mathcal{L}_0^{(i)}(I) = \mathcal{L}_0(I, \dots, I, \mathbf{m}, \dots, \mathbf{m})$, with I repeated i times and $\mathbf{m}$ repeated $n - i$ times, $i = 1, \dots, n$. The number $\mathcal{L}_J^{(i)}(I)$ is defined analogously, for all $i = 1, \dots, n$, and any ideal J of R of finite colength.

The following result is proven in [6, Corollary 4.11].

Theorem 2.2. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring and let I, J be ideals of R of finite colength. Let us suppose that the residue field $k = R/\mathbf{m}$ is infinite. Then $\mathcal{L}_J^{(1)}(I) \leq \cdots \leq \mathcal{L}_J^{(n)}(I)$.*

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$. If $k \in \mathbb{Z}_{\geq 0}^n$, then we write x^k to denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$. If $h \in \mathcal{O}_n$ and the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$, then the *support* of h, denoted by $\mathrm{supp}(h)$, is defined as the set of those $k \in \mathbb{Z}_{\geq 0}^n$ such that $a_k \neq 0$. If $h \neq 0$, then we define the *Newton polyhedron of* h, denoted by $\Gamma_+(h)$, as the convex hull in $\mathbb{R}_{\geq 0}^n$ of $\{k + v : k \in \mathrm{supp}(h),\ v \in \mathbb{R}_{\geq 0}^n\}$. If $h = 0$, then we set $\Gamma_+(h) = \emptyset$. If I is an ideal of $\mathcal{O}_n$, then the *Newton polyhedron of* I, denoted by $\Gamma_+(I)$, is defined as the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_s)$, where we assume that $g_1, \dots, g_s$ is a generating system of I. We see immediately that the definition of $\Gamma_+(I)$ does not depend on the chosen generating system of I.

Let us fix a subset $\mathtt{L} \subseteq \{1, \dots, n\}$, $\mathtt{L} \neq \emptyset$. Then we denote by $\mathbb{R}_{\mathtt{L}}^n$ the set of those $k \in \mathbb{R}^n$ such that $k_j = 0$ for all $j \notin \mathtt{L}$. If A denotes any subset of $\mathbb{R}^n$, then we set $A^{\mathtt{L}} = A \cap \mathbb{R}_{\mathtt{L}}^n$. The cardinal of $\mathtt{L}$ will be denoted by $|\mathtt{L}|$.

If I is a monomial ideal of $\mathcal{O}_n$, then we denote by $I^{\mathtt{L}}$ the ideal of $\mathcal{O}_n$ generated by the monomials $x^k \in I$ such that $k \in \mathbb{R}_{\mathtt{L}}^n$. If $\mathrm{supp}(I) \cap \mathbb{R}_{\mathtt{L}}^n = \emptyset$, then we set $I^{\mathtt{L}} = 0$. If I is a monomial ideal of $\mathcal{O}_n$ of finite colength, then we have $I^{\mathtt{L}} \neq 0$ for all $\mathtt{L} \subseteq \{1, \dots, n\}$, $\mathtt{L} \neq \emptyset$.

The next result gives a description of the sequence $\mathcal{L}_0^*(I)$ in terms of $\Gamma_+(I)$ when I is a monomial ideal of finite colength of $\mathcal{O}_n$.

Theorem 2.3 [6]. *Let I be a monomial ideal of $\mathcal{O}_n$ of finite colength. Let $i \in \{1, \dots, n\}$. Then*

$$\mathcal{L}_0^{(i)}(I) = \max\{\mathrm{ord}(I^{\mathtt{L}}) : \mathtt{L} \subseteq \{1, \dots, n\},\ |\mathtt{L}| = n - i + 1\}.$$

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The following result is motivated by [12, Théorème 1.1] and, in turn, the case $J = \mathbf{m}$ is the motivation of the problem considered in this article. This result can be seen as a particular case of [6, Theorem 4.7] (see [6, Corollary 4.8]). We also refer to [12, Remarque 4.3(3)] for the deduction of inequality (2.1) using different techniques in a slightly different context.

PROPOSITION 2.4 [6]. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring and let I and J be ideals of R of finite colength. Then*

$$\frac{e(I)}{e(J)} \leq \mathcal{L}_J^{(1)}(I) \cdots \mathcal{L}_J^{(n)}(I). \tag{2.1}$$

In the main result, Theorem 3.5, we characterise when equality holds in (2.1) in the case when I is a monomial ideal of $\mathcal{O}_n$ and J is the maximal ideal of $\mathcal{O}_n$. As we will see, Theorem 3.5 can be considered as a purely combinatorial result.

3. Main result

In this section we expose the concepts and results from [21] that we need in the proof of the main result.

If A is a subset of $\mathbb{R}^n$, then we denote by $\mathrm{Conv}(A)$ the convex hull of A in $\mathbb{R}^n$. If $P \subseteq \mathbb{R}^n$, then we say that P is a *polytope* when there exists a finite subset $A \subseteq \mathbb{R}^n$ such that $P = \mathrm{Conv}(A)$. If A is contained in $\mathbb{Z}^n$, then $\mathrm{Conv}(A)$ is said to be a *lattice polytope*.

If P is a polytope in $\mathbb{R}^n$, then the *dimension* of P is defined as the minimum dimension of an affine subspace containing P.

If P is any subset of $\mathbb{R}^n$, then we denote by $\mathbb{C}[P]$ the family of polynomial maps $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\mathrm{supp}(h) \subseteq P$. Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n-tuple of subsets of $\mathbb{R}^n$. We denote by $\mathbb{C}_n[\mathbf{P}]$ the set of polynomial maps $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ such that $\mathrm{supp}(F_i) \subseteq P_i$ for all $i = 1, \dots, n$. We can identify $\mathbb{C}_n[\mathbf{P}]$ with a finite-dimensional vector space $\mathbb{C}^N$, for a sufficiently large positive integer N, by associating to each map $F \in \mathbb{C}_n[\mathbf{P}]$ the vector formed by the coefficients of F. Under this identification, we say that a property holds for a generic $F \in \mathbb{C}_n[\mathbf{P}]$ when the property holds in a dense Zariski open subset of $\mathbb{C}^N$.

Given a lattice polytope $P \subseteq \mathbb{R}^n$, then we say that P is *cornered* when, for all $j = 1, \dots, n$, there exists some $k \in P$ such that $k_j = 0$ (see [21, page 119]). If $P \subseteq \mathbb{R}^n_{\geq 0}$ and f_P denotes the polynomial obtained as the sum of all terms x^k such that $k \in P \cap \mathbb{Z}^n_{\geq 0}$, then we observe that P is cornered if and only if f_P is not divisible by x_j for all $j = 1, \dots, n$.

Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n-tuple of lattice polytopes in $\mathbb{R}^n$; then $\mathbf{P}$ is said to be *cornered* when P_i is cornered for all $i = 1, \dots, n$. We say that $\mathbf{P}$ is *nice* when $F^{-1}(0)$ is finite for a generic map $F \in \mathbb{C}_n[\mathbf{P}]$.

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that $F^{-1}(0)$ is finite, then we denote by $m(F)$ the number of roots of F counted with multiplicities. That is, if $I(F)$ denotes the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ generated by the component functions of F, then, applying [7, Ch. 4, Corollary 2.5],

$$m(F) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \ldots, x_n]}{I(F)}.$$

If $K \subseteq \mathbb{R}^n$, then we denote by $\mathrm{Vol}_n(K)$ the n-dimensional volume of K. Let $C_1, \ldots, C_n$ be n polytopes of $\mathbb{R}^n$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$. Let $\lambda_1 C_1 + \cdots + \lambda_n C_n = \{\lambda_1 k_1 + \cdots + \lambda_n k_n : k_i \in C_i,\ i = 1, \ldots, n\}$. It is a classical result from convex geometry that $\mathrm{Vol}_n(\lambda_1 C_1 + \cdots + \lambda_n C_n)$ is a homogeneous polynomial of degree n in the variables $\lambda_1, \ldots, \lambda_n$ (see for instance [7, page 337]). The n-dimensional mixed volume of $C_1, \ldots, C_n$ is defined as the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial $\mathrm{Vol}_n(\lambda_1 C_1 + \cdots + \lambda_n C_n)$. We denote this number by $M(C_1, \ldots, C_n)$. Let us recall some elementary properties of this number (taken from [20, page 112]; see also [7, Ch. 7, Section 4]):

(1) $M(C_1, \ldots, C_n)$ is symmetric and linear in each variable;
(2) $M(C_1, \ldots, C_n) \geq 0$ and $M(C_1, \ldots, C_n) = 0$ if and only if $\dim(\sum_{i \in I} C_i) < |I|$ for some nonempty subset $I \subseteq \{1, \ldots, n\}$, where $|I|$ denotes the cardinal of I;
(3) $M(C_1, \ldots, C_n) \in \mathbb{Z}_{\geq 0}$, if C_i is a lattice polytope, for all $i = 1, \ldots, n$;
(4) $M(C, \ldots, C) = n!\, \mathrm{Vol}_n(C)$ for any polytope $C \subseteq \mathbb{R}^n$.

We refer to [7, 10, 20] for more information about $M(C_1, \ldots, C_n)$.

If P is a polytope in $\mathbb{R}^n$, then we denote $\mathrm{Conv}(P \cup \{0\})$ by P^0. If $\mathbf{P} = (P_1, \ldots, P_n)$ is an n-tuple of polytopes of $\mathbb{R}^n$, then we define $\mathbf{P}^0 = (P_1^0, \ldots, P_n^0)$. In particular, it makes sense to speak about the mixed volumes $M(\mathbf{P})$ and $M(\mathbf{P}^0)$. Let us remark that $\mathbf{P}^0$ is always cornered. By [16, Theorem 2.4], if $F \in \mathbb{C}_n[\mathbf{P}]$, $\mathbf{P}$ is a lattice polytope and $F^{-1}(0)$ is finite, then $m(F) \leq M(\mathbf{P}^0)$. As remarked in [21, page 119], the conditions 'nice' and 'cornered' on $\mathbf{P}$ are independent conditions. The following result tells us that both properties together in $\mathbf{P}$ imply $m(F) = M(\mathbf{P}^0)$ for a generic $F \in \mathbb{C}_n[\mathbf{P}]$.

Theorem 3.1 [21, page 119]. *Let $\mathbf{P} = (P_1, \ldots, P_n)$ be an n-tuple of lattice polytopes of $\mathbb{R}^n_{\geq 0}$. Let us suppose that $\mathbf{P}$ is nice and cornered. Then a generic polynomial map $F \in \mathbb{C}_n[\mathbf{P}]$ has exactly $M(\mathbf{P}^0)$ roots in $\mathbb{C}^n$, counting multiplicities.*

The previous theorem is proven in [21] in a more general context (for polynomial maps with coefficients in a given algebraically closed field of any characteristic).

Let $\mathbf{P} = (P_1, \ldots, P_n)$ and $\mathbf{Q} = (Q_1, \ldots, Q_n)$ be n-tuples of polytopes in $\mathbb{R}^n_{\geq 0}$. We write $\mathbf{P} \subseteq \mathbf{Q}$ to denote that $P_i \subseteq Q_i$ for all $i = 1, \ldots, n$. We also define the n-tuples of subsets $\mathbf{P} \cap \mathbf{Q} = (P_1 \cap Q_1, \ldots, P_n \cap Q_n)$ and $\mathbf{Q} \smallsetminus \mathbf{P} = (Q_1 \smallsetminus P_1, \ldots, Q_n \smallsetminus P_n)$. If $\mathrm{L} \subseteq \{1, \ldots, n\}$, $\mathrm{L} \neq \emptyset$, then we set $\mathbf{P}^{\mathrm{L}} = (P_1 \cap \mathbb{R}^n_{\mathrm{L}}, \ldots, P_n \cap \mathbb{R}^n_{\mathrm{L}})$.

Next we recall a particular case of a definition introduced in [21, page 120].

Definition 3.2. Let $\mathbf{P}$ and $\mathbf{Q}$ be n-tuples of polytopes in $\mathbb{R}^n_{\geq 0}$ such that $\mathbf{Q}$ is nice and cornered. We say that $\mathbf{P}$ *counts* $\mathbf{Q}$ when:

(1) $\mathbf{P} \subseteq \mathbf{Q}$;
(2) $\mathbf{P}$ is nice;

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(3) for any $F \in \mathbb{C}_n[\mathbf{Q}\smallsetminus\mathbf{P}]$, the map $F + F'$ has a finite zero set and $m(F + F') = M(\mathbf{Q}^0)$ for a generic $F' \in \mathbb{C}_n[\mathbf{P}]$.

In particular, if $\mathbf{P}$ counts $\mathbf{Q}$, then $m(F) = M(\mathbf{Q}^0)$ for a generic $F \in \mathbb{C}_n[\mathbf{P}]$ and therefore $M(\mathbf{P}^0) = M(\mathbf{Q}^0)$ provided that $\mathbf{P}$ is also cornered, by Theorem 3.1.

Definition 3.3 [21, page 124]. Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n-tuple of polytopes in $\mathbb{R}^n$. The *support* of $\mathbf{P}$ is defined as the set of indices $i \in \{1, \dots, n\}$ such that $P_i \neq \emptyset$. We denote this set by $\mathrm{supp}(\mathbf{P})$. Let $J \subseteq \{1, \dots, n\}$. Then J is said to be *essential* for $\mathbf{P}$ when the following conditions hold:

(1) $J \subseteq \mathrm{supp}(\mathbf{P})$;
(2) $\dim(\sum_{j\in J} P_j) = |J| - 1$;
(3) for all nonempty proper subsets $J' \subset J$, we have $\dim(\sum_{j\in J'} P_j) \geq |J'|$.

Given a closed subset $P \subseteq \mathbb{R}^n_{\geq 0}$ and a vector $w \in \mathbb{R}^n$, we define $\ell(w, P) = \min\{\langle w, k\rangle : k \in P\}$, where $\langle\,,\rangle$ stands for the standard inner product in $\mathbb{R}^n$. If $\ell(w, P) > -\infty$, then we denote by P^w, or by $\Delta(w, P)$, the subset of P formed by those $k \in P$ such that $\langle w, k\rangle = \ell(w, P)$. The sets of the form $\Delta(w, P)$, for some $w \in \mathbb{R}^n$, are called *faces* of P. If $\mathbf{P} = (P_1, \dots, P_n)$ is an n-tuple of lattice polytopes contained in $\mathbb{R}^n_{\geq 0}$, then we denote the n-tuple $(P_1^w, \dots, P_n^w)$ by $\mathbf{P}^w$.

Next we state a particular case of [21, Theorem 7] that we need for our purposes (we remark that this theorem is stated for polynomials with coefficients in any algebraically closed field). Given two n-tuples of polytopes $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbb{R}^n$ such that $\mathbf{P} \subseteq \mathbf{Q}$, this result gives a purely combinatorial characterisation of when $\mathbf{P}$ counts $\mathbf{Q}$.

Theorem 3.4 [21, page 127]. *Let $\mathbf{P}$ and $\mathbf{Q}$ be n-tuples of lattice polytopes contained in $\mathbb{R}^n_{\geq 0}$ such that $\mathbf{P} \subseteq \mathbf{Q}$. Let us suppose that $\mathbf{Q}$ is nice and cornered and $M(\mathbf{Q}^0) > 0$. Then $\mathbf{P}$ counts $\mathbf{Q}$ if and only if* $\mathrm{supp}(\mathbf{P} \cap \mathbf{Q}^w)$ *contains an essential subset for $\mathbf{Q}^w$ for all $w \in \mathbb{R}^n\smallsetminus\mathbb{R}_{\geq 0}$.*

If I is a monomial ideal of $\mathcal{O}_n$ of finite colength, then we define

$$a_i(I) = \max\{\mathrm{ord}(I^{\mathrm{L}}) : \mathrm{L} \subseteq \{1, \dots, n\}, |\mathrm{L}| = n - i + 1\}$$

for any $i \in \{1, \dots, n\}$. Observe that $a_1(I) \leq \cdots \leq a_n(I)$ and that the definition of $a_i(I)$ depends only on $\Gamma_+(I)$. Therefore, $a_i(I) = a_i(\overline{I})$ for all $i = 1, \dots, n$. We also define the vector $\mathbf{a}(I) = (a_1(I), \dots, a_n(I))$. For example, if $I = \langle xyz, x^a, y^b, z^c\rangle \subseteq \mathcal{O}_3$, where $3 < a \leq b \leq c$, then $\mathbf{a}(I) = (3, b, c)$.

We recall that $a_i(I) = \mathcal{L}_0^{(i)}(I)$ for all $i = 1, \dots, n$, by Theorem 2.3; however, this equality is not used in the main result.

If $k \in \mathbb{R}^n$, then we denote by $|k|$ the sum of the coordinates of k.

Theorem 3.5. *Let I be a monomial ideal of finite colength of $\mathcal{O}_n$. Then*

$$e(I) \leq a_1(I) \cdots a_n(I) \tag{3.1}$$

and equality holds if and only if there exist polynomials $g_1, \dots, g_n \in \mathbb{C}[x_1, \dots, x_n]$ such that g_i is homogeneous of degree $a_i(I)$, for all $i = 1, \dots, n$, and $\overline{I} = \overline{\langle g_1, \dots, g_n\rangle}$.

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Proof. Since $e(I) = e(\overline{I})$ and $a_i(I) = a_i(\overline{I})$, for all $i = 1, \dots, n$, we can assume that I is integrally closed. Then $I = \langle x^k : k \in \Gamma_+(I)\rangle$ (see for instance [14, Proposition 1.4.6]). Let $a_i = a_i(I)$ for all $i = 1, \dots, n$. Let us denote by D_i the convex hull in $\mathbb{R}^n$ of the set

$$\{k \in \mathrm{supp}(I^{\mathrm{L}}) : |k| = a_i, \mathrm{L} \subseteq \{1, \dots, n\}, |\mathrm{L}| = n - i + 1\}$$

for all $i = 1, \dots, n$. By the definition of a_i, we have $a_i = \mathrm{ord}(I^{\mathrm{L}})$ for some $\mathrm{L} \subseteq \{1, \dots, n\}$ such that $|\mathrm{L}| = n - i + 1$ for all $i = 1, \dots, n$. In particular, $D_i \neq \emptyset$ for all $i = 1, \dots, n$. Let $\mathbf{D}$ denote the n-tuple of polytopes $(D_1, \dots, D_n)$.

If $\alpha \in \mathbb{R}_{\geq 0}$, let $\Delta(\alpha)$ denote the convex hull in $\mathbb{R}^n$ of the set $\{k \in \mathbb{Z}^n_{\geq 0} : |k| = \alpha\}$ and let $\mathbf{\Delta}$ denote the n-tuple of polytopes $(\Delta(a_1), \dots, \Delta(a_n))$. It is clear that Δ is nice and cornered and $M(\mathbf{\Delta}^0) = a_1 \cdots a_n > 0$. Clearly, we have $\mathbf{D} \subseteq \mathbf{\Delta}$. We claim that $\mathbf{D}$ counts $\mathbf{\Delta}$. To see this, we will apply Theorem 3.4.

Let us fix a vector $w = (w_1, \dots, w_n) \in \mathbb{R}^n \smallsetminus \mathbb{R}^n_{\geq 0}$ and let $w_0 = \min\{w_1, \dots, w_n\}$. Let L_w denote the set of indices $\{i : w_i = w_0\}$. It is immediate that $\ell(w, \Delta(a_j)) = a_j w_0$ and $\Delta(a_j)^w = \Delta(a_j) \cap \mathbb{R}^n_{\mathrm{L}_w}$ for all $w \in \mathbb{R}^n \smallsetminus \mathbb{R}^n_{\geq 0}$ and all $j = 1, \dots, n$. Then $\mathbf{\Delta}^w = \mathbf{\Delta}^{\mathrm{L}_w}$ for all $w \in \mathbb{R}^n \smallsetminus \mathbb{R}^n_{\geq 0}$. In particular, we have the equality

$$\{\mathbf{\Delta}^w : w \in \mathbb{R}^n \smallsetminus \mathbb{R}^n_{\geq 0}\} = \{\mathbf{\Delta}^{\mathrm{L}} : \mathrm{L} \subseteq \{1, \dots, n\}, \mathrm{L} \neq \emptyset\}.$$

Fix a subset $\mathrm{L} \subseteq \{1, \dots, n\}$, $\mathrm{L} \neq \emptyset$. Let $\alpha = |\mathrm{L}|$ and consider the set of indices $J_{\mathrm{L}} = \{n + 1 - \alpha, \dots, n\}$. Let us show that $J_{\mathrm{L}} \subseteq \mathrm{supp}(\mathbf{D} \cap \mathbf{\Delta}^{\mathrm{L}})$ and J_{L} is an essential set for $\mathbf{\Delta}^{\mathrm{L}}$.

If $i \in J_{\mathrm{L}}$, then $\alpha \geq n - i + 1$ and thus $\mathrm{ord}(I^{\mathrm{L}}) \leq \mathrm{ord}(I^{\mathrm{L}'}) \leq a_i$ for all $\mathrm{L}' \subseteq \mathrm{L}$ such that $|\mathrm{L}'| = n - i + 1$. In particular, if $\mathrm{L}' \subseteq \mathrm{L}$ is any subset such that $|\mathrm{L}'| = n - i + 1$, there exists some $k \in \mathrm{supp}(I^{\mathrm{L}'}) \subseteq \mathrm{supp}(I^{\mathrm{L}})$ such that $|k| = a_i$. Then $D_i \cap \Delta(a_i)^{\mathrm{L}} \neq \emptyset$ for all $i \in J_{\mathrm{L}}$. That is, we have $J_{\mathrm{L}} \subseteq \mathrm{supp}(\mathbf{D} \cap \mathbf{\Delta}^{\mathrm{L}})$. We observe that $\dim \Delta(a)^{\mathrm{L}} = |\mathrm{L}| - 1$ for all $a \in \mathbb{R}_{\geq 0}$. Moreover, $\sum_{j \in J_{\mathrm{L}}} \Delta(a_j)^{\mathrm{L}} = (\Delta(\sum_{j \in J_{\mathrm{L}}} a_j))^{\mathrm{L}}$. In particular, we have $\dim \sum_{j \in J_{\mathrm{L}}} \Delta(a_j)^{\mathrm{L}} = |\mathrm{L}| - 1$. Then we observe that J_{L} satisfies conditions (2) and (3) of the definition of essential subset for $\mathbf{\Delta}^{\mathrm{L}}$ (see Definition 3.3). Thus, we deduce that $\mathbf{D}$ counts $\mathbf{\Delta}$, by Theorem 3.4.

In particular, there exist homogeneous polynomials $g_i \in \mathbb{C}[D_i]$, $i = 1, \dots, n$, such that, for the map $G = (g_1, \dots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$, $G^{-1}(0)$ is finite and $m(G) = M(\mathbf{\Delta}^0) = a_1 \cdots a_n$. Since g_i is homogeneous, for all $i = 1, \dots, n$, and $G^{-1}(0)$ is finite, we conclude that $G^{-1}(0) = \{0\}$. Let $I(G)$ be the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. Then $I(G)$ has finite colength and $e(I(G)) = a_1 \cdots a_n$. If we assume that I is monomial and integrally closed, then $I(G) \subseteq I$. This implies that $a_1 \cdots a_n = e(I(G)) \geq e(I)$.

Then, by Rees's multiplicity theorem (see for instance [11, page 147] or [14, page 222]), the equality $e(I(G)) = e(I)$ holds if and only if $I = \overline{I} = \overline{\langle g_1, \dots, g_n\rangle}$. □

Let G denote a homogeneous polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ such that $G^{-1}(0) = \{0\}$. Denote by $I(G)$ the ideal of $\mathcal{O}_n$ generated by the component functions of G. We remark that the integral closure of $I(G)$ is not always a monomial ideal, as is shown by the map $G : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ given by $g(x, y) = (xy + x^2, y^3)$.

Let I be a monomial ideal of $\mathcal{O}_n$ of finite colength and let $\mathbf{v} = (1, \dots, 1) \in \mathbb{R}^n$. Then we denote the face $\Delta(\mathbf{v}, \Gamma_+(I))$ by $\Delta_0(I)$. Let us observe that the elements $k \in \Gamma_+(I)$ such that $|k| = \mathrm{ord}(I)$ are contained in $\Delta_0(I)$.

REMARK 3.6. Let I be a monomial ideal of $\mathcal{O}_n$ of finite colength satisfying the condition $e(I) = a_1 \cdots a_n$, where $a_i = a_i(I)$, $i = 1, \dots, n$. Then, as we have seen in the proof of Theorem 3.5, the ideal $\overline{I}$ admits a reduction $\langle g_1, \dots, g_n \rangle$, where g_i is a homogeneous polynomial of $\mathbb{C}[x_1, \dots, x_n]$ of degree a_i for all $i = 1, \dots, n$. Let $d = \dim \Delta_0(I)$. Then, as a consequence of [1, Theorem 2.10], where all the reductions of monomial ideals are characterised, it follows that $a_1 = \cdots = a_d = a_{d+1}$. In particular, if $a_1 < a_2$, then $\dim \Delta_0(I) = 0$, that is, the face $\Delta(\mathbf{v}, \Gamma_+(I))$ is a vertex.

Let us also observe that, by [1, Theorem 2.10], the condition $e(I) = a_1 \cdots a_n$ forces the face $\Delta_0(I)$ to intersect all faces of $\Gamma_+(I)$ of dimension n. We conjecture that it is possible to obtain a characterisation of the condition $e(I) = \prod_{i=1}^n a_i(I)$ in terms of some property of the tree determined by the vertices and edges of $\Gamma_+(I)$.

EXAMPLE 3.7. Let us consider the ideal of $\mathcal{O}_3$ given by $I = \langle x^a, y^b, z^c, xy, xz, yz \rangle$, where $2 \leq a \leq b \leq c$. Then $\mathbf{a}(I) = (2, 2, c)$. Moreover, $e(I) = 2 + a + b + c$. Observe that $e(I) \leq 4c$ and equality holds if and only if $a = b = c = 2$.

Here we illustrate Remark 3.6. Let us observe that the face $\Delta_0(I)$ contains the convex hull of the supports of the monomials xy, xz, yz. Hence, $\dim \Delta_0(I) = 2$. Thus, I does not satisfy the relation $e(I) = a_1(I)a_2(I)a_3(I)$ if $c > 2$, by Remark 3.6.

EXAMPLE 3.8. Let I be the ideal of $\mathcal{O}_n$ generated by $x^k, x_1^{a_1}, \dots, x_n^{a_n}$, where $k \in \mathbb{Z}^n_{\geq 0}$, $k \neq 0$, and $a_1, \dots, a_n$ are integers such that $|k| \leq a_1 \leq \cdots \leq a_n$. We recall that $|k|$ denotes the sum of the coordinates of k. Then we have $\mathbf{a}(I) = (k_1 + \cdots + k_n, a_2, \dots, a_n)$ and $e(I) = k_1 a_2 \cdots a_n + \cdots + a_1 \cdots a_{n-1} k_n$. Therefore, $e(I) = \prod_{i=1}^n a_i(I)$ if and only if $a_1 = \cdots = a_n$.

Let $\operatorname{lct}(I)$ denote the log canonical threshold of an ideal I of $\mathcal{O}_n$ and let $\mu(I)$ denote the inverse $1/\operatorname{lct}(I)$, which is also known as the Arnold index of I. By a result of de Fernex *et al.* [8, Theorem 1.4], if I is an ideal of $\mathcal{O}_n$ of finite colength, then $e(I) \geq (n\mu(I))^n$ and equality holds if and only if $\overline{I} = \mathbf{m}_n^{\operatorname{ord}(I)}$. In particular, using (1.3),

$$(n\mu(I))^n \leq e(I) \leq \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I).$$

We recall that, by a result of Howald [13], if I is a monomial ideal, then

$$\mu(I) = \min\{\alpha > 0 : \alpha \mathbf{v} \in \Gamma_+(I)\}, \tag{3.2}$$

where $\mathbf{v} = (1, \dots, 1) \in \mathbb{R}^n$. Let us denote the number on the right-hand side of (3.2) by $\alpha(I)$. Then, as a consequence of (3.1) and [8, Theorem 1.4], we obtain the following conclusion, which is a combinatorial result.

COROLLARY 3.9. *Let I be a monomial ideal of $\mathcal{O}_n$ of finite colength. Then*

$$a_1(I) \cdots a_n(I) \geq (n\, \alpha(I))^n$$

and equality holds if and only if $\overline{I} = \mathbf{m}_n^{\operatorname{ord}(I)}$.

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