A CLASS OF C-TOTALLY REAL SUBMANIFOLDS OF SASAKIAN SPACE FORMS

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Abstract

Recently, Chen defined an invariant $\delta_M$ of a Riemannian manifold $M$. Sharp inequalities for this Riemannian invariant were obtained for submanifolds in real, complex and Sasakian space forms, in terms of their mean curvature. In the present paper, we investigate certain C-totally real submanifolds of a Sasakian space form $M^{2m+1}(c)$ satisfying Chen's equality.


1. Introduction

We consider C-totally real submanifolds $M^n$ of a Sasakian space form $M^{2m+1}(c)$; let $H$ denote the mean curvature vector field of $M^n$ in $M^{2m+1}(c)$. Precise definitions of the concepts used are given in Sections 2 and 3.

In [7] a general best possible inequality was obtained between the main intrinsic invariants of the submanifold $M^n$ on one side, namely its sectional curvature function $K$ and its scalar curvature function $\tau$, and its main extrinsic invariant on the other side, namely its mean curvature function $|H|$. More precisely, in the Sasakian case, Chen’s inequality, relating $K$, $\tau$ and $H$, reads:

$$\inf K \geq \tau - \frac{n^2(n-2)}{2(n-1)} |H|^2 - \frac{(n+1)(n-2)(c+3)}{8}.$$
[7] also classifies the C-totally real submanifolds $M^n$ of $\tilde{M}^{2n+1}(c)$ with constant scalar curvature for which Chen's inequality becomes an equality.

In [6] a similar inequality for $\delta_M$ was established for totally real submanifolds of a complex space form; [6] and [4] contain also a classification of certain such submanifolds satisfying the equality.

In the present paper, we enlarge the investigation of [7] to the class of C-totally real submanifolds having nonconstant scalar curvature. Following [5], we consider C-totally real submanifolds $M^n$ in $\tilde{M}^{2n+1}(c)$, satisfying Chen's equality, under some additional integrability condition. This extra condition then appropriately singles out, and conversely characterizes in some sense, a specific class of C-totally real submanifolds of $\tilde{M}^{2n+1}(c)$ capturing the particular example with nonconstant scalar curvature, that fell outside the range of the classification result of [7]. This condition is stated in terms of some distribution, introduced in this context in [2]. More precisely, we prove the following theorem.

**Theorem 1.** Let $\tilde{M}^{2n+1}(c)$ be a Sasakian space form and $M^n$ an $n$-dimensional $(n > 2)$ C-totally real submanifold with nonconstant scalar curvature such that the subspaces

$$\mathcal{D}(p) = \{X \in T_pM^n; h(X, Y) = 0, \forall Y \in T_pM^n\}, \ p \in M^n,$$

define a differentiable subbundle and its complementary orthogonal subbundle $\mathcal{D}^\perp$ is involutive. Then $M^n$ satisfies

$$\delta_M = \tau - \inf K = \frac{(n - 2)(n + 1)(c + 3)}{8},$$

if and only if $M^n$ is locally congruent to an immersion

$$\psi : (0, \frac{1}{2}\pi) \times_{\cos} M^2 \times_{\sin} S^{n-3} \to S^{2n+1}, \quad \psi(t, p, q) = (\cos t)p + (\sin t)q,$$

where $M^2$ is a C-totally real minimal surface of $S^5$.

We remark that the example of a C-totally real submanifold with nonconstant scalar curvature satisfying Chen's equality given in [7] is included as a particular case of the above theorem, for $n = 3$.

2. C-totally real submanifolds of a Sasakian space form

Let $\tilde{M}^{2m+1}$ be an odd dimensional Riemannian manifold of class $C^\infty$ with Riemannian metric tensor field $g$. 

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Let $\phi$ be a $(1,1)$-tensor field, $\xi$ a vector field, and $\eta$ a 1-form on $\tilde{M}^{2m+1}$, such that

\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi).
\]

If, in addition, $d\eta(X, Y) = g(\phi X, Y)$ for all vector fields $X$, $Y$ on $\tilde{M}^{2m+1}$, then $\tilde{M}^{2m+1}$ is said to have a contact metric structure $(\phi, \xi, \eta, g)$, and $\tilde{M}^{2m+1}$ is called a contact metric manifold.

If moreover the structure is normal, that is if $[\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y)\xi$, then the contact metric structure is called a Sasakian structure (normal contact metric structure) and $\tilde{M}^{2m+1}$ is called a Sasakian manifold. For more details and background, see the standard references [1, 10].

A plane section $\sigma$ in $T_p\tilde{M}^{2m+1}$ of a Sasakian manifold $\tilde{M}^{2m+1}$ is called a $\phi$-section if it is spanned by $X$ and $\phi X$, where $X$ is a unit tangent vector field orthogonal to $\xi$. The sectional curvature $\tilde{K}(\sigma)$ with respect to a $\phi$-section $\sigma$ is called a $\phi$-sectional curvature. If a Sasakian manifold $\tilde{M}^{2m+1}$ has constant $\phi$-sectional curvature $c$, $\tilde{M}^{2m+1}$ is called a Sasakian space form and is denoted by $\tilde{M}^{2m+1}(c)$.

The curvature tensor $\tilde{R}$ of a Sasakian space form $\tilde{M}^{2m+1}(c)$ is given by [1]:

\[
\tilde{R}(X, Y)Z = \frac{c+3}{4}(g(Y, Z)X - g(X, Z)Y)
\]

\[+ \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi
\]

\[+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z),
\]

for any tangent vector fields $X$, $Y$, $Z$ to $\tilde{M}^{2m+1}(c)$.

An $n$-dimensional submanifold $M^n$ of a Sasakian space form $\tilde{M}^{2m+1}(c)$ is called a C-totally real submanifold of $\tilde{M}^{2m+1}(c)$ if $\xi$ is a normal vector field on $M^n$. A direct consequence of this definition is that $\phi(TM^n) \subset T^\perp M^n$, which means that $M^n$ is an anti-invariant submanifold of $\tilde{M}^{2m+1}(c)$, (hence their name of ‘contact’-totally real submanifolds); see for example [9].

The Gauss equation implies that

\[
R(X, Y, Z, W) = \frac{1}{4}(c+3)(g(Y, Z)g(X, W) - g(X, Z)g(Y, W))
\]

\[+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),
\]

for all vector fields $X$, $Y$, $Z$, $W$ tangent to $M^n$, where $h$ denotes the second fundamental form and $R$ the curvature tensor of $M^n$.

It is easily seen that

\[
2\tau = n^2|H|^2 - \|h\|^2 + \frac{n(n-1)(c+3)}{4}.
\]
Let $M^n$ be an $n$-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of the plane section $\pi \subset T_pM^n$, $p \in M^n$. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space $T_pM^n$, the scalar curvature $\tau$ at $p$ is defined by

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

For each point $p \in M^n$, we put

$$(\inf K)(p) = \inf\{K(\pi); \pi \subset T_pM^n, \dim \pi = 2\}.$$

The function $\inf K$ is a well-defined function on $M^n$. Let $\delta_M$ denote the difference between the scalar curvature and $\inf K$, that is

$$\delta_M(p) = \tau(p) - (\inf K)(p);$$

$\delta_M$ is a well-defined Riemannian invariant, which is trivial when $n = 2$. The invariant $\delta_M$ was introduced by Chen in [2], where he gave a sharp inequality for $\delta_M$ for submanifolds in real space forms and also obtained a classification of the minimal submanifolds satisfying the equality-case (see also [3]).

We now state the inequality of Chen for the situation where the ambient space is a Sasakian space form [7].

**Theorem 2.** Let $M^n$ be an $n$-dimensional $(n \geq 2)$ $C$-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\tilde{M}^{2m+1}(c)$. Then

$$\delta_M \geq \frac{n^2}{2} \left( \frac{n^2}{n-1} |H|^2 + \frac{1}{4}(n+1)(c+3) \right).$$

Moreover, the equality holds at a point $p \in M^n$ if and only if there exist a tangent basis $\{e_1, \ldots, e_n\} \subset T_pM^n$ and a normal basis $\{e_{n+1}, \ldots, e_{2m}, \xi\} \subset T^\perp_pM^n$ such that the shape operators take the following forms

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & \mu I_{n-2} \end{pmatrix}, \quad a + b = \mu,$$

$$A_r = \begin{pmatrix} h_{11}' & h_{12}' & 0 & \ldots & 0 \\ h_{12}' & -h_{11}' & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad r \in \{n+2, \ldots, 2m\},$$

and $A_{n+1}$ is trivial.
4. Submanifolds with maximal dimension

We recall the following results, which we will need in the proof of Theorem 1.

**Proposition 3.** Let $M^n$, $(n > 2)$, be a $C$-totally real submanifold of a Sasakian space form $M^{2m+1}(c)$ which satisfies Chen's equality (4). Then for all $X$ tangent to $M^n$, $\phi X$ is perpendicular to $H$.

From now on, we restrict our attention to the totally real submanifolds $M^n$ of a Sasakian space form $\tilde{M}^{2m+1}(c)$ with lowest possible codimension or equivalently with maximal dimension, that is, we assume that $m = n$.

In this case, under the assumptions of Proposition 3, it follows that the mean curvature vector field $H$ is in the direction of $\xi$ along $M^n$. Hence, we have the following corollary.

**Corollary 4.** Every $C$-totally real submanifold $M^n$ $(n > 2)$, of a Sasakian space form $M^{2n+1}(c)$ which satisfies Chen's equality is minimal.

For a proof of these, as well as of the following Proposition 5, we refer to [7].

**Proposition 5.** Let $M^n$ be an $n$-dimensional $(n > 2)$ minimal $C$-totally real submanifold of a $(2n + 1)$-dimensional Sasakian space form $\tilde{M}^{2n+1}(c)$. Then

$$\delta_M \leq \frac{(n - 2)(n + 1)(c + 3)}{8},$$

and the equality holds at a point $p \in M^n$ if and only if there exists a tangent basis $\{e_1, \ldots, e_n\} \subset T_pM^n$ such that

$$h(e_1, e_1) = \lambda \phi e_1, \quad h(e_1, e_2) = -\lambda \phi e_2, \quad h(e_2, e_2) = -\lambda \phi e_1, \quad h(e_i, e_j) = 0, \quad i, j > 2,$$

where $\lambda \geq 0$ is given by

$$4\lambda^2 = \frac{n(n - 1)(c + 3)}{4} - 2\tau.$$

Next, we prove Theorem 1. Before doing so, we remark that the conditions under which this theorem is stated, can be formulated in a slightly more explicit form. Indeed, let $M^n$ be a minimal $C$-totally real submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$. For each $p \in M^n$, we put

$$\mathcal{D}(p) = \{X \in T_pM^n; \ h(X, Y) = 0, \ \forall Y \in T_pM^n\}.$$
The geometric meaning of $\mathcal{D}$ is clear, namely $\mathcal{D}$ is the kernel of the second fundamental form $h$. In [2], it was shown that if $\dim \mathcal{D}(p)$ is constant, then it is completely integrable and its dimension is either $n$ or $n - 2$.

In view of this last result, we can restate Theorem 1 in the following equivalent form, which is better suited for technical application.

**THEOREM 2.** Let $\tilde{M}^{2n+1}(c)$ be a Sasakian space form and $M^n$ an $n$-dimensional ($n > 2$) $C$-totally real submanifold with nonconstant scalar curvature such that:

(i) $\delta_M = \frac{1}{8}(n - 2)(n + 1)(c + 3)$;

(ii) the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are both completely integrable.

Then $M^n$ is, up to a homothety, locally congruent to an immersion

$$\psi : (0, \frac{\pi}{2}) \times \cos, M^2 \times \sin, S^n \to S^{2n+1}, \quad \psi(t, p, q) = (\cos t)p + (\sin t)q,$$

where $M^2$ is a $C$-totally real minimal surface of $S^5$.

**PROOF.** By Corollary 4, we know that $M^n$ is actually a minimal submanifold. Hence, by Proposition 5, there exists at every point $p \in M^n$ an orthonormal basis $\{e_1, \ldots, e_n\} \subset T_p M^n$ such that

$$h(e_i, e_1) = \lambda \phi e_1, \quad h(e_i, e_2) = -\lambda \phi e_2, \quad h(e_2, e_2) = -\lambda \phi e_1, \quad h(e_i, e_j) = 0, \quad i, j > 2,$$

with $\lambda \neq 0$. We remark that in contrast to the situation studied in [7], $\lambda$ need not be a constant. Following the line of proof of the Lemmas 4.2 and 4.3 of [6], we can extend $\{e_1, \ldots, e_n\}$ to vector fields $\{E_1, \ldots, E_n\}$, which satisfy the above relations on a neighborhood of the point $p \in M^n$.

We observe that $M^n$ cannot be totally geodesic. Indeed, if $M^n$ were totally geodesic, (2) shows that in this case $M^n$ should in fact be a real space form. This would however imply that its scalar curvature is constant, which is excluded by assumption. Since we know from [2] that $\dim \mathcal{D} = n$ only occurs when $M^n$ is totally geodesic, we conclude that the dimension of $\mathcal{D}$ is $n - 2$.

Hence, we may assume that locally, $\mathcal{D}^\perp$ is spanned by $\{E_1, E_2\}$. So, there exists $E_3 \in \mathcal{D}$ (unique up to sign) such that

$$\nabla_{E_i} E_1 = b E_2 + a E_3,$$

where $a, b$ are $C^\infty$-functions with $a \neq 0$.

We must have $c \neq -3$. Otherwise, $\tilde{M}^{2n+1}$ is locally Euclidean and from the Gauss equation it follows that $R(E_1, E_3, E_1, E_3) = 0$. This implies that $M^n$ is totally geodesic, which leads to a contradiction, as already remarked above. By a homothety, we can arrange that the ambient space has normalized $c = 1$. 

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We denote $\gamma_{ij}^k = g(\nabla_E E_j, E_k)$. Using the Codazzi equation, we find

$$\gamma_{ij}^1 = \gamma_{ij}^2 = 0, \quad \gamma_{ij}^1 = \gamma_{ij}^i, \quad \gamma_{ij}^2 = \gamma_{ij}^i = 0, \quad \gamma_{ij}^2 = -\frac{1}{3} \gamma_{ij}^i; \quad i, j \geq 3. $$

Indeed, let $i, j \geq 3$. The equation $$(\nabla_E, h)(E_1, E_j) = (\nabla_E, h)(E_1, E_j)$$ is equivalent to $h(E_1, \nabla_E E_j) = 0$, which implies $\gamma_{ij}^1 = 0$. Analogously, $\gamma_{ij}^2 = 0$.

Similarly, $$(\nabla_E, h)(E_1, E_j) = (\nabla_E, h)(E_2, E_1)$$ leads to $\gamma_{ij}^1 = -\gamma_{ij}^i$. Since moreover $\mathcal{D}^\perp$ is involutive, we also have that $\gamma_{ij}^1 = \gamma_{ij}^i$. Therefore $\gamma_{ij}^1 = 0$. Finally, from $$(\nabla_E, h)(E_1, E_1) = (\nabla_E, h)(E_1, E_1)$$ it follows that $E_i(\lambda) = \lambda \gamma_{ij}^1$, which together with $$(\hat{\nabla}_{E_1, h})(E_2, E_2) = (\hat{\nabla}_{E_1, h})(E_2, E_2)$$ implies $\gamma_{ij}^2 = \gamma_{ij}^1$.

In a similar way, we obtain

$$(6) \quad E_1(\lambda) = -3\lambda \gamma_{ij}^1, \quad E_2(\lambda) = 3\lambda \gamma_{ij}^2. $$

Denoting $d = \gamma_{21}^2$, by the above relations we may write

$$\nabla_E E_1 = d E_2, \quad \nabla_E E_2 = -d E_1 + a E_3, \quad \nabla_E E_1 = \nabla_E E_2 = 0. $$

Here, we clearly see that the assumption for the scalar curvature $\tau$ to be nonconstant is essential for the present proof. Indeed with $\tau$ constant, (2) shows that $\lambda$ would also be constant in this case. However, the above would then imply that $a = b = d = 0$. But with $a = b = d = 0$, the following final part of the proof is no longer applicable.

In order to finish the proof, it suffices to check Hiepko’s condition from [8].

We denote by $\mathcal{T}_1 = \text{span}\{E_3\}$ and $\mathcal{T}_2 = \text{span}\{E_4, \ldots, E_n\}$. So, it is sufficient to prove that:

(a) $\mathcal{T}_1$ is totally geodesic;

(b) $\mathcal{T}_2$ is spherical and $\mathcal{D}$ is totally geodesic;

(c) $\mathcal{D}^\perp$ is spherical and $\mathcal{T}_1 \oplus \mathcal{D}^\perp$ is totally geodesic in $M^n$.

Indeed, we have $0 = R(E_1, E_3, E_i, E_1) = -a \gamma_{ij}^3$, $\forall i \geq 4$, which implies $\gamma_{ij}^3 = 0$. So, we have that $\mathcal{T}_1$ is totally geodesic, thus proving (a).

Next, we prove (b). For $i, j \geq 4$

$$\delta_{ij} = R(E_i, E_1, E_j, E_1) = g(\nabla_E (b E_2 + a E_3) + \nabla_{E_i, E_1}, E_j, E_1) = a \gamma_{ij}^3. $$

Then $\nabla_E E_j = -(1/a)\delta_{ij} E_3 + Y, \quad Y \in \mathcal{T}_2$. $\mathcal{T}_2$ is spherical if and only if $a$ is a constant. But $E_i(a) = R(E_i, E_1, E_3, E_1) = 0$.

For (c), obviously $\mathcal{T}_1 \oplus \mathcal{D}^\perp$ is totally geodesic. It remains to show that $\mathcal{D}^\perp$ is spherical. Let $p, q \in \{1, 2\}$; then $\nabla_E E_q = a \delta_{pq} E_3 + Z, \quad Z \in \mathcal{D}^\perp$. It follows that $\mathcal{D}^\perp$ is totally umbilical and its mean curvature vector $a E_3$ is parallel. Thus $\mathcal{D}^\perp$ is spherical.
Using Hiepko’s result [8], it follows that $M^n$ is locally isometric to a warped product manifold

$$M^n = M_0 \times_{\rho_1} M_1 \times_{\rho_2} M_2.$$ 

Recall that by a result of [2], $\mathcal{D}$ is integrable and has dimension $n - 2$. Since now both $\mathcal{D}_1$ and $\mathcal{D}$ are totally geodesic in $S^{2n+1}$ and $\mathcal{D}_2$ is totally umbilical in $S^{2n+1}$, $\dim M_0 = 1$ and $\dim M_2 = n - 3$. So, in fact, $M_2$ is locally a totally geodesic sphere of dimension $n - 3 : M_2 = S^{n-3}$. Then, by counting dimensions, we see that $M_1$ being spherical is lying in $S^5$; since $M^n$ is minimal, $M_1$ is minimal in $S^5$ too. The warping functions can be determined from the equations (6), but we do not need explicit calculations. As the decomposition of $S^{2n+1}$ into a warped product whose first factor is 1-dimensional is unique up to isometries (see [5]), following a similar argument as in [4], we can assume that

$$\rho_1 = \cos t, \quad \rho_2 = \sin t.$$ 

Therefore, we obtain that $M^n$ is indeed immersed as desired. 

\[ \Box \]

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