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PAPER

Constructing linear bicategories

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Abstract

Linearly distributive categories were introduced to model the tensor/par fragment of linear logic, without resorting to the use of negation. *Linear bicategories* are the bicategorical version of linearly distributive categories. Essentially, a linear bicategory has two forms of composition, each determining the structure of a bicategory, and the two compositions are related by a linear distribution. After the initial paper on the subject, there was little further work as there seemed to be a lack of examples. The main goal of this paper is to demonstrate that there are in fact a great many examples, which are obtained by considering quantales and quantaloids and by extending familiar constructions from the (ordinary) bicategorical setting. It is standard in the field of *monoidal topology* that the category of quantale-valued relations is a bicategory. Here, we begin by showing that a quantale is *Girard* if and only if the corresponding bicategory is a Girard quantaloid, which is an example of a linear bicategory. The *tropical* and *arctic semiring* structures fit together into a Girard quantale, so this construction is likely to have multiple applications. More generally, we define LD-quantales, which are suplattices with two quantale structures related by a linear distribution, and their bicategorical analog, linear quantaloids. We show that Q-Rel is a linear quantaloid if and only if Q is a LD-quantale. We then consider several standard constructions from enriched bicategory theory and show that these lift to the linear quantaloid setting and produce new examples of linear bicategories. In particular, we consider linear $\mathscr{Q}$-categories, matrices in $\mathscr{Q}$, and linear monads in $\mathscr{Q}$, where $\mathscr{Q}$ is a linear quantaloid. We develop non-locally posetal examples as well: $\mathscr{Q}$uant, the bicategory of quantales, modules, and module homomorphisms; and $\mathscr{Q}$tld, the bicategory of quantaloids, modules, and module homomorphisms. These turn out to be cyclic $*$-autonomous bicategories, which are in essence a closed version of linear bicategories.

Keywords: Categorical linear logic; linear bicategories; $*$-autonomous; quantales; quantaloids

1. Introduction

The idea behind the theory of linearly distributive categories (LDC) as introduced by Cockett and Seely (1997) is that (the multiplicative fragment of) linear logic is best modeled by taking the two multiplicative connectives, $\otimes$ (tensor) and $\oplus$ (par), as primitive. One then obtains a category with two monoidal structures, related by *linear distributions*. A linear distribution is a natural transformation, not an isomorphism typically, of the following form or a symmetric equivalent:

$$A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

Dedicated to the memory of our friend Phil Scott

This approach to the model theory of linear logic differs from the original approach, using the $*$-autonomous categories of Barr (1979), which take tensor and negation as primitive and then define the par by de Morgan duality.

Just as monoidal categories can be viewed as one-object bicategories, one can ask for the bicategorical version of linearly distributive categories. These are the *linear bicategories* of Cockett et al. (2000), which provide a natural semantics for non-commutative linear logic. The primary goal of this paper is to give new classes of linear bicategories arising from several different sources.

Before diving into the formal definitions, we describe the example that led to the more general discussion below. The structures we consider are two ordered semiring structures on the extended integers $\mathbb{Z}_\infty = \mathbb{Z} \cup \{+\infty, -\infty\}$, as examined by Golan (2003). (One could just as well consider the extended reals.) This set, in fact, has two semiring structures, and these are typically called the *tropical* and *arctic* semirings. They are of great use in the theory of synchronization as considered in Baccelli et al. (1992). See Droste and Kuich (2009) and Droste et al. (2009) for how extensively these structures arise.

In both, multiplication is given by the usual addition of integers. But we must be careful in defining $\infty + -\infty$. In the first structure, we define $-\infty +_1 \infty = -\infty = \infty +_1 -\infty$. The addition to this structure is given by max. This gives $\mathbb{Z}_\infty$ the structure of an ordered semiring with $\mathbb{Z}_\infty$ equipped with its usual order.

For the second structure, we again have that the multiplication is given by addition. But analogously, we now must have $-\infty +_2 \infty = \infty = \infty +_2 -\infty$. The addition for this structure is given by min. This gives $\mathbb{Z}_\infty$ the structure of an ordered semiring with $\mathbb{Z}_\infty$ equipped with the opposite of its usual order.

If X and Y are sets, we will define a $\mathbb{Z}_\infty$*-relation* from X to Y to be a function $R\colon X \times Y \to \mathbb{Z}_\infty$. As in the category of relations, we consider this as a morphism $R\colon X \nrightarrow Y$. Then the two semiring constructions above allow us to define two distinct relational compositions.

Explicitly, given $X \overset{A}{\nrightarrow} Y \overset{B}{\nrightarrow} Z$, we define

$$A \otimes B(x, z) = \bigvee_{y \in Y} (A(x, y) +_1 B(y, z)) \qquad A \oplus B(x, z) = \bigwedge_{y \in Y} (A(x, y) +_2 B(y, z))$$

We are of course using the fact that both $\mathbb{Z}_\infty$ and $\mathbb{Z}_\infty^{op}$ are not just semirings with the above structures but are in fact *quantales*. We are thus following the program of *monoidal topology* as described in Hofmann et al. (2014).

The current project began with the observation that the two compositions described above are related by a *linear distribution* and, in fact, determine a locally posetal linear bicategory. On the other hand, we have the observation that $\mathbb{Z}_\infty$ with the above operations is a Girard quantale, as investigated by Yetter (1990) and Rosenthal (1990), and the two structures are related by the Girard duality.

We first introduce the quantale analog of linearly distributive categories and the quantaloidal analog of linear bicategories in Section 4, which we call LD-*quantales* and *linear quantaloids*, respectively. These definitions will underlie most of our main results. All Girard quantales are LD-quantales, and all Girard quantaloids are linear quantaloids.

Our first result in Section 5 is that the category Q-Rel is a Girard quantaloid, defined by Rosenthal (1992), and therefore, a locally posetal linear bicategory if and only if Q is a Girard quantale. This extends naturally to the case where Q is an arbitrary LD-quantale. We then provide concrete examples of Q-Rel as a linear quantaloid.

In Section 6, following the general theory of enriching in a bicategory and the work of Rosenthal (1992), we introduce the quantaloid $\mathcal{Q}$-Mod, whose 0-cells are $\mathcal{Q}$-categories, 1-cells are $\mathcal{Q}$-modules, and 2-cells are pointwise inequalities, where $\mathcal{Q}$ is a Girard quantaloid. $\mathcal{Q}$-Mod is then itself a Girard quantaloid. This leads to considering enrichment in a linear quantaloid $\mathcal{Q}$ and the introduction of the bicategory $\mathcal{Q}$-Mod of linear $\mathcal{Q}$-categories and linear $\mathcal{Q}$-modules. It is shown to be a linear quantaloid if and only if $\mathcal{Q}$ is itself linear. This result proceeds by first proving the corresponding theorems for linear monads in $\mathcal{Q}$ and matrices in $\mathcal{Q}$. Given these new constructions, we provide more examples of locally posetal linear bicategories, using the linear quantaloids presented in the previous section.

We finally develop non-local poset examples as well. This is done in Section 7. We begin by considering $\mathcal{L}$oc, the bicategory whose objects are locales, 1-cells are bimodules, and two-cells are bimodule homomorphisms, which we use to illustrate a more general notion. This turns out to be what Cockett et al. (2000) refer to as cyclic $*$-autonomous bicategories, which are linear bicategories. We show that a number of classic examples of bicategories fit into this framework. In particular, the bicategories of quantales and of quantaloids (with their respective modules) are linear bicategories.

Remark 1.1. *There are unfortunate notational conflicts between linear logic notation and the usual notation for ordered structures, as well as within the linear logic community. We now give our choice for notation for the remainder of the paper, chosen to be in line with the notation of Cockett and Seely (1997) and Cockett et al. (2000).*

- For partially ordered sets, we will denote the top element by $\mathbf{1}$ and the bottom element by $\mathbf{0}$, if they exist.
- We will denote quantales by Q and quantaloids by $\mathcal{Q}$.
- The composition of arrows will be written in diagrammatic order.
- In a monoidal category, we will use $\otimes$ to denote the tensor product and $\top$ to denote the unit, including in the case of quantales, as opposed to & used by Mulvey (1986). Moreover, we will use $\otimes$ to denote composition in a bicategory and $\top_X$ or $\top\!\!\!\top_X$ to denote the identity 1-cell on X, in particular in the case of quantaloids.
- In a $*$-autonomous or linearly distributive category, there is a second monoidal structure, which we will denote by $\oplus$, rather than the $⅋$ of Girard (1987). This includes in the case of Girard quantales and LD-quantales. The unit of this second monoidal product will be denoted by $\perp$. In the context of linear bicategories, including in Girard and linear quantaloids, $\oplus$ will also denote the second composition and $\perp_X$ or $\perp\!\!\!\perp_X$ will be the identity 1-cell on X. Note that $\perp$, $\perp_X$, and $\perp\!\!\!\perp_X$ will also be used to denote cyclic dualizing elements and families in Girard quantales and quantaloids.

2. Preliminaries

2.1 Quantales and quantaloids

See Niefield and Rosenthal (1988) and Rosenthal (1990, 1996) for a detailed discussion about quantales and quantaloids.

Definition 2.1.

- *A* quantale, *introduced by Mulvey (1986), is a partially ordered set Q with all suprema and an associative multiplication $\otimes\colon Q \times Q \to Q$ such that for all subsets $P \subseteq Q$ and all elements $a \in Q$, we have*

$$\Big(\bigvee P\Big) \otimes a = \bigvee_{p \in P} p \otimes a \qquad \textit{and} \qquad a \otimes \Big(\bigvee P\Big) = \bigvee_{p \in P} a \otimes p$$

Note that Q necessarily satisfies $a \otimes \mathbf{0} = \mathbf{0} = \mathbf{0} \otimes a$.

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- *Since the operations* $(-) \otimes a$ *and* $a \otimes (-)$ *preserve all sups, they have right adjoints for all* $a \in Q$, *known as the left and right* residuations. *We denote them by* $(-) \multimapinv a$ *and* $a \multimap (-)$ *respectively and they are defined by:*

$$c \multimapinv a = \bigvee \{b \in Q \mid b \otimes a \leq c\} \qquad \text{and} \qquad a \multimap c = \bigvee \{b \in Q \mid a \otimes b \leq c\}$$

- *An element* $\top \in Q$ *is a* unit *if for all* $a \in Q$, $\top \otimes a = a \otimes \top$, *in which case* Q *is called* unital. *The following definitions and result are due to Yetter (1990):*
- *An element* $\perp \in Q$ *is a* cyclic dualizing element *if for all* $a \in Q$, *we have*

$$\perp \multimapinv a = a \multimap \perp \qquad \text{and} \qquad (a \multimap \perp) \multimap \perp = a$$

A Girard quantale *is a pair* $(Q, \perp)$ *where* Q *is a quantale and* $\perp$ *is a chosen cyclic dualizing element. We denote* $a \multimap \perp$ *as* $a^\perp$.

- *If* Q *is a Girard quantale, it has a second multiplication defined by the linear logic version of de Morgan duality:*

Lemma 2.2. *Let* $(Q, \perp)$ *be a Girard quantale; then it is unital with* $\top = \perp^\perp$ *and the operation* $(-)^\perp$ *is a contravariant isomorphism.* Q^{op} *is thus a unital quantale with multiplication*

$$a \oplus b = (b^\perp \otimes a^\perp)^\perp$$

and unit $\perp$. *Evidently, this operation satisfies:*

$$\left(\bigwedge P\right) \oplus a = \bigwedge_{p \in P} p \oplus a \qquad \textit{and} \qquad a \oplus \left(\bigwedge P\right) = \bigwedge_{p \in P} a \oplus p$$

Example 2.3.

(1) *Every locale (or frame)* L, *that is a complete lattice satisfying the infinite distributive law,*

$$a \wedge \left(\bigvee b_i\right) = \bigvee (a \wedge b_i) \ \forall a, b_i \in L$$

is a unital quantale with $\otimes = \wedge$ *and* $\top = 1$. *As discussed by Niefield and Rosenthal (1988), a quantale is a locale if and only if it is commutative, right-sided, and idempotent. Following is a list of important locales we will consider in this paper:*

(a) *Truth value two-chain* $\Omega = \{\mathbf{0}, \mathbf{1}\}$

(b) *Totally ordered 3-chain* $3 = \{\mathbf{0}, 1/2, \mathbf{1}\}$, *with residuations given by*

$$c \multimapinv a = \begin{cases} \mathbf{1} & \textit{if } a \leq c \\ c & \textit{if } a > c \end{cases}$$

(c) *Extended real half-line with opposite ordering* $\mathsf{P}_{\max} = ([0, \infty]^{op}, \max, 0)$, *with residuations defined by Lawvere (1973) to be*

$$c \multimapinv a = \begin{cases} c & \textit{if } a < c \\ 0 & \textit{if } a \geq c \end{cases}$$

(d) *Lattice of open sets* $\mathcal{O}(X)$ *in a topological space* X

(2) *The set of relations* $\mathsf{Rel}(X)$ *on a set* X *is a unital quantale with standard relational composition as its operation, that is given relations* $R, S \colon X \nrightarrow X$,

$$(x, x'') \in R \otimes S \quad \text{if and only if} \quad \exists x' \quad (x, x') \in R \ \text{ and } \ (x', x'') \in S$$

and with the diagonal relation Δ_X *as its unit.*

(3) *The extended real half-line with opposite ordering can be equipped with other quantale structures besides* max. *In particular, consider* $\mathsf{P}_+ = ([0, \infty]^{op}, +, 0)$, *with its operation standard*

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addition extended by $a+\infty=\infty+a=\infty$*. It is often called Lawvere's quantale of positive real numbers, as it was first introduced by Lawvere (1973). Residuation is given by "truncated subtraction":*

$$c \multimapinv a = \begin{cases} c-a & \text{if } a \le c < \infty \\ 0 & \text{if } a \ge c \\ \infty & \text{if } a < c = \infty \end{cases}$$

(4) *The unit interval with multiplication* $([0,1],\cdot,1)$ *is isomorphic to* $\mathsf{P}_+ = ([0,\infty]^{op},+,0)$ *under the map* $x \mapsto -\ln(x)$*, which is a unital homomorphism of quantales (function preserving arbitrary sups, quantale operation* $\otimes$*, and the unit* $\top$*), and its inverse* $y \mapsto \exp(-y)$*. Residuations are given "truncated division," as defined by Hofmann and Reis (2013):*

$$c \multimapinv a = \begin{cases} c/a & \text{if } 0 \neq a > c \\ 1 & \text{otherwise} \end{cases}$$

Definition 2.4.

- *A* quantaloid, *introduced by Abramsky and Vickers (1993), is a category* $\mathcal{Q}$ *enriched over the category of complete lattices and sup-preserving maps. In other words, a quantaloid* $\mathcal{Q}$ *is a locally small category such that*

  (1) *each hom-set* $\mathcal{Q}(a,b)$ *is a complete lattice, and*
  (2) *composition of morphisms in* $\mathcal{Q}$ *preserves sups in both variables.*

- *As in the case of quantales, for all arrows* $f\colon a \to b \in \mathcal{Q}$*, the functors* $(-)\otimes f\colon \mathcal{Q}(a',a) \to \mathcal{Q}(a',b)$ *and* $f\otimes(-)\colon \mathcal{Q}(b,b') \to \mathcal{Q}(a,b')$ *preserve all sups and thus have right adjoints, also known as left and right* residuations*, denoted by* $(-)\multimapinv f\colon \mathcal{Q}(a',b) \to \mathcal{Q}(a',a)$ *and* $f \multimap (-)\colon \mathcal{Q}(a,b') \to \mathcal{Q}(b,b')$ *respectively. The following definitions and results are due to Rosenthal (1992):*

- *A family of 1-cells* $\mathcal{D} = \{\perp_a\colon a \to a \mid a \in \mathcal{Q}\}$ *is a* cyclic family *if* $f \multimap \perp_a = \perp_b \multimapinv f$*, for all* $f\colon a \to b$*, and let* $f^\perp$ *denote their common value. Then* $\mathcal{D}$ *is called a* cyclic dualizing family *if* $f^{\perp\perp} = f$*, for all* f*.*
  A Girard quantaloid *is a quantaloid* $\mathcal{Q}$ *together with a* cyclic dualizing family $\mathcal{D}$*.*

- *Mirroring Lemma 2.2, if* $\mathcal{Q}$ *is Girard, it has a second composition* $\oplus$ *given by the linear logic version of the Morgan duality, such that* $\mathcal{Q}^{co}$ *is also a quantaloid (where* $-^{co}$ *denotes the reversal of 2-cells).*

Remark 2.5. *Quickly, we introduce a bit of notation that will be used throughout this paper. Suppose* $(\mathcal{V},\otimes,\top)$ *monoidal category, then let* $\mathscr{B}(\mathcal{V})$ *denote its suspension, namely the bicategory with one object* $\star$ *whose 1-cells and 2-cells are the objects and morphisms of* $\mathcal{V}$*, respectively, with composition given by the tensor product* $\otimes$ *and* $\top_\star$ *given by the unit* $\top$ *of* $\mathcal{V}$*.*

The next preliminary subsections will outline constructions that give rise to various examples of quantaloids, but we outline here three key examples.

Example 2.6. *(Rosenthal 1996)*

(1) $\mathscr{B}(\mathrm{Q})$*, the suspension of a unital quantale, is a quantaloid with one object. Note that a Girard quantaloid with one object is a Girard quantale.*

(2) Rel*, the locally posetal bicategory of sets and relations, is a quantaloid with hom-sets ordered under inclusion and standard relational composition: if we have* $R\colon X \nrightarrow Y$ *and* $S\colon Y \nrightarrow Z$*, then define* $R\otimes S\colon X \nrightarrow Z$ *by*

$$(x,z) \in R\otimes S \quad \text{if and only if} \quad \exists y \quad (x,y) \in R \ \text{ and } \ (y,z) \in S$$

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(3) *Ord, the locally posetal bicategory of preordered sets and order ideals, is a quantaloid with standard relational composition: recall that a preordered set is a set X endowed with a reflexive and transitive relation $\leq_X$ and order ideals are relations $R\colon X \nrightarrow Y$ such that*

$$x \leq_X x', \quad x'Ry \quad \Longrightarrow \quad xRy \quad \text{and} \quad y \leq_Y y', \quad xRy \quad \Longrightarrow \quad xRy'$$

Definition 2.7. *(Rosenthal 1991, Def 2.2) If $\mathcal{Q}$ and $\mathcal{Q}'$ are quantaloids, then a* quantaloid homomorphism *is a functor $F\colon \mathcal{Q} \to \mathcal{Q}'$ such that on hom-sets it induces a suplattice morphism $\mathcal{Q}(a, b) \to \mathcal{Q}'(F(a), F(b))$ for all $a, b \in \mathcal{Q}$.*

2.1.1 *The category* Q-Rel

A relation $R : X \nrightarrow Y$ assigns a truth value to each pair in $X \times Y$; as such, it can be understood as a function from $X \times Y$ to the two-chain quantale. Rel can thus be generalized to arbitrary quantales by considering quantale-valued relations as follows, giving rise to a multitude of quantaloid examples.

Definition 2.8. *If Q is a unital quantale, we can form the category* Q-Rel *whose objects are sets and arrows $R\colon X \nrightarrow Y$ are functions $R\colon X \times Y \to Q$, called Q-relations. Given $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$, the composition $R \otimes S : X \nrightarrow Z$ is defined by*

$$R \otimes S(x, z) = \bigvee_{y \in Y} R(x, y) \otimes S(y, z)$$

Note that the use of $\otimes$ on the left refers to composition in Q-Rel *and on the right refers to multiplication in Q. Identities are given by*

$$\top_X(x, x') = \begin{cases} \top & \text{if } x = x' \\ \mathbf{0} & \text{if } x \neq x' \end{cases}$$

Lemma 2.9. *(Hofmann et al. 2014, Sections 3, 1.1) If Q is a unital quantale, then* Q-Rel *is a quantaloid under pointwise ordering, so in particular it is a locally posetal bicategory.*

As Q-Rel is a quantaloid, given a Q-relation $R\colon X \nrightarrow Y$, there exist residuation functors $(-) \multimapinv R$ and $R \multimap (-)$, defined for $U\colon W \nrightarrow Y$ and $T\colon X \nrightarrow Z$ by

$$U \multimapinv R(w, z) = \bigwedge_{y \in Y} U(w, y) \multimapinv R(x, y) \qquad R \multimap T(y, z) = \bigwedge_{x \in X} R(x, y) \multimap T(x, z)$$

Example 2.10.

(1) Rel $\cong \Omega$-*Rel, the quantaloid of sets and relations.*

(2) P_+-*Rel, a quantaloid of sets and extended distance relations $D\colon X \times Y \to [0, \infty]$, with composition $\otimes$ defined, for $D_1 : X \times Y \to [0, \infty]$ and $D_2 : Y \times Z \to [0, \infty]$, by*

$$(D_1 \otimes D_2)(x, z) = \bigwedge_{y \in Y} D_1(x, y) + D_2(y, z)$$

and identities $\top_X$ defined by

$$\top_X(x, x') = \begin{cases} 0 & \text{if } x = x' \\ \infty & \text{if } x \neq x' \end{cases}$$

Alternatively, one can consider $[0, 1]$-Rel, *isomorphic to* P_+-*Rel as the map* $[0, 1] \cong \mathsf{P}_+$ *extends to an isomorphism of quantaloids, with composition $\otimes$ defined, for $D_1 : X \times Y \to [0, 1]$ and*

$D_2\colon Y \times Z \to [0, 1]$, by

$$(D_1 \otimes D_2)(x, z) = \bigvee_{y \in Y} D_1(x, y) \cdot D_2(y, z)$$

and identities $\top_X$ defined by

$$\top_X(x, x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

(3) $\mathsf{P}_{\max}$-Rel*, another quantaloid of sets and extended distance relations $D\colon X \times Y \to [0, \infty]$, but with a different composition $\otimes$ defined for $D_1 : X \times Y \to [0, \infty]$ and $D_2\colon Y \times Z \to [0, \infty]$, by*

$$(D_1 \otimes D_2)(x, z) = \bigwedge_{y \in Y} \mathsf{max}(D_1(x, y), D_2(y, z))$$

The last two examples will connect to the notion of Lawvere metric spaces. This will be discussed further in Example 2.31.

Note that there is a quantaloid embedding (quantaloid homomorphism, which is a monomorphism) from the suspension of the quantale $\mathcal{B}(Q)$ into Q-Rel:

$$\mathcal{B}(Q) \hookrightarrow Q\text{-}\mathsf{Rel} : \qquad \star \mapsto 1 = \{*\} \qquad a \mapsto R_a\colon 1 \nrightarrow 1 \quad \text{where} \quad R_a(*, *) = a$$

2.2 Modules, matrices, and monads

We assume the reader is familiar with the theory of bicategories, introduced by Bénabou (1967). Appropriate references are Betti et al. (1983) and Leinster (1998). In this section, we introduce three standard constructions in bicategory theory, but first, we must review the notion of a biclosed bicategory.

2.2.1 Biclosed bicategories

Recall the definitions of right extensions and right liftings in the sense of Street (2014).

Definition 2.11. *Let $\mathscr{B}$ be a bicategory. A* right extension *of $B\colon X \to Z$ along $A\colon X \to Y$ in $\mathscr{B}$ is a 1-cell, which we denote* as $A \multimap B$*, also sometimes denoted by* $X\mathrm{Mod}(A, B)$*, together with a 2-cell*

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, "B"'] \arrow[dr, phantom, "\overset{\varepsilon_{X,A}}{\Leftarrow}"] & Y \arrow[dl, dashed, "A \multimap B"] \\ Z & \end{tikzcd}$$

inducing a bijection, called currying*, between 2-cells $C \to A \multimap B$ and $A \otimes C \to B$, for all $C\colon Y \to Z$.*

A right lifting *of $C\colon Z \to Y$ through $A\colon X \to Y$ is a right extension in $\mathcal{B}^{op}$, that is a 1-cell which we denote $C \multimapinv A$, also denoted by* $\mathrm{Mod}Y(A, C)$*, together with a 2-cell*

$$\begin{tikzcd} & X \arrow[d, "A"] \\ Z \arrow[ur, dashed, "C \multimapinv A"] \arrow[r, "C"'] \arrow[ur, phantom, "\Downarrow \varepsilon_{C,Y}"', shift right=3] & Y \end{tikzcd}$$

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inducing a bijection, also called currying, *between 2-cells* $B \to C \multimapinv A$ *and* $B \otimes A \to C$, *for all* $B\colon Z \to X$.
A bicategory $\mathscr{B}$ *is called* biclosed *if it admits all right extensions and right liftings.*

Remark 2.12. *We have chosen to use the more traditional notation and terminology from bicategory theory.* Cockett et al. (2000) *chose their terminology to be in agreement with Lambek's non-commutative linear logic rather than the theory of bicategories. We note that our notion of right extension is the same as their notion of* right hom, *and our notion of right lifting is their* left hom.

Example 2.13. *Suppose* $(\mathscr{V}, \otimes, \top)$ *is a symmetric monoidal closed category and consider its suspension* $\mathscr{B}(\mathscr{V})$. *Taking* $A \multimap B = \mathrm{Hom}(A, B)$, *often alternatively denoted by* $[A, B]$, *and* $C \multimapinv A = \mathrm{Hom}(A, C)$, *we get:* $\mathscr{B}(\mathscr{V})$ *is a bicategory.*

Proposition 2.14. *If* $\mathscr{B}$ *is a biclosed bicategory, then the induced 2-cells*

$$(A \otimes C) \multimap B \to C \multimap (A \multimap B) \qquad \text{and} \qquad D \multimapinv (A \otimes C) \to (D \multimapinv C) \multimapinv A$$

are invertible and natural in $A\colon X \to Y$, $B\colon X \to W$, $C\colon Y \to Z$, *and* $D\colon W \to Z$.

Proof. This follows from the universal properties of the extensions and liftings. □

2.2.2 $\mathscr{B}$-categories and modules

The theory of enriched categories generalizes categories by replacing hom-sets by hom-objects in some monoidal category $\mathscr{V}$. Given that monoidal categories $\mathscr{V}$ are one-object bicategories, it was natural to consider enriching a category in a bicategory $\mathscr{B}$. Moreover, $\mathscr{V}$-categories can be readily discussed in the language of bicategories, since they can be alternatively described as lax functors from sets viewed as indiscrete bicategories to the suspension $\mathscr{B}(\mathscr{V})$.

As such, the theory of $\mathscr{B}$-categories and $\mathscr{B}$-modules was introduced, first for locally posetal bicategories by Walters (1981) and then for a larger class of bicategories by Street (2005), generalizing $\mathscr{V}$-categories and their modules, the latter often known as $\mathscr{V}$-profunctors or $\mathscr{V}$-distributors.

Definition 2.15. *Let* $\mathscr{B}$ *be a biclosed bicategory.*

- *A* $\mathscr{B}$-category *M consists of the following data:*
  - *for each 0-cell* $A \in \mathscr{B}$, *a set* M_A *"over* A,*"*
  - *for each pair of elements* $x \in M_A, x' \in M_B$ *over 0-cells* A, B, *respectively, a 1-cell* $M(x, x')\colon A \to B$ *in* $\mathscr{B}$,
  - *for each triple of elements* $x \in M_A, x' \in M_B, x'' \in M_C$ *over 0-cells* A, B, C, *respectively, 2-cells* $\eta\colon 1_A \to M(x, x)$ *and* $\mu\colon M(x, x') \otimes M(x', x'') \to M(x, x'')$ *in* $\mathscr{B}$

  satisfying the axioms of left and right identities and associativity. Alternatively, a $\mathscr{B}$*-category is a lax functor from the union of sets* M_A *viewed as an indiscrete bicategory to* $\mathscr{B}$.
- *A* $\mathscr{B}$-module $\Theta\colon M \nrightarrow N$ *assigns:*
  - *to each pair* $x \in M_A$ *and* $y \in N_B$ *over 0-cells* A *and* B, *a 1-cell* $\Theta(x, y)\colon A \to B$ *in* $\mathscr{B}$,
  - *to each triple* $x, x' \in M_A$ *and* $y \in N_B$ *over 0-cells* A *and* B, *a left action 2-cell* $\rho\colon M(x, x') \otimes \Theta(x', y) \to \Theta(x, y)$ *in* $\mathscr{B}$,

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  - *to each triple $x \in M_A$ and $y, y' \in N_B$ over 0-cells A and B, a right action 2-cell $\lambda\colon \Theta(x,y) \otimes N(y,y') \to \Theta(x,y')$ in $\mathscr{B}$,*

  satisfying five compatibility axioms.
- *Given $\mathscr{B}$-modules $\Theta, \Phi\colon M \nrightarrow N$, a morphism of $\mathscr{B}$-modules $\Theta \to \Phi$ is a family of 2-cells $\alpha\colon \Theta(x,y) \to \Phi(x,y)$ in $\mathscr{B}$ compatible with the left and right actions λ, ϕ.*
- *Given $\mathscr{B}$-modules $\Theta\colon M \nrightarrow N$ and $\Pi : N \nrightarrow P$, the composite $\Theta \otimes \Pi\colon M \nrightarrow N$ is defined, for $x \in M_A$ and $z \in P_C$ over A and C, by $\Theta \otimes \Pi(x,z)$ as a colimit in $\mathscr{B}(A,C)$.*
- *By restricting to certain bicategories $\mathscr{B}$, we can define the bicategory $\mathscr{B}$-Mod of $\mathscr{B}$-categories and $\mathscr{B}$-modules. For more details, see Street (2005).*

Suppose $(\mathscr{V}, \otimes, \top)$ is a complete and cocomplete symmetric monoidal closed category. Then, we can have $\mathscr{B}(\mathscr{V})$-$\mathsf{Mod} \cong \mathscr{V}$-Prof, the bicategory of $\mathscr{V}$-categories, $\mathscr{V}$-profunctors, otherwise known as $\mathscr{V}$-distributors, and $\mathscr{V}$-transformations.

Note that composition $A \otimes_R B$ of $\mathscr{V}$-profunctors $A\colon Q \nrightarrow R$ and $B\colon R \nrightarrow S$ can be described by the coend

$$(A \otimes_R B)(q,s) = \int^r A(q,r) \otimes B(r,s)$$

and $Q(-,-)\colon Q \nrightarrow Q$ is the identity 1-cell.

2.2.3 $\mathscr{B}$-matrices

The work of categories enriched in bicategories was further developed by Betti et al. (1983), which introduced the bicategory of $\mathscr{B}$-matrices as a stepping stone to the study of $\mathscr{B}$-Mod.

Definition 2.16. *Let $\mathscr{B}$ be a locally small-cocomplete bicategory with a small set of objects $\mathscr{B}_0$.*

- *Given a family $X = (X_A)_{A \in \mathcal{B}_0}$ of small sets indexed by $\mathscr{B}_0$, an element $x \in X_A$ is said to be an* element of X over A.
- *Given a pair of families $X = (X_A)_{A \in \mathscr{B}_0}$ and $Y = (Y_A)_{A \in \mathscr{B}_0}$, a $\mathscr{B}$-*matrix *$S\colon X \nrightarrow Y$ assigns to each pair x, y of elements over $A, B \in \mathscr{B}_0$ a 1-cell $S(x,y)\colon A \to B$ in $\mathscr{B}$. Composition of $\mathscr{B}$-matrices $\mathscr{B}$-matrices $S\colon X \nrightarrow Y$ and $T\colon Y \nrightarrow Z$ is by matrix multiplication, that is*

$$(S \otimes T)(x,z) = \coprod_{y \in Y} S(x,y) \otimes T(y,z)$$

- *A* morphism of *$\mathscr{B}$-*matrices *$\alpha\colon S \to S'$ is a family of 2-cells $\alpha_{x,y}\colon S(x,y) \to S'(x,y)$ in $\mathscr{B}$.*
- *Define $\mathsf{Matr}\mathscr{B}$ to be the bicategory with families of small sets indexed by $\mathscr{B}_0$ as 0-cells, $\mathscr{B}$-matrices as 1-cells, and $\mathscr{B}$-matrix morphisms as 2-cells.*

Suppose $(\mathscr{V}, \otimes, \top)$ is a symmetric monoidal closed category with set-indexed products and coproducts. We can consider $\mathsf{Matr}\mathscr{B}(\mathscr{V}) \cong \mathscr{V}$-$\mathsf{Matr}$ the bicategory of sets, $\mathscr{V}$-matrices, and $\mathscr{V}$-matrix morphisms.

Recall from Carboni et al. (1987) that a $\mathscr{V}$-matrix $A\colon X \nrightarrow Y$ is a function $A\colon X \times Y \to \mathsf{ob}\mathscr{V}$. A morphism $f\colon A \to B$ of $\mathscr{V}$-matrices $X \nrightarrow Y$ is a family

$$\{f(x,y)\colon A(x,y) \to B(x,y) \mid (x,y) \in X \times Y\}$$

of morphisms of $\mathscr{V}$. The composition of $A\colon X \nrightarrow Y$ and $B\colon Y \nrightarrow Z$ is given by matrix multiplication

$$(A \otimes B)(x,z) = \coprod_{y \in Y} A(x,y) \otimes B(y,z)$$

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and the identity $X \nrightarrow X$ is

$$\top_X(x,x') = \begin{cases} \top & x = x' \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where $\top$ is the unit for $\otimes$ and $\mathbf{0}$ is initial in $\mathscr{V}$.

Given $\mathscr{V}$-matrices $A\colon X \nrightarrow Y$, $B\colon X \nrightarrow Z$, and $C\colon Z \nrightarrow Y$, taking

$$A \multimap B(y,z) = \prod_{x\in X} \mathscr{V}(A(x,y), B(x,z)) \qquad C \multimapinv A(z,x) = \prod_{y\in Y} \mathscr{V}(A(x,y), C(z,y))$$

we get the well-known result within folklore, see Blute et al. (1996) and Lack (2010):

Lemma 2.17. *If $\mathscr{V}$ is a symmetric monoidal closed category with set-indexed products and coproducts, then $\mathscr{V}$-Matr is a biclosed bicategory.*

2.2.4 Monads in $\mathscr{B}$ and modules

Carboni et al. (1987) demonstrated that $\mathscr{B}$-Mod can be broken into the constructions of $\mathsf{Matr}\mathscr{B}$ and of $\mathsf{Mon}\mathscr{B}$, the bicategory of monads and modules in $\mathscr{B}$ as defined below.

Definition 2.18. *Let $\mathscr{B}$ be a bicategory with local coequalizers stable under composition.*

- *A* monad *in $\mathscr{B}$ is a 1-cell $Q\colon X \to X$ together with two 2-cells $e\colon 1_X \to Q$ and $m\colon Q\otimes Q \to Q$ satisfying the usual associativity and identity axioms. Note that a monad (X, Q) (denoted by just Q when X is understood) is a monoid in the monoidal category $\mathcal{B}(X, X)$.*
- *A* (Q, R)-module *$A\colon (X, Q) \nrightarrow (Y, R)$ is a 1-cell $A\colon X \to Y$ together with action 2-cells $\lambda\colon Q \otimes A \to A$ and $\rho\colon A\otimes R \to A$ satisfying (one-sided) associative and unit laws, and a diagram expressing the commutativity of the two actions.*
- *A* (Q, R)-module morphism *is a 2-cell $f\colon A \to B$ satisfying compatibility conditions with respect to the actions.*
- *Let* $\mathsf{Mon}\mathscr{B}$ *denote the bicategory of monads, modules, and module morphisms, with the composition $A\otimes_R B$ of $A\colon (X, Q) \nrightarrow (Y, R)$ and $B\colon (Y, R) \nrightarrow (Z, S)$ given by the local coequalizer*

$$A\otimes R\otimes B \underset{A\otimes\lambda_B}{\overset{\rho_A\otimes B}{\rightrightarrows}} A\otimes B \to A\otimes_R B$$

  in $\mathcal{B}(X, Z)$, and identity 1-cell $\mathbb{T}_{(X,Q)} = Q\colon (X, Q) \nrightarrow (X, Q)$.

Note that identity 1-cell $\top_X\colon X \to X$ in $\mathscr{B}$ is a trivial monad, every 1-cell $A\colon X \to Y$ is an $(\top_X, \top_Y)$-module, and every 2-cell $f\colon A \to B$ is a $(\top_X, \top_Y)$-module homomorphism. Moreover, $X \mapsto \top_X$ defines a pseudo-functor, which is a left pseudo-adjoint to the forgetful (lax) functor $\mathsf{Mon}\mathscr{B} \to \mathscr{B}$.

Suppose further that $\mathscr{B}$ is biclosed with local equalizers stable under composition. Given $A\colon (X, Q) \nrightarrow (Y, R)$, $B\colon (X, Q) \nrightarrow (Z, S)$, and $C\colon (Z, S) \nrightarrow (Y, R)$, the right extension $A \multimap_Q B$ in $\mathsf{Mon}\mathscr{B}$ of B along A is obtained by taking the local equalizer.

$$A \multimap_Q B \to A \multimap B \underset{\hat{\lambda}_B}{\overset{\lambda_A^*}{\rightrightarrows}} (Q\otimes A) \multimap B$$

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where $\lambda_A^* = \lambda_A \multimap B$ and $\hat{\lambda}_B$ is the curry of

$$(Q \otimes A) \otimes (A \multimap B) \xrightarrow{\sim} Q \otimes (A \otimes (A \multimap B)) \xrightarrow{Q \otimes \epsilon_{X,A}} Q \otimes B \xrightarrow{\lambda_B} B$$

Similarly, the right lifting $C \multimapinv_R A$ in $\mathsf{Mon}\mathscr{B}$ of C through A is

$$C \multimapinv_R A \to C \multimapinv A \overset{\rho_A^*}{\underset{\hat{\rho}_B}{\rightrightarrows}} C \multimapinv (A \otimes R)$$

Note that given morphisms $A\colon X \to Y$, $B\colon X \to Z$ and $C\colon Z \to Y$ in $\mathscr{B}$, the 2-cells

$$A \multimap_{\top_X} B \to A \multimap B \text{ and } C \multimapinv_{\top_Y} A \to C \multimapinv A$$

defining the extensions and liftings in $\mathsf{Mon}\mathscr{B}$, are invertible, and without loss of generality, we can take them to be equalities.

Moreover, given the above definitions, the following is a well-known result; see Betti et al. (1983) and Lack (2010):

Lemma 2.19. *If $\mathscr{B}$ is a biclosed bicategory with local equalizers and coequalizers stable under composition, then $\mathsf{Mon}\mathscr{B}$ is a biclosed bicategory.*

By examining the definitions of the three constructions, it is immediate that:

Proposition 2.20. *(Carboni et al. 1987) If $\mathscr{B}$ is a distributive bicategory (locally cocomplete bicategory with colimits preserved by composition on both sides), $\mathscr{B}$-Mod is biequivalent to $\mathsf{MonMatr}\mathscr{B}$.*

Furthermore, the constructions are idempotent.

Proposition 2.21. *(Carboni et al. 1987) If $\mathscr{B}$ is a distributive bicategory, then ($\mathscr{B}$-Mod)-Mod is biequivalent to $\mathscr{B}$-Mod. If $\mathscr{B}$ is a bicategory with local coequalizers stable under, then $\mathsf{Mon}(\mathsf{Mon}\mathscr{B})$ is biequivalent to $\mathsf{Mon}\mathscr{B}$. If is a bicategory with small local coproducts stable under composition, then $\mathsf{Matr}(\mathsf{Matr}\mathscr{B})$ is biequivalent to $\mathsf{Matr}\mathscr{B}$.*

Thus, by Lemmas 2.17 and 2.19, and calculating the liftings and extensions by the ends

$$(A \multimap B)(r,s) = \int_q \mathrm{Hom}(A(q,r), B(q,s)) \qquad (C \multimapinv A)(s,q) = \int_q \mathrm{Hom}(A(q,r), B(q,s))$$

for $\mathscr{V}$-profunctors $A\colon Q \nrightarrow R$, $B\colon Q \nrightarrow S$, and $C\colon S \nrightarrow R$, we get:

Lemma 2.22. *Suppose $\mathscr{V}$ is a complete and cocomplete symmetric monoidal closed category, then $\mathscr{V}$-Prof is a biclosed bicategory.*

2.2.5 Enrichment in a quantaloid $\mathcal{Q}$

The previous bicategorical constructions can of course be considered in the special case of a quantaloid $\mathcal{Q}$, as done by Rosenthal (1992). We take the time to describe them in detail, as a main result of this paper is their generalization to the linear context.

Definition 2.23. *Let $\mathcal{Q}$ be a quantaloid.*

- *A $\mathcal{Q}$-category is a pair $M = (X, \rho)$ where:*
  - *X is a set,*
  - *ρ is a function $\rho\colon X \to \mathsf{ob}\mathcal{Q}$, and*

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– *there is a function, called the* enrichment, *assigning to each pair* $(x,x') \in X \times X$ *a morphism* $M(x,x') \colon \rho(x) \to \rho(x')$ *such that* $\forall x,x',x'' \in X$

$$\top_{\rho(x)} \leq M(x,x) \qquad M(x,x') \otimes M(x',x'') \leq M(x,x'')$$

Alternatively, a $\mathcal{Q}$*-category is a lax functor from set* X *to* $\mathcal{Q}$*, when* X *is viewed as an indiscrete bicategory.*

- *Consider* $\mathcal{Q}$*-categories* $M = (X, \rho_M)$ *and* $N = (Y, \rho_N)$*. A* $\mathcal{Q}$-module $\Theta \colon M \nrightarrow N$ *consists of an assignment of a morphism* $\Theta(x,y) \colon \rho_M(x) \to \rho_N(y)$ *to every pair* $(x,y) \in X \times Y$ *such that* $\forall x,x' \in X, y,y' \in Y$

$$\Theta(x,y) \otimes N(y,y') \leq \Theta(x,y') \qquad M(x,x') \otimes \Theta(x',y) \leq \Theta(x,y)$$

- *Define the category* $\mathcal{Q}$-Mod *whose objects are* $\mathcal{Q}$*-categories and arrows are* $\mathcal{Q}$*-modules. Given* $M \overset{\Theta}{\nrightarrow} N \overset{\Pi}{\nrightarrow} P$*, composition composition* $\Theta \otimes \Pi \colon M \nrightarrow P$ *is defined by*

$$(\Theta \otimes \Pi)(x,z) = \bigvee_{y \in Y} \Theta(x,y) \otimes \Pi(y,z)$$

Identity 1-cells $\mathbb{T}_M \colon M \nrightarrow M$ *are defined by* $\mathbb{T}_M(x,x') = M(x,x')$*. Note that the use of* $\otimes$ *on the left refers to composition in* $\mathcal{Q}$-Mod *and on the right refers to composition in* $\mathcal{Q}$*.*

Remark 2.24. *We have chosen to denote the bicategory of* $\mathcal{Q}$*-categories and* $\mathcal{Q}$*-modules as* $\mathcal{Q}$-Mod *in agreement with previously introduced notation, while it is called* $\mathsf{Bim}(\mathcal{Q})$ *by Rosenthal (1992).*

Definition 2.25. *Let* $\mathcal{Q}$ *be a quantaloid. Define the category* $\mathsf{Matr}\mathcal{Q}$ *whose objects are small families of objects in* $\mathcal{Q}$*, that is pairs* (X,γ) *of a set* X *and a function* $\gamma \colon X \to \mathrm{ob}\mathcal{Q}$ *and arrows are* $\mathcal{Q}$*-matrices* $r \colon (X,\gamma) \nrightarrow (Y,\phi)$*, that is families of morphisms* $(r_{x,y} \colon \gamma(x) \to \phi(y))_{(x,y) \in X \times Y}$ *in* $\mathcal{Q}$*. Given* $\mathcal{Q}$*-matrices* $(X,\gamma) \overset{r}{\nrightarrow} (Y,\phi) \overset{s}{\nrightarrow} (Z,\chi)$*, composition* $r \otimes s \colon (X,\gamma) \nrightarrow (Z,\chi)$ *is defined by*

$$(r \otimes s)_{x,z} = \bigvee_{y \in Y} r_{x,y} \otimes s_{y,z}$$

Identity 1-cells $\mathbb{T}_{(X,\gamma)} \colon (X,\gamma) \nrightarrow (X,\gamma)$ *are defined by* $\mathbb{T}_{(X,\gamma)_{x,x'}} = \begin{cases} \top_{\gamma(x)} & \text{if } x = x' \\ \mathbf{0}_{\gamma(x),\phi(x')} & \text{if } x \neq x' \end{cases}$*.*

Example 2.26.

(1) *It is a standard result that* $\mathsf{Matr}(\mathscr{B}(\Omega))$*, the category of* $\mathscr{B}(\Omega)$*-matrices, is isomorphic to* Rel*.*

(2) *Consider* $\mathsf{Matr}(\mathscr{B}(\mathrm{Q}))$*. The objects are pairs* (X,γ) *of a set* X *and a trivial function mapping the set to* $\star$*. The* $\mathscr{B}(\mathrm{Q})$*-matrices are families* $(r_{x,y} \colon \star \to \star)_{(x,y) \in X \times Y}$*, a collection of elements in* Q*, that is a function* $X \times Y \to \mathrm{Q}$*, and the composition of* $\mathscr{B}(\mathrm{Q})$*-matrices is exactly the one of* Q*-relations. As such,* $\mathsf{Matr}(\mathscr{B}(\mathrm{Q})) \cong \mathrm{Q}$-$\mathsf{Rel}$*, generalizing the first example.*

Definition 2.27. *Let* $\mathcal{Q}$ *be a quantaloid.*

- *A* monad (a,m) *in* $\mathcal{Q}$ *is an object* a *equipped with a morphism* $m \colon a \to a$ *such that*

$$\top_a \leq m \qquad m \otimes m \leq m$$

Alternatively, a monad in $\mathcal{Q}$ *is a lax functor from the terminal bicategory 1 to* $\mathcal{Q}$*.*

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- *A* (m, n)-module $f\colon (a, m) \nrightarrow (b, n)$ *is a morphism* $f\colon a \to b$ *in such that*

$$m \otimes f \leq f \qquad f \otimes n \leq f$$

- $\mathsf{Mon}\mathscr{Q}$ *is the category of monads and monad modules. in* $\mathscr{Q}$. *Composition is directly inherited from* $\mathscr{Q}$ *and the identity 1-cell* $\mathbb{T}_{(a,m)}\colon (a, m) \nrightarrow (a, m)$ *is* $m : a \to a$.

Example 2.28.

(1) *It is immediate that monads in* Rel *are preordered sets and monad modules are order ideals; thus,* $\mathsf{Mon(Rel)} \cong \mathsf{Ord}$.

(2) *The above example can be generalized to* Q-Rel. *Consider* Mon(Q-Rel)*: monads in* Q-Rel *are Q-categories* (X, m) *consisting of a set* X *equipped with a relation* $m\colon X \nrightarrow X$ *which is Q-reflexive and Q-transitive, meaning*

$$\top \leq m(x, x) \qquad \text{and} \qquad m(x, x') \otimes m(x', x'') \leq m(x, x'') \qquad \forall x, x', x'' \in X$$

while monad modules are Q-profunctors, relations $S\colon (X, m) \nrightarrow (Y, n)$ *such that*

$$m(x, x') \otimes S(x', y) \leq S(x, y) \quad \text{and} \quad S(x, y') \otimes n(y', y) \leq S(x, y) \quad \forall x, x' \in X, y, y' \in Y$$

These constructions are more than just categories:

Lemma 2.29. *(Rosenthal 1992)* $\mathscr{Q}$-Mod, $\mathsf{Matr}\mathscr{Q}$, *and* $\mathsf{Mon}\mathscr{Q}$ *are quantaloids, so in particular, they are locally posetal bicategories.*

As $\mathscr{Q}$-Mod, $\mathsf{Matr}\mathscr{Q}$, and $\mathsf{Mon}\mathscr{Q}$ are quantaloids, there exist residuation functors. In the case of $\mathscr{Q}$-Mod and Mod and $\mathsf{Matr}\mathscr{Q}$, the formulas are similar to Q-Rel, in other words, given by the infimum of the pointwise residuations, while $\mathsf{Mon}\mathscr{Q}$ inherits residuations directly from $\mathscr{Q}$.

Propositions 2.20 and 2.21 apply to the case of quantaloids:

Proposition 2.30. *(Rosenthal 1992)* $\mathscr{Q}$-Mod *is biequivalent to* $\mathsf{MonMatr}\mathscr{Q}$, *and the constructions are idempotent.*

Thus, we see that $\mathsf{Ord} \cong \mathscr{B}(\Omega)$-$\mathsf{Mod} \cong \mathsf{Rel}$-$\mathsf{Mod}$ and Q-Prof $\cong \mathscr{B}$(Q)-$\mathsf{Mod} \cong$ (Q-Rel)-Mod.

Example 2.31.

(1) *Consider* Q $= \mathsf{P}_+$, *then* P_+-Prof *is the quantaloid of Lawvere metric spaces and* P_+*-bimodules, in the sense of Lawvere (1973). Objects are sets* X *equipped with extended distance functions* $m\colon X \times X \to [0, \infty]$ *which satisfy the axiom of point inequality and the triangle inequality:*

$$m(x, x) \leq 0 \quad \text{and} \quad m(x, x'') \leq m(x, x') + m(x', x'') \qquad \forall x, x', x'' \in X$$

In other words, the objects are extended quasi-pseudo-metric spaces, or simply Lawvere metric spaces. The arrows $(X, m) \nrightarrow (Y, n)$ *are real-valued functions* $F\colon X \times Y \to [0, \infty]$ *satisfying*

$$F(x, y) \leq m(x, x') + F(x', y) \quad \text{and} \quad F(x, y) \leq F(x, y') + n(y', y) \quad \forall x, x' \in X, y, y' \in Y$$

The standard category Met *of Lawvere metric spaces and non-expansive maps is equivalent to the category of* P_+*-categories and* P_+*-functors. Every* P_+*-functor gives rise to a pair of adjoint* P_+*-bimodules; therefore,* P_+-Prof *is a quantaloid of Lawvere metric spaces with a more general notion of morphisms.*

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Alternatively, take $Q = [0, 1]$, *then* $[0, 1]$-Prof $\cong \mathsf{P}_+$-Prof. *Objects are sets* X *equipped with a functions* $m\colon X \times X \to [0, 1]$ *satisfying*

$$1 \leq m(x, x) \quad \textit{and} \quad m(x, x') \cdot m(x', x'') \leq m(x, x'') \quad \forall x, x', x'' \in X$$

while arrows $(X, m) \nrightarrow (Y, n)$ *are functions* $F\colon X \times Y \to [0, 1]$ *satisfying*

$$m(x, x') \cdot F(x', y) \leq F(x, y) \quad \text{and} \quad F(x, y') \cdot n(y', y) \leq f(x, y) \quad \forall x, x' \in X, y, y' \in Y$$

(2) *If instead, we consider* $Q = \mathsf{P}_{\max}$, *then* $\mathsf{P}_{\max}$-Prof *is the quantaloid of Lawvere ultrametric spaces and* $\mathsf{P}_{\max}$-*bimodules. The metric spaces* (X, m) *satisfy the strengthened triangle inequality:*

$$m(x, x'') \leq \max(m(x, x'), m(x', x'')) \qquad \forall x, x', x'' \in X$$

and the arrows $(X, m) \nrightarrow (Y, n)$ *are real-valued functions* $F\colon X \times Y \to [0, \infty]$ *satisfying*

$$F(x, y) \leq \max(m(x, x'), F(x', y)) \quad \text{and} \quad F(x, y) \leq \max(F(x, y'), n(y', y)) \quad \forall x, x' \in X, y, y' \in Y$$

For more details, see Lawvere (1973).

Note the following quantaloid embeddings of $\mathscr{Q}$:

$$\mathscr{Q} \hookrightarrow \mathscr{Q}\text{-}\mathsf{Mod}: \qquad a \mapsto M_a = (1, \rho_a) \quad \text{where} \quad 1 = \{*\}, \quad \rho_a(\,*\,) = a \quad \text{and} \quad M_a(*, *) = \top_a$$
$$f\colon a \to b \mapsto \Theta_f\colon M_a \nrightarrow M_b \quad \text{where} \quad \Theta_f(*, *) = f$$

$$\mathscr{Q} \hookrightarrow \mathsf{Matr}\mathscr{Q}: \qquad a \mapsto (1, \gamma_a) \quad \text{where} \quad 1 = \{*\}, \quad \gamma_a(\,*\,) = a$$
$$f\colon a \to b \mapsto \Theta_f\colon M_a \nrightarrow M_b \quad \text{where} \quad \Theta_f(*, *) = f$$

$$\mathscr{Q} \hookrightarrow \mathsf{Mon}\mathscr{Q}: \qquad a \mapsto (a, \top_a) \ \text{ the trivial monad}$$
$$f\colon a \to b \mapsto f\colon (a, \top_a) \nrightarrow (b, \top_b)$$

3. Linear Bicategories

We introduce the theory of *linear bicategories* as defined by Cockett et al. (2000). The material in this subsection is entirely from that paper, unless otherwise specified.

Linear bicategories are an extension of the theory of *linearly distributive categories*, due to Cockett and Seely (1997). Linearly distributive categories axiomatize the multiplicative fragment of linear logic in a way that is closer to the syntax. So the two binary connectives, $\otimes$ and $\oplus$, are taken as primitives, and negation can be added if one wishes.

Linear logic and, in particular, its multiplicative fragment were first introduced by Girard (1987) as a commutative substructural logic. In terms of categorical semantics, this means the connectives $\otimes$ and $\oplus$ are modeled by symmetric monoidal structures. As the theory of linear logic continued to be developed, non-commutative variants began being explored. Given that bicategorical composition is intrinsically non-commutative, bicategorical structures inherently provide categorical semantics for non-commutative logics. As such, linear bicategories were introduced to provide natural semantics for non-commutative multiplicative linear logic.

As usual with bicategories, one begins with a class of *0-cells,* which we will denote $\mathscr{B}_0 = \{X, Y, Z, \ldots\}$. Then for every pair of 0-cells, one has a category $\mathscr{B}(X, Y)$. The objects of $\mathscr{B}(X, Y)$ are called *1-cells,* and the arrows are called *2-cells*. But now we have two composition functors:

$$\otimes, \oplus\colon \mathscr{B}(X, Y) \times \mathscr{B}(Y, Z) \longrightarrow \mathscr{B}(X, Z)$$

Each of these compositions gives a bicategory structure. Thus, for each composition, we have all of the morphisms and coherences that this entails. In particular, we must have identity 1-cells for each of the two compositions:

$$\top_X : X \to X \quad \textit{and} \quad \bot_X : X \to X$$

These two bicategory structures are related by linear distributions as follows. Given 0-cells X, Y, Z, W, we have two functors

$$-\otimes(-\oplus-), (-\otimes-)\oplus- : \mathscr{B}(X,Y)\times\mathscr{B}(Y,Z)\times\mathscr{B}(Z,W) \longrightarrow \mathscr{B}(X,W)$$

and we require a natural transformation between them, which is not necessarily an isomorphism:

$$\delta^L : -\otimes(-\oplus-) \Longrightarrow (-\otimes-)\oplus-$$

Symmetrically, we also need a natural transformation

$$\delta^R : (-\oplus-)\otimes- \longrightarrow -\oplus(-\otimes-).$$

All of this structure must satisfy coherence requirements detailed in (Cockett et al. 2000, Def 2.1).

One of the main goals of this paper will be to provide new examples of linear bicategories, but we mention here the quintessential example Rel, as it will be the example on which ours are based.

Rel is a locally posetal bicategory with its first composition being the standard one. But we have a second composition: for $R\colon X \twoheadrightarrow Y$ and $S\colon Y \twoheadrightarrow Z$, define $R\oplus S\colon X \twoheadrightarrow Z$ by

$$(x,z)\in R\oplus S \quad \text{if and only if} \quad \forall y \quad (x,y)\in R \quad \text{or} \quad (y,z)\in S$$

We quickly mention here that the appropriate notion of a functor between linear bicategories was developed, although it will only make a brief appearance in this article.

Definition 3.1. *(Cockett et al. 2000, Def 2.4) Let $\mathscr{B}$ and $\mathscr{B}'$ be linear bicategories. A* linear functor *$F = (F_\otimes, F_\oplus)\colon \mathscr{B}\to\mathscr{B}'$ consists of:*

- *a lax monoidal functor $(F_\otimes, m_\top, m_\otimes)\colon (\mathscr{B},\otimes,\top)\to(\mathscr{B}',\otimes,\top)$, equipped with*

$$m_\top : \top \to F_\otimes(\top) \qquad m_{\otimes A,B}\colon F_\otimes(A)\otimes F_\otimes(B)\to F_\otimes(A\otimes B)$$

- *a colax monoidal functor $(F_\oplus, n_\bot, n_\oplus)\colon (\mathscr{B},\oplus,\bot)\to(\mathscr{B}',\oplus,\bot)$, equipped with*

$$n_\bot : F_\oplus(\bot)\to\bot \qquad n_{\oplus A,B}\colon F_\oplus(A\oplus B)\to F_\oplus(A)\oplus F_\oplus(B)$$

- *four natural transformations, known as* linear strengths,

$$\begin{aligned} \nu^R_{\otimes A,B}&\colon F_\otimes(A\oplus B)\to F_\oplus(A)\oplus F_\otimes(B) \qquad & \nu^L_{\otimes A,B}&\colon F_\otimes(A\oplus B)\to F_\otimes(A)\oplus F_\oplus(B)\\ \nu^R_{\oplus A,B}&\colon F_\otimes(A)\otimes F_\oplus(B)\to F_\oplus(A\otimes B) \qquad & \nu^L_{\oplus A,B}&\colon F_\oplus(A)\otimes F_\otimes(B)\to F_\oplus(A\otimes B)\end{aligned}$$

subject to various coherence conditions.

3.1 Cyclic ∗-autonomous bicategories

Prior to the introduction of linearly distributive categories, the categorical semantics for linear logic were investigated by Seely (1989), who demonstrated that ∗-autonomous categories provided the appropriate model. The notion of a ∗-autonomous category was originally introduced independently of linear logic by Barr (1979), who was trying to capture some of the dualities present in various categories of topological vector spaces. Barr's definition of ∗-autonomous category was a symmetric monoidal closed category with a dualizing object.

As previously mentioned, non-commutative variants of linear logic began to be studied. Of note is cyclic linear logic, introduced by Yetter (1990), which considers non-commutative connectives $\otimes$ and $\oplus$ with one notion of linear negation. The quantale semantics of cyclic linear

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logic were shown to be given by Girard quantales. These were generalized to *cyclic $*$-autonomous categories* by Rosenthal (1994a): non-symmetric biclosed categories with a cyclic dualizing object. Given that bicategorical structures naturally provide semantics for non-commutative connectives, Koslowski (2001) introduced their bicategorical analog as follows.

Definition 3.2. *(Koslowski 2001, Def 1.03) A bicategory $\mathscr{B}$ is* cyclic$*$-autonomous *if*

- *for any pair of 0-cells X, Y, there is an adjoint equivalence*
$$(-)^*_{X,Y} \dashv ((-)^*_{Y,X})^{op} \colon \mathscr{B}(X,Y) \simeq \mathscr{B}(Y,X)^{op}$$
- *for any 1-cell $A : X \to Y$, the 1-cell A^* is the right extension of $\top^*_X$ along A, such that these extensions are natural in A.*

As discussed in (Koslowski 2001, Sec 1) and in (Cockett et al. 2000, Sec 3.4), we can equivalently define cyclic $*$-autonomous bicategories by introducing the concept of dualizing 1-cells.

Definition 3.3. *Suppose $\mathcal{D} = \{\perp_X \colon X \to X \mid X \in \mathscr{B}\}$ is a family of 1-cells in a biclosed bicategory $\mathscr{B}$. Given $A \colon X \to Y$, there are canonical 2-cells*

$$A \xrightarrow{\delta_{A,Y}} (\perp_Y \multimapinv A) \multimap \perp_Y \quad \textit{and} \quad A \xrightarrow{\delta_{X,A}} \perp_X \multimapinv (A \multimap \perp_X)$$

- *A family $\mathcal{D}$ is called* dualizing *if the 2-cells $\delta_{X,A}$ and $\delta_{A,Y}$ are invertible, for all $A \colon X \to Y$.*
- *A dualizing family $\mathcal{D}$ is called* cyclic *if there are invertible 2-cells $\theta_A \colon \perp_Y \multimapinv A \xrightarrow{\sim} A \multimap \perp_X$, natural in $A \colon X \to Y$, such that the following diagram commutes*

$$\begin{tikzcd}
 & \perp_X \multimapinv (A \multimap \perp_X) \arrow[d, "\perp_X \multimapinv \theta_A"] \\
A \arrow[ur, "\delta_{X,A}"] \arrow[dr, "\delta_{A,Y}"'] & \perp_X \multimapinv (\perp_Y \multimapinv A) \arrow[d, "\theta_{\perp_Y \multimapinv A}"] \\
 & (\perp_Y \multimapinv A) \multimap \perp_Y
\end{tikzcd}$$

In this case, we let $A^\perp = A \multimap \perp_X$.

Proposition 3.4. *$\mathscr{B}$ is a cyclic $*$-autonomous bicategory if and only if it admits a cyclic dualizing family of 1-cells.*

Example 3.5. *Suppose $\mathscr{V}$ is a cyclic $*$-autonomous category with cyclic dualizing $\perp$; one can show that $\mathcal{D} = \{\perp \colon \star \longrightarrow \star\}$ is a cyclic dualizing family for its suspension $\mathscr{B}(V)$. So $\mathscr{B}(V)$ is a cyclic $*$-autonomous bicategory.*

In particular, consider $\mathscr{V} = \mathrm{Sup}$, the category of suplattices and suplattice homomorphisms (functions that preserve all joins). Sup is a $*$-autonomous category with cyclic dualizing object $A \multimapinv \Omega^{op} \cong \Omega^{op} \multimap A \cong A^\circ$, where A° denotes the opposite poset of A, and $A \cong (A^\circ)^\circ$, for all A. Therefore, $\mathscr{B}(\mathrm{Sup})$ is a cyclic $*$-autonomous bicategory with cyclic dualizing family $\mathcal{D} = \{\Omega^{op} \colon \star \to \star\}$.

Recall that linearly distributive categories capture $*$-autonomous categories by adding linear negation. Thus, $*$-autonomous categories and, in particular, cyclic $*$-autonomous categories are examples of the former. Therefore, it will come as no surprise that every cyclic $*$-autonomous bicategory is a linear bicategory by the de Morgan equations. It is remarked in Cockett et al. (2000) without a detailed proof, as it would largely follow the same computation described in (Cockett

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and Seely 1997, Sec 4), in the case of linearly distributive categories and $*$-autonomous categories. We describe below some of the key details.

Lemma 3.6. *If $\mathscr{B}$ is a cyclic $*$-autonomous bicategory, then there are invertible 2-cells*

$$A^\perp \multimap B \xrightarrow{\sim} (B^\perp \otimes A^\perp)^\perp \quad \textit{and} \quad B \circ\!- C^\perp \xrightarrow{\sim} (C^\perp \otimes B^\perp)^\perp$$

for all $A\colon X \to Y$, $B\colon Y \to Z$ and $C\colon Z \to W$.

Proof. Suppose $\mathcal{D} = \{\perp_X \colon X \to X\}$ is a cyclic dualizing family for $\mathscr{B}$. Then composition with $\delta_{B,Z}$ induces the transpose invertible 2-cell

$$A^\perp \multimap B \xrightarrow{\sim} A^\perp \multimap ((\perp_Z \circ\!- B) \multimap \perp_Z)$$

Since $\perp_Z \circ\!- B \cong B^\perp$ and by Proposition 2.14, it follows that

$$A^\perp \multimap ((\perp_Z \circ\!- B) \multimap \perp_Z) \xrightarrow{\sim} (B^\perp \otimes A^\perp) \multimap \perp_Z = (B^\perp \otimes A^\perp)^\perp$$

and the desired 2-cell follows. Similarly for the second result. □

Proposition 3.7. *(Cockett et al. 2000, Prop 3.13) Every cyclic $*$-autonomous bicategory is a linear bicategory.*

Proof. Suppose $\mathcal{D} = \{\perp_X \colon X \to X\}$ is a cyclic dualizing family for $\mathscr{B}$. Given $A\colon X \to Y$ and $B\colon Y \to Z$, define $A \oplus B = (B^\perp \otimes A^\perp)^\perp$.

Then $\oplus$ is associative and has identity 1-cells $\perp_X = \top_X^\perp \colon X \to X$:

$$\begin{aligned} A \oplus (B \oplus C) &= A \oplus (C^\perp \otimes B^\perp)^\perp \cong ((C^\perp \otimes B^\perp) \otimes A^\perp)^\perp \\ &\cong (C^\perp \otimes (B^\perp \otimes A^\perp))^\perp \cong (C^\perp \otimes (A \oplus B)^\perp)^\perp \\ &= (A \oplus B) \oplus C \end{aligned}$$

and $\top_X^\perp \oplus A \cong A \cong A \oplus \top_Y^\perp$.

To see that $\mathscr{B}$ is a linear bicategory, we will define the left linear distribution

$$A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

Since $A \otimes (B \oplus C) \cong A \otimes (B^\perp \multimap C)$ and $(A \otimes B) \oplus C \cong (A \otimes B)^\perp \multimap C \cong (B^\perp \circ\!- A) \multimap C$, by Lemma 3.6; it suffices to define a 2-cell

$$A \otimes (B^\perp \multimap C) \to (B^\perp \circ\!- A) \multimap C$$

or equivalently, its transpose

$$(B^\perp \circ\!- A) \otimes (A \otimes (B^\perp \multimap C)) \to C$$

For this, we can use the associator and the evaluation maps

$$(B^\perp \circ\!- A) \otimes (A \otimes (B^\perp \multimap C)) \to ((B^\perp \circ\!- A) \otimes A) \otimes (B^\perp \multimap C) \to B^\perp \otimes (B^\perp \multimap C) \to C$$

Similarly, we get a 2-cell $(A \oplus B) \otimes C \to A \oplus (B \otimes C)$, as desired. It remains to show the coherence conditions, in particular, the ones involving the linear distributions. □

The last section of this article will be a discussion of several examples of non-posetal cyclic $*$-autonomous bicategories, including $\mathscr{Q}$uant and $\mathscr{Q}$tld.

Cyclic $*$-autonomous bicategories are unique as linear bicategories in that every 1-cell $A\colon X \to Y$ has a canonical linear negation $A^\perp \colon Y \to X$. This is encoded by the notion of cyclic linear adjoints, in the following sense:

Definition 3.8. *(Cockett et al. 2000, Def 3.1) A* linear adjunction $A \dashv\!\!\mid B\colon X \to Y$ *consists of a pair of 1-cells* $A\colon X \to Y$ *and* $B\colon Y \to X$*, equipped with 2-cells* $\tau\colon \top_X \to A \oplus B$*, known as the unit, an* $\gamma\colon B \otimes A \to \bot_Y$*, known as the conunit, satisfying the snake equations.*

Definition 3.9. *(Cockett et al. 2000, Def 4.1) A* cyclic linear adjunction $A \Vert\!\!\mapsto\!\!\dashv\!\Vert B$ *is a pair of linear adjoints* $A \dashv\!\!\mid B$ *and* $B \dashv\!\!\mid A$*.*

Proposition 3.10. *(Cockett et al. 2000, Cor 3.4) Any two right (respectively left) linear adjoints to a 1-cell are isomorphic, in the sense that there is a unique 2-cell mediating the isomorphism.*

Then,

Proposition 3.11. *(Cockett et al. 2000, Rem 4.10) Consider a cyclic* $*$*-autonomous bicategory, then every 1-cell* $A\colon X \to Y$ *has a unique (up to isomorphism) cyclic linear adjoint given by* $A^{\perp}\colon Y \to X$*.*

4. LD-Quantales and Linear Quantaloids

It is immediate that Girard quantaloids are locally posetal cyclic $*$-autonomous bicategories. Mirroring the alternative approach to categorical linear logic of linearly distributive categories and linear bicategories, we define what the analogous structure would be for quantaloids below.

Definition 4.1. *A* linear quantaloid *is a locally small category* $\mathcal{Q}$ *whose hom-sets are complete lattices with binary operations* $\otimes$ *and* $\oplus$ *and families of distinguished morphisms* $\{\top_a \mid a \in \mathrm{ob}\mathcal{Q}\}$ *and* $\{\bot_a \mid a \in \mathrm{ob}\mathcal{Q}\}$ *such that*

- $(\mathcal{Q}, \otimes, \top_a)$ *and* $(\mathcal{Q}^{co}, \oplus, \bot_a)$ *are quantaloids,*
- *for all* $f\colon a \to b, g\colon b \to c, h\colon c \to d \in \mathcal{Q}$*,*

$$f \otimes (g \oplus h) \leq (f \otimes g) \oplus h \quad \text{and} \quad (f \oplus g) \otimes h \leq f \oplus (g \otimes h)$$

Every Girard quantaloid is a linear quantaloid, and we have the following obvious observation.

Lemma 4.2. *A linear quantaloid is a linear bicategory.*

Remark 4.3. *Given linear quantaloids* $\mathcal{Q}$ *and* $\mathcal{Q}'$*, we say they are isomorphic if there exists an invertible functor* $F\colon \mathcal{Q} \to \mathcal{Q}'$ *such that* $F\colon (\mathcal{Q}, \otimes, \top_X) \to (\mathcal{Q}', \otimes, \top_X)$ *and* $F\colon (\mathcal{Q}^{co}, \oplus, \bot_X) \to (\mathcal{Q}'^{co}, \oplus, \bot_X)$ *are quantaloid homomorphisms. It is then immediate that* $F = (F, F)\colon (\mathcal{Q}, \otimes, \top_X, \oplus, \bot_X) \to (\mathcal{Q}', \otimes, \top_X, \oplus, \bot_X)$ *is an invertible linear functor when equipped with identity linear strengths.*

A main result of the present work will be new examples of linear bicategories, which are linear quantaloids. To construct these, we need the definition of the analogous linear structure for quantales. These are the LD-quantales as defined below.

Definition 4.4. *A* LD*-quantale* $(Q, \otimes, \top, \oplus, \bot)$ *is a complete lattice* Q *with operations* $\otimes$ *and* $\oplus$ *and elements* $\top$ *and* $\bot$ *such that*

- $(Q, \otimes, \top)$ *and* $(Q^{op}, \oplus, \bot)$ *are quantales, and*

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- *for all $a, b, c \in Q$,*

$$a \otimes (b \oplus c) \leq (a \otimes b) \oplus c \quad \text{and} \quad (a \oplus b) \otimes c \leq a \oplus (b \otimes c)$$

Clearly, a Girard quantale is a LD-quantale.

The notion of a LD-quantale is not truly a new one. It has previously appeared in some form in the literature, following the introduction of linearly distributive categories. In particular, it has been considered within the field of algebraic logic when discussing ordered algebras, wherein linear distributivity was called *hemi-distributivity* by Dunn and Hardegree (2001).

Indeed, within this context, a LD-quantale refers to a lattice-ordered bimonoid $\langle \mathbb{A}, \wedge, \vee, \cdot, 1, +, 0 \rangle$, where all joins and meets are admissible in the multiplicative pomonoid $\mathbb{A}.$ and in the additive pomonoid $\mathbb{A}_{+}$, respectively, as defined by Galatos and Přenosil (2023).

Now, every locale is a quantale and Rosenthal (1992) remarks that a locale is a Girard quantale if and only if it is a Boolean algebra with $\mathbf{0}$ its dualizing element. We can extend this remark to LD-quantales, but first we must introduce a slightly less well-known type of lattice.

Definition 4.5. *(Reyes and Zolfaghari 1996, Def 1.1)*

- *A* Heyting *algebra is a bounded distributive lattice $\mathcal{L}$ with an implication operator $\rightarrow \, : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ with the following property: $\forall a, b, c \in \mathcal{L}$,*

$$a \leq b \rightarrow c \iff a \wedge b \leq c$$

- *A* co-Heyting *algebra is a bounded distributive lattice $\mathcal{L}$ with a subtraction operator $\setminus \, : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ with the following property: $\forall a, b, c \in \mathcal{L}$,*

$$a \backslash b \leq c \iff a \leq b \vee c$$

- *A* bi-Heyting *algebra is a bounded distributive lattice that is both a Heyting and a co-Heyting algebra.*

Then:

Proposition 4.6. *A locale is a LD-quantale $(\mathcal{L}, \wedge, \mathbf{1}, \vee, \mathbf{0})$ if and only if it is a complete bi-Heyting algebra.*

Proof. Note that a complete Heyting algebra is the same notion as a locale. Moreover, a locale is a LD-quantale if and only if the opposite infinitary law holds,

$$\Big(\bigwedge_{i \in I} a_i\Big) \vee b = \bigwedge_{i \in I} (a_i \vee b)$$

This is equivalent to requiring a right adjoint to $(-) \vee b \colon \mathcal{L}^{op} \rightarrow \mathcal{L}^{op}$ for all b; in other words, $\mathcal{L}$ has a subtraction operation and is a co-Heyting algebra. □

Example 4.7.

(1) $\mathsf{Rel}(X)$, *the poset of relations on a set X, is a Girard quantale with cyclic dualizing element Δ_X^c, by Proposition 1.2 in Rosenthal (1992), and therefore is a LD-quantale.*

(2) *Consider the following locales, which are LD-quantales.*

- *Two-chain Ω is a Girard quantale and therefore a LD-quantale.*
- *Three-chain 3 is the smallest locale, which is not Boolean as the law of excluded middle doesn't hold:*

$$1/2 \vee (0 \multimapinv 1/2) = 1/2 \vee 0 = 1/2 \neq 1$$

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As such, it is not a Girard quantale, but it is nonetheless a LD-quantale, as it is a bi-Heyting algebra.

- $\mathsf{P}_{\max}$ *is a LD-quantale, as it is a bi-Heyting algebra, with* $\oplus = \min$ *and* $\perp = \infty$*, but it is not a Girard quantale, as* ∞ *is not a dualizing element:*

$$\text{if} \quad a \neq \infty, \quad a \multimap \infty = \infty \implies (a \multimap \infty) \multimap \infty = 0$$

- *Reyes and Zolfaghari (1996) demonstrate that given an oriented irreflexive multigraph, the lattice of its subgraphs is a bi-Heyting algebra and therefore a* LD*-quantale.*
- Borceux et al. (2006) *defined the notion of a bi-Heyting topos to be a topos such that the lattice of its subobjects is precisely a bi-Heyting algebra. An important example is the topos of presheafs* $[C^{op}, Set]$ *for any small category* C*. As such, its lattice of subobjects is a* LD*-quantale.*

(3) *Consider the unit interval with multiplication* $([0, 1], \cdot, 1)$*. It becomes a LD-quantale* $([0, 1], \cdot, \oplus)$ *when considering truncated addition* $a \oplus b = \min(a + b, 1)$ *for its par structure with unit* $\perp = 0$*.* $([0, 1]^{op}, \oplus, 0)$ *is a quantale since*

$$\bigwedge_\alpha \min(a + b_\alpha, 1) = \min\big(a + \bigwedge_\alpha b_\alpha, 1\big)$$

Linear distributivities hold as

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \leq (a \cdot b) + c \qquad \forall a \in [0, 1]$$

It is not however a Girard quantale as 0 *is not a dualizing element:*

$$\text{if} \quad a \neq 0, \quad a \multimap 0 = 0 \implies (a \multimap 0) \multimap 0 = 1$$

Since $([0, 1], \cdot, 1) \cong \mathsf{P}_+$*, we can use the isomorphims to define a par structure on* P_+*:* $a \oplus b = \max(-\ln(\exp(-a) + \exp(-b)), 0)$ *and* $\perp = \infty$*. Then,* $\mathsf{P}_+ = ([0, \infty]^{op}, +, \oplus)$ *is a LD-quantale isomorphic to* $([0, 1], \cdot, \oplus)$*.*

Remark 4.8. *Given a locale* L*, its booleanization* $\mathsf{Bool}(L) = \{a \in L \mid (a \multimap \mathbf{0}) \multimap \mathbf{0} = a\}$ *is a complete Boolean algebra by Banaschewski and Pultr (1996). Therefore, viewing any complete bi-Heyting algebra as a LD-quantale, there is always a sub-locale* $\mathsf{Bool}(L)$*, which is a Girard quantale.*

The following is another example of a LD-quantale, which requires a little more discussion.

4.1 Shift monoids

Cockett and Seely (1997) present *shift monoids* to provide examples of discrete linearly distributive categories with invertible linear distributions.

Definition 4.9.

- *A* shift monoid *consists of a 4-tuple* $\mathcal{M} = (M, +, \top, a)$ *where* $(M, +, \top)$ *is a commutative monoid and a is an invertible element of* M*.*
- *If* $\mathcal{M}$ *is a shift monoid, define a second multiplication by*

$$x \cdot y = x + y - a$$

Then this is a second monoid structure on M *with unit given by* a*.*

It is trivial to see that $x \cdot (y + z) = (x \cdot y) + z$, and so every shift monoid is a discrete linearly distributive category $(M, +, \cdot)$. We modify the notion of shift monoid as follows in order to construct LD-quantales. Recall that a commutative monoid M is *cancellative* if for all $a, b, c \in M$,

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one has

$$a + b = a + c \Rightarrow b = c$$

Proposition 4.10.

- *Let $(M, +, \top)$ be a cancellative commutative monoid. We view M as a discrete poset and then add top and bottom elements, which we denote as $\mathbf{1}$ and $\mathbf{0}$. Denote this set as M^+. We then extend the addition on M:*

$$\mathbf{1} + b = \mathbf{1} \quad \text{if } b \in M \text{ or } b = \mathbf{1} \qquad \mathbf{0} + b = \mathbf{0} \quad \text{for all } b \in M^+$$

  Then $(M^+, +, \top)$ is a commutative unital quantale.
- *Let $\mathcal{M} = (M, +, \top, a)$ be, furthermore, a cancellative commutative shift monoid, then extend the second operation in the dual way:*

$$\mathbf{0} \cdot b = \mathbf{0} \quad \text{if } b \in M \text{ or } b = \mathbf{0} \qquad \mathbf{1} \cdot b = \mathbf{1} \quad \text{for all } b \in M^+$$

  Then $(M^+, +, \cdot)$ is a LD-quantale.

Remark 4.11. *The cancellative property is needed to ensure that $+$ preserves all suprema in M^+.*

5. Q-Rel as a Linear Bicategory

We now demonstrate that if Q is a Girard quantale, or more generally a LD-quantale, Q-Rel determines a linear quantaloid, providing new examples of linear bicategories.

Proposition 5.1. *Q-Rel is a Girard quantaloid with cyclic dualizing family $\mathcal{D} = \{\perp_X : X \nrightarrow X\}$ if and only if Q is a Girard quantale with cyclic dualizing element $\perp$, where $\perp = \perp_1(*, *)$, $\perp_1 : 1 \nrightarrow 1$ being the constant map between singleton sets $1 = \{*\}$, and*

$$\perp_X(x, x') = \begin{cases} \perp & x = x' \\ \mathbf{1} & x \neq x' \end{cases} \tag{1}$$

Proof. Suppose Q-Rel is a Girard quantaloid. Given $a \in Q$, let $R_a : 1 \nrightarrow 1$ denote the a-valued constant relation. Since $R_a \multimap \perp_1 = \perp_1 \circ\!\!- R_a$ and $R_a^{\perp\perp} = R_a$, it follows that $\perp = \perp_1(*, *)$ is a cyclic dualizing element for Q, and so Q is a Girard quantale.
Consider a set X. Let $x' \in X$ and define Q-relation $R : 1 \nrightarrow X$ by

$$R(*, x) = \begin{cases} \top & x = x' \\ \mathbf{1} & x \neq x' \end{cases}$$

Recall that, in Q, $\mathbf{1} \multimap a = \mathbf{1} = a \circ\!\!- \mathbf{1}$ and $\top \multimap a = a = a \circ\!\!- \top$.
Now $(\perp_X \circ\!\!- R)(x, *) = \bigwedge_{\bar{x} \in X} \perp_X(x, \bar{x}) \circ\!\!- R(*, \bar{x}) = \perp_X(x, x')$ and

$$(R \multimap \perp_1)(x, *) = \bigwedge_{* \in 1} R(*, x) \multimap \perp_1(*, *) = R(*, x) \multimap \perp = \begin{cases} \perp & x = x' \\ \mathbf{1} & x \neq x' \end{cases}$$

As $\mathcal{D}$ is cyclic, $\perp_X \circ\!\!- R = R \multimap \perp_1$ and consequently, consequently, (1) holds.

Suppose Q is a Girard quantale with cyclic dualizing element $\perp$. Define a family of Q-relations $\mathcal{D}$ by (1). Consider $R : X \nrightarrow Y$, then

$$(R \multimap \perp_X)(y, x) = \bigwedge_{\bar{x}} R(\bar{x}, y) \multimap \perp_X(\bar{x}, x) = R(x, y) \multimap \perp = R(x, y)^\perp$$

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$$(\perp_Y \multimapinv R)(y,x) = \bigwedge_{\bar{y}} \perp_Y(y,\bar{y}) \multimapinv R(x,\bar{y}) = \perp \multimapinv R(x,y) = R(x,y)^{\perp}$$

and so $\mathcal{D}$ is a dualizing family for Q-Rel as $R \multimap \perp_X = \perp_Y \multimapinv R$. $\mathcal{D}$ being cyclic immediately, and as such, Q-Rel is a Girard quantaloid. □

Consequently, if Q is a Girard quantale, there is a second categorical structure on Q-Rel determining a linear bicategory. Indeed, its second composition is obtained as the de Morgan dual of its standard composition

$$R \oplus S(x,z) = \Big(\bigvee_{y\in Y} S(y,z)^{\perp} \otimes R(x,y)^{\perp}\Big)^{\perp} = \bigwedge_{y\in Y} (S(y,z)^{\perp} \otimes R(x,y)^{\perp})^{\perp} = \bigwedge_{y\in Y} R(x,y) \oplus S(y,z)$$

with identities $\perp_X$.

This generalizes further. Suppose $(Q, \otimes, \top)$ and $(Q^{op}, \oplus, \perp)$ are quantales. Two notions of composition

$$\otimes, \oplus \colon \mathscr{B}(X,Y) \times \mathscr{B}(Y,Z) \longrightarrow \mathscr{B}(X,Z)$$

are defined as follows: given $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$,

$$R \otimes S(x,z) = \bigvee_{y\in Y} (R(x,y) \otimes S(y,z)) \qquad\qquad R \oplus S(x,z) = \bigwedge_{y\in Y} (R(x,y) \oplus S(y,z))$$

The identity 1-cells are given by:

$$\top_X(x,x') = \begin{cases} \top & \text{if } x = x' \\ \mathbf{0} & \text{if } x \neq x' \end{cases} \qquad\qquad \perp_X(x,x') = \begin{cases} \perp & \text{if } x = x' \\ \mathbf{1} & \text{if } x \neq x' \end{cases}$$

and we get:

Theorem 5.2. *Q is a* LD*-quantale if and only if* Q-Rel *is a linear quantaloid.*

Proof. By Lemma 2.9, (Q-Rel, $\otimes$, $\top_X$) and (Q-Rel^{co}, $\oplus$, $\perp_X$), which can be alternatively described as (Q^{op}-Rel, $\oplus$, $\perp_X$), are quantaloids as $(Q, \otimes, \top)$ and $(Q^{op}, \oplus, \perp)$ are quantales, inheriting their structure from Q pointwise.

Suppose (Q, $\otimes$, $\oplus$) is a LD-quantale. Given

$$W \overset{R}{\nrightarrow} X \overset{S}{\nrightarrow} Y \overset{T}{\nrightarrow} Z$$

we have $R \otimes (S \oplus T) \leq (R \otimes S) \oplus T$ if and only if, for all w, z,

$$\bigvee_x [R(w,x) \otimes (S \oplus T)(x,z)] \leq \bigwedge_y [(R \otimes S)(w,y) \oplus T(y,z)]$$

if and only if $R(w,x) \otimes (S \oplus T)(x,z) \leq (R \otimes S)(w,y) \oplus T(y,z)$, for all w, x, y, z. But,

$$\begin{aligned} R(w,x) \otimes (S \oplus T)(x,z) &= R(w,x) \otimes \bigwedge_y [S(x,y) \oplus T(y,z)] \\ &\leq R(w,x) \otimes [S(x,y) \oplus T(y,z)] \\ &\leq [R(w,x) \otimes S(x,y)] \oplus T(y,z) \\ &\leq \bigvee_x [R(w,x) \otimes S(x,y)] \oplus T(y,z) \\ &\leq (R \otimes S)(w,y) \oplus T(y,z) \end{aligned}$$

The other inequality follows similarly, and we conclude that Q-Rel is a linear quantaloid.

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Conversely, suppose Q-Rel is a linear quantaloid. Then a, b, c in Q induces $1 \overset{R_a}{\nrightarrow} 1 \overset{R_b}{\nrightarrow} 1 \overset{R_c}{\nrightarrow} 1$ in Q-Rel, and $R_a \otimes (R_b \oplus R_c) \leq (R_a \otimes R_b) \oplus R_c$ implies $a \otimes (b \oplus c) \leq (a \otimes b) \oplus c$. Thus, Q is a LD-quantale. $\square$

Example 5.3.

(1) Ω-Rel $\cong$ Rel *is a Girard quantaloid, with its par composition given by de Morgan duality, the same as previously introduced in the preliminaries section.*

(2) 3-*Rel* *is a linear quantaloid, which is not a Girard quantaloid, as* 3 *is a complete bi-Heyting algebra, but not a Boolean algebra.*

(3) $\mathsf{P}_{\max}$-*Rel, the quantaloid of sets and "extended" distance relations, is a linear quantaloid, which is not Girard, with its par composition defined for* $D_1 : X \nrightarrow Y$ *and* $D_2 : Y \nrightarrow X$ *by*

$$(D_1 \oplus D_2)(x, z) = \bigvee_{y \in Y} \min(D_1(x, y), D_2(y, z))$$

and par identities given by

$$\perp_X(x, x') = \begin{cases} \infty & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

(4) $[0, 1]$-*Rel* *is another linear quantaloid of sets and relations, which is not Girard, with its par composition defined, for* $D_1 : X \nrightarrow Y$ *and* $D_2 : Y \nrightarrow X$ *by*

$$(D_1 \oplus D_2)(x, z) = \bigwedge_{y \in Y} \min(D_1(x, y) + D_2(y, z), 1)$$

and par identities given by

$$\perp_X(x, x') = \begin{cases} 0 & \text{if } x = x' \\ 1 & \text{if } x \neq x' \end{cases}$$

Alternatively, P_+-*Rel with its par composition defined, for* $D_1 : X \nrightarrow Y$ *and* $D_2 : Y \nrightarrow X$ *by*

$$(D_1 \oplus D_2)(x, z) = \bigvee_{y \in Y} \max(-\ln(\exp(-D_1(x, y)) + \exp(-D_2(y, z))), 0)$$

and par identities given by

$$\perp_X(x, x') = \begin{cases} \infty & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

6. Enriched in a Linear Quantaloid(2)

Rosenthal (1992) demonstrated that the definition of a Girard quantaloid was closed under various constructions; in particular, if is a Girard quantaloid, then $\mathcal{Q}$-Mod is as well. This can be easily extended into a necessary and sufficient condition.

Proposition 6.1. $\mathcal{Q}$-Mod *is a Girard quantaloid with a cyclic dualizing family* $\{\perp\!\!\!\perp_M : M \nrightarrow M\}$ *if and only if* $\mathcal{Q}$ *is a Girard quantaloid with cyclic dualizing family* $\{\perp_a : a \to a\}$*, where* $\perp_a = \perp\!\!\!\perp_{M_a}(*, *)$*,* $M_a = (1, \rho_a)$ *being the* $\mathcal{Q}$*-category defined by* $\rho_a(*) = a$ *and* $M_a(*, *) = \top_a$*, and*

$$\perp\!\!\!\perp_M(x, x') = M(x', x) \multimap \perp_{\rho(x')}$$

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Proof. Suppose $\mathcal{Q}$-Mod is a Girard quantaloid with cyclic dualizing family $\{\perp\!\!\!\perp_M : M \nrightarrow M\}$. Given an object a in $\mathcal{Q}$, define the $\mathcal{Q}$-category $M_a = (1, \rho_a)$ as indicated above and a family of morphisms $\mathcal{D} = \{\perp_a : a \to a\}$ in $\mathcal{Q}$ by $\perp_a = \perp\!\!\!\perp_{M_a}(*, *)$.

Given a morphism $f : a \to b$ in $\mathcal{Q}$, consider its image $\Theta_f : M_a \nrightarrow M_b$ under the embedding of $\mathcal{Q}$ into $\mathcal{Q}$-Mod. Then,

$$f \multimap \perp_a = \Theta_f(*, *) \multimap \perp\!\!\!\perp_{M_a}(*, *) = \bigwedge_{x \in \{*\}} \Theta_f(x, *) \multimap \perp\!\!\!\perp_{M_a}(x, *) = (\Theta_f \multimap \perp\!\!\!\perp_{M_a})(*, *) = \Theta_f^{\perp}(*, *)$$

$$\perp_b \circ\!\!- f = \perp\!\!\!\perp_{M_b}(*, *) \circ\!\!- \Theta_f(*, *) = \bigwedge_{x \in \{*\}} \perp\!\!\!\perp_{M_b}(*, x) \circ\!\!- \Theta_f(*, x) = (\perp\!\!\!\perp_{M_b} \circ\!\!- \Theta_f)(*, *) = \Theta_f^{\perp}(*, *)$$

implying $\mathcal{D}$ is a cyclic family of morphisms in $\mathcal{Q}$ and $\mathcal{D}$ being dualizing follows similarly.

The sufficiency proof is shown in Theorem 3.1 in Rosenthal (1992). $\square$

Notice that if $\mathcal{Q}$ is a Girard quantaloid, then for each $\mathcal{Q}$-category $M = (X, \rho)$ and $(x, x') \in X$, there is a morphism $M(x', x)^{\perp} : \rho(x) \to \rho(x')$ satisfying $\forall x, x', x'' \in X$

$$M(x, x)^{\perp} \leq \perp_{\rho(x)} \qquad M(x, x'')^{\perp} \leq M(x', x'')^{\perp} \oplus M(x, x')^{\perp}$$

Therefore the map $M^{\perp} : (x, x') \mapsto M(x', x)^{\perp}$, coupled with (X, ρ) is a $\mathcal{Q}^{co}$-category, where $\mathcal{Q}^{co}$ is the quantaloid with the de Morgan dual composition $\oplus$ and identities $\perp_X$.
$M^{\perp}$ is moreover a $\mathcal{Q}$-module $M \nrightarrow M$: $\forall x, x', x'' \in X$

$$M(x', x)^{\perp} \otimes M(x', x'') \leq M(x'', x)^{\perp} \qquad M(x, x') \otimes M(x'', x')^{\perp} \leq M(x'', x)^{\perp}$$

While M becomes a $\mathcal{Q}^{co}$-module $M^{\perp} \nrightarrow M^{\perp}$: $\forall x, x', x'' \in X$

$$M(x, x'') \leq M(x', x)^{\perp} \oplus M(x', x'') \qquad M(x, x'') \leq M(x, x') \oplus M(x'', x')^{\perp}$$

All together, this entails that $(M, M^{\perp}) : X \to \mathcal{Q}$ is a linear functor when X is viewed as a degenerate indiscrete linear bicategory.

Each $\mathcal{Q}$-module $\Theta : M \nrightarrow N$ interacts coherently with this structure, and it determines a $\mathcal{Q}^{co}$-module $M^{\perp} \nrightarrow N^{\perp}$: $\forall x, x' \in X, y, y' \in Y$

$$\Theta(x, y) \leq M(x', x)^{\perp} \oplus \Theta(x', y) \qquad \Theta(x, y) \leq \Theta(x, y') \oplus N(y, y')^{\perp}$$

As with every Girard quantaloid, we can define a second composition on $\mathcal{Q}$-Mod. Given $M \overset{\Theta}{\nrightarrow} N \overset{\Pi}{\nrightarrow} P$, $\Theta \otimes \Pi : M \nrightarrow P$ and $\Theta \oplus \Pi : M \nrightarrow P$ are defined by

$$(\Theta \otimes \Pi)(x, z) = \bigvee_{y \in Y} \Theta(x, y) \otimes \Pi(y, z) \quad \text{and} \quad (\Theta \oplus \Pi)(x, z) = \bigwedge_{y \in Y} \Theta(x, y) \oplus \Pi(y, z)$$

The identities 1-cells for $\otimes$ and $\oplus$ are given by $\top\!\!\!\top_M$ and $\perp\!\!\!\perp_M^{\perp}$ respectively, where

$$\top\!\!\!\top_M(x, x') = M(x, x') \quad \text{and} \quad \perp\!\!\!\perp_M(x, x') = M(x', x)^{\perp}$$

The above discussion motivates the following new definitions, which generalize $\mathcal{Q}$-Mod to the case where $\mathcal{Q}$ is a linear quantaloid.
Let $(\mathcal{Q}, \otimes, \top_a)$ and $(\mathcal{Q}^{co}, \oplus, \perp_a)$ be quantaloids.

Definition 6.2. *A* linear $\mathcal{Q}$-category *is a pair* $M = (X, \rho)$ *where:*

- *X is a set.*
- *ρ is a function $X \to ob\mathcal{Q}$,*
- *there is a function, called the* $\otimes$-enrichment, *assigning a morphism $M_{\otimes}(x, x') : \rho(x) \to \rho(x')$ in $\mathcal{Q}$ to each pair $(x, x') \in X \times X$ such that $\forall x, x', x'' \in X$*

$$\top_{\rho(x)} \leq M_{\otimes}(x, x) \qquad M_{\otimes}(x, x') \otimes M_{\otimes}(x', x'') \leq M_{\otimes}(x, x'')$$

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- *there is a function, called the* $\oplus$-enrichment, *assigning a morphism* $M_\oplus(x, x') \colon \rho(x) \to \rho(x')$ *in* $\mathcal{Q}$ *to each pair* $(x, x') \in X \times X$ *such that* $\forall x, x', x'' \in X$
$$M_\oplus(x, x) \le \bot_{\rho(x)} \qquad M_\oplus(x, x'') \le M_\oplus(x, x') \oplus M_\oplus(x', x'')$$
satisfying the following inequalities $\forall x, x', x'' \in X$,
$$M_\otimes(x, x'') \le M_\oplus(x, x') \oplus M_\otimes(x', x'') \qquad M_\otimes(x, x'') \le M_\otimes(x, x') \oplus M_\oplus(x', x'')$$
$$M_\otimes(x, x') \otimes M_\oplus(x', x'') \le M_\oplus(x, x'') \qquad M_\oplus(x, x') \otimes M_\otimes(x', x'') \le M_\oplus(x, x'')$$

More succinctly, if $\mathcal{Q}$ *is a linear quantaloid, a linear* $\mathcal{Q}$*-category is a linear functor from X to* $\mathcal{Q}$*, where X is viewed as a degenerate indiscrete linear bicategory.*

Note that given a linear $\mathcal{Q}$-category $M = (X, \rho)$, $M_\otimes = (X, \rho)$ is a $\mathcal{Q}$-category and $M_\oplus = (X, \rho)$ is a $\mathcal{Q}^{co}$-category. Moreover, the $\otimes$-enrichment and $\oplus$-enrichment together assign cyclic linear adjoints in $\mathcal{Q}$ as follows.

Proposition 6.3. *Given a linear quantaloid* $\mathcal{Q}$*, consider a linear* $\mathcal{Q}$*-category* $M = (X, \rho)$ *and a pair* $(x, x') \in X \times X$*. Then, the* $\otimes$*-enrichment and* $\oplus$*-enrichment provide a cyclic linear adjunction* $M_\otimes(x, x') \Vdash\!\!\!\mapsto\!\!\Vdash M_\oplus(x', x) \colon \rho(x) \to \rho(x')$.

Proof. To show $M_\otimes(x, x') \dashv\!\!\!\mid M_\oplus(x', x)$, we need only provide the unit and co-unit 2-cells, which are inequalities in this context.
$$\top_{\rho(x)} \le M_\otimes(x, x) \le M_\otimes(x, x') \oplus M_\oplus(x', x)$$
$$M_\oplus(x', x) \otimes M_\otimes(x, x') \le M_\oplus(x', x') \le \bot_{\rho(x')}$$
Similarly, $M_\oplus(x', x) \dashv\!\!\!\mid M_\otimes(x, x')$. □

Definition 6.4. *Consider linear* $\mathcal{Q}$*-categories* $M = (X, \rho_M)$ *and* $N = (Y, \rho_N)$*. A* linear $\mathcal{Q}$-module $\mathcal{Q}$*-module* $\Theta \colon M \nrightarrow N$ *consists of an assignment of a morphism* $\Theta(x, y) \colon \rho_M(x) \to \rho_N(y)$ *to every pair* $(x, y) \in X \times Y$ *such that* $\forall x, x' \in X, y, y' \in Y$,
$$\Theta(x, y) \otimes N_\otimes(y, y') \le \Theta(x, y') \qquad M_\otimes(x, x') \otimes \Theta(x', y) \le \Theta(x, y)$$
$$\Theta(x, y) \le M_\oplus(x, x') \oplus \Theta(x', y) \qquad \Theta(x, y) \le \Theta(x, y') \oplus N_\oplus(y', y)$$

The above definition of a linear $\mathcal{Q}$-module can be simplified, as the four inequalities are not independent: the top two imply the bottom two, and vice versa.

Proposition 6.5. *Consider linear* $\mathcal{Q}$*-categories* $M = (X, \rho_M)$ *and* $N = (Y, \rho_N)$*, then* $\Theta \colon M \nrightarrow N$ *is a* linear$\mathcal{Q}$-module *if one of the following conditions holds:*

(1) Θ *is a* $\mathcal{Q}$*-module* $M_\otimes \nrightarrow N_\otimes$*, and*
(2) Θ *is a* $\mathcal{Q}^{co}$*-module* $M_\oplus \nrightarrow N_\oplus$.

Proof. Suppose Θ is a $\mathcal{Q}$-module $M_\otimes \nrightarrow N_\otimes$; in other words, $\Theta(x, y) \otimes N_\otimes(y, y') \le \Theta(x, y')$ and $M_\otimes(x, x') \otimes \Theta(x', y) \le \Theta(x, y)$ hold $\forall x, x' \in X, y, y' \in Y$. Then,
$$\begin{aligned}\Theta(x, y) = \Theta(x, y) \otimes \top_{\rho_N(y)} &\le \Theta(x, y) \otimes N_\otimes(y, y) \le \Theta(x, y) \otimes (N_\otimes(y, y') \oplus N_\oplus(y', y)) \\ &\le (\Theta(x, y) \otimes N_\otimes(y, y')) \oplus N_\oplus(y', y) \le \Theta(x, y') \oplus N_\oplus(y', y)\end{aligned}$$

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$$\begin{aligned}\Theta(x,y) = \top_{\rho_X(x)} \otimes \Theta(x,y) \leq M_\otimes(x,x) \otimes \Theta(x,y) &\leq (M_\oplus(x,x') \oplus M_\otimes(x',x)) \otimes \Theta(x,y) \\ \leq M_\oplus(x,x') \oplus (M_\otimes(x',x) \otimes \Theta(x,y)) &\leq M_\oplus(x,x') \oplus \Theta(x',y)\end{aligned}$$

The other direction follows similarly. □

Remark 6.6. *It was shown by Cockett, Koslowski, and Seely that representable poly-bicategories are linear bicategories, and poly-functors between them are linear functors. When viewing linear $\mathcal{Q}$-categories as poly-functors between representable poly-bicategories, a linear $\mathcal{Q}$-module is a* poly-module *between the poly-functors, as defined in Def 4.1* Cockett et al. (2003).

We can now define bicategory structures attached to the above data.

Definition 6.7. *Define $\mathcal{Q}$-*Mod *to consist of the following data:*

- *0-cells are linear $\mathcal{Q}$-categories*
- *given a pair of $\mathcal{Q}$-categories M and N, there is a category with 1-cells, the linear $\mathcal{Q}$-modules $M \nrightarrow N$, and 2-cells, the pointwise inequalities, that is*
$$\Theta \leq \Pi \Leftrightarrow \forall (x,y) \in X \times Y, \quad \Theta(x,y) \leq \Pi(x,y)$$
- *given $\mathcal{Q}$-categories M, N, and P, there are two composition functors $\otimes, \oplus$ defined for $\Theta\colon M \nrightarrow N$ and $\Pi\colon N \nrightarrow P$ by*
$$(\Theta \otimes \Pi)(x,z) = \bigvee_{y \in Y} \Theta(x,y) \otimes \Pi(y,z) \quad \textit{and} \quad (\Theta \oplus \Pi)(x,z) = \bigwedge_{y \in Y} \Theta(x,y) \oplus \Pi(y,z)$$
- *given every $\mathcal{Q}$-category M, there are identity 1-cells $\top\!\!\!\top_M, \perp\!\!\!\perp_M\colon M \nrightarrow M$ defined by*
$$\top\!\!\!\top_M(x,x') = M_\otimes(x,x') \quad \textit{and} \quad \perp\!\!\!\perp_M(x,x') = M_\oplus(x,x')$$

And we can show:

Theorem 6.8. $(\mathcal{Q}\text{-}\mathsf{Mod}, \otimes, \oplus)$ *is a linear quantaloid if and only if* $(\mathcal{Q}, \otimes, \oplus)$ *is a linear quantaloid.*

This will be an immediate consequence of similar results for linear $\mathcal{Q}$-matrices and linear monads in $\mathcal{Q}$, so we will delay proving the above theorem.

Another construction that was shown to be Girard by Rosenthal (1992) is the quantaloid of $\mathcal{Q}$-matrices:

Proposition 6.9. $\mathsf{Matr}\mathcal{Q}$ *is a Girard quantaloid with cyclic dualizing family $\{\perp\!\!\!\perp_{(X,\gamma)} : (X,\gamma) \nrightarrow (X,\gamma)\}$ if and only if $\mathcal{Q}$ is a Girard quantaloid with cyclic dualizing family $\{\perp_a\colon a \to a\}$, where $\perp_a = \perp\!\!\!\perp_{(1,\gamma_a)_{*,*}}$, $(1,\gamma_a)$ consisting of the singleton set 1 and the function $\gamma_a\colon * \mapsto a$, and function $\gamma_a\colon * \mapsto a$, and*

$$\perp\!\!\!\perp_{(X,\gamma)_{x,x'}} = \begin{cases} \perp_{\gamma(x)} & x = x' \\ \mathbf{1}_{\gamma(x),\gamma(x')} & x \neq x' \end{cases}$$

Proof. The forward direction follows similarly to the proof in the case of $\mathcal{Q}$-Mod, while the backwards direction was proved in (Rosenthal 1992, Thm 3.2). □

If $(\mathcal{Q}, \otimes, \top_a)$ and $(\mathcal{Q}^{co}, \oplus, \perp_a)$ are quantaloids, then $\mathsf{Matr}\mathcal{Q}$ inherits a second bicategorical structure. The two notions of composition are defined as follows given: $\mathcal{Q}$-matrices $(X,\gamma) \overset{r}{\nrightarrow}$

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$(Y, \phi) \overset{s}{\nrightarrow} (Z, \chi)$;$r \otimes s, r \otimes s, r \oplus s \colon (X, \gamma) \nrightarrow (Z, \chi)$ are defined by

$$(r \otimes s)_{x,z} = \bigvee_{y \in Y} r_{x,y} \otimes s_{y,z} \quad \textit{and} \quad (r \oplus s)_{x,z} = \bigwedge_{y \in Y} r_{x,y} \oplus s_{y,z}$$

Identity 1-cells $\top\!\!\top_{(X,\gamma)}, \perp\!\!\perp_{(X,\gamma)} \colon (X, \gamma) \nrightarrow (X, \gamma)$ are defined by

$$\top\!\!\top_{(X,\gamma)_{x,x'}} = \begin{cases} \top_{\gamma(x)} & \text{if } x = x' \\ \mathbf{0}_{\gamma(x),\phi(x')} & \text{if } x \neq x' \end{cases} \quad \textit{and} \quad \perp\!\!\perp_{(X,\gamma)_{x,x'}} = \begin{cases} \perp_{\gamma(x)} & \text{if } x = x' \\ \mathbf{1}_{\gamma(x),\phi(x')} & \text{if } x \neq x' \end{cases}$$

Theorem 6.10. $\mathsf{Matr}\mathcal{Q}$ *is a linear quantaloid if and only if* $\mathcal{Q}$ *is a linear quantaloid.*

Proof. The proof is identical to that of Theorem 5.2, replacing objects in a quantale Q by morphisms in $\mathcal{Q}$. □

Remark 6.11. *As in the case of ordinary quantales, it is immediate that for an LD-quantale* Q, *the linear quantaloid* $\mathsf{Matr}\mathcal{B}(Q)$ *is isomorphic to* Q-Rel.

Finally, as one might expect, taking monads and modules in a Girard quantaloid remains Girard.

Proposition 6.12. $\mathsf{Mon}\mathcal{Q}$ *is a Girard quantaloid with cyclic dualizing family* $\{\perp\!\!\perp_{(a,m)} \colon (a, m) \nrightarrow (a, m)\}$ *if and only if* $\mathcal{Q}$ *is a Girard quantaloid with cyclic dualizing family* $\{\perp_a \colon a \to a\}$, *where* $\perp_a = \perp\!\!\perp_{(a,\top_a)}$, $(a, \top_a)$ *being the trivial monad on* a, *and*

$$\perp\!\!\perp_{(a,m)} = m \multimap \perp_a$$

Proof. The forward direction is immediate from the trivial monad embedding of $\mathcal{Q}$ into $\mathsf{Mon}\mathcal{Q}$ the backwards direction is proven by Theorem 3.3 in Rosenthal (1992). □

As in the case of $\mathcal{Q}$-Mod, this can be generalized to the linear setting.
Let $(\mathcal{Q}, \otimes, \top_a)$ and $(\mathcal{Q}^{co}, \oplus, \perp_a)$ be quantaloids.

Definition 6.13. *(Cockett et al. 2000, Def 4.13) A* linear monad $(a, m) = (a, m_\otimes, m_\oplus)$ *in* $\mathcal{Q}$ *is a pair of compatible* $\otimes$*-monad* $(a, m_\otimes)$ *and* $\oplus$*-comonad* $(a, m_\oplus)$, *meaning it consists of*

- *an object* a
- *a morphism* $m_\otimes : a \to a$ *such that*

$$\top_a \leq m_\otimes \quad \textit{and} \quad m_\otimes \otimes m_\otimes \leq m_\otimes$$

- *a morphism* $m_\oplus : a \to a$ *such that*

$$m_\oplus \leq \perp_a \quad \textit{and} \quad m_\oplus \leq m_\oplus \oplus m_\oplus$$

satisfying the following inequalities:

$$m_\otimes \leq m_\oplus \oplus m_\otimes \qquad m_\otimes \leq m_\otimes \oplus m_\oplus$$

$$m_\otimes \otimes m_\oplus \leq m_\oplus \qquad m_\oplus \otimes m_\otimes \leq m_\oplus$$

Definition 6.14. *Let* (a, m) *and* (b, n) *be linear monads in* $\mathcal{Q}$. *A* linear (m, n)-module *(or monad module)* $f \colon (a, m) \nrightarrow (b, n)$ *is a morphism* $f \colon a \to b$ *in* $\mathcal{Q}$, *which is* $\otimes$*-monad module*

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$f\colon (a, m_\otimes) \nrightarrow (b, n_\otimes)$ and a $\oplus$-comonad module $f\colon (a, m_\oplus) \nrightarrow (b, n_\oplus)$, that is satisfies the following inequalities:

$$f \otimes n_\otimes \leq f \qquad m_\otimes \otimes f \leq f$$

$$f \leq m_\oplus \oplus f \qquad f \leq f \oplus n_\oplus$$

For the same reasons as in the case of linear $\mathcal{Q}$-modules, the $\otimes$ and $\oplus$ monad maps are cyclic linear adjoints, and the above definition can be simplified. A morphism $f\colon a \to b$ in $\mathcal{Q}$ is $\otimes$-monad module $f\colon (a, m_\otimes) \nrightarrow (b, n_\otimes)$ if and only if it is a $\oplus$-comonad module $f\colon (a, m_\oplus) \nrightarrow (b, n_\oplus)$.

Definition 6.15. *Define* $\mathsf{Mon}\mathcal{Q}$ *to consist of the following data:*

- *0-cells are linear monads in $\mathcal{Q}$*
- *given a pair of linear monads (a, m) and (b, n), there is a category with 1-cells, the linear (m, n)-modules, and 2-cells are inherited from $\mathcal{Q}$*
- *given linear monads (a, m), (b, n) and (c, p), there are two composition functors $\otimes, \oplus$, which are inherited from $\mathcal{Q}$*
- *given every linear monad (a, m), there are identity 1-cells $\top\!\!\top_{(a,m)}, \perp\!\!\perp_{(a,m)}\colon (a, m) \nrightarrow (a, m)$ defined by*

$$\top\!\!\top_{(a,m)} = m_\otimes \quad \text{and} \quad \perp\!\!\perp_{(a,m)} = m_\oplus$$

Lemma 6.16. *If $\mathcal{Q}$ is a linear quantaloid, then* $(\mathsf{Mon}\mathcal{Q}, \otimes, \top\!\!\top_{(a,m)})$ and $(\mathsf{Mon}\mathcal{Q}^{co}, \oplus, \perp\!\!\perp_{(a,m)})$ *are quantaloids.*

Proof. $(\mathsf{Mon}\mathcal{Q}, \otimes, \top\!\!\top_{(a,m)})$ is a category with a well-defined composition $\otimes$: given linear monad modules $f\colon (a, m) \nrightarrow (b, n)$ and $g\colon (b, n) \nrightarrow (c, p)$, it is immediate that $f \otimes g\colon a \to c$ is a $\otimes$-monad module since f, g are $\otimes$-monad modules and, by linear distributivity, $f \otimes g\colon a \to c$ is a $\oplus$-comonad module as follows:

$$f \otimes g \leq (m_\oplus \oplus f) \otimes g \leq m_\oplus \oplus (f \otimes g) \quad \text{and} \quad f \otimes g \leq f \otimes (g \oplus p_\oplus) \leq (f \otimes g) \oplus p_\oplus$$

as f, g are $\oplus$-comonad modules. Identities $\top\!\!\top_{(a,m)}\colon (a, m) \nrightarrow (a, m)$ are also well-defined as $m_\otimes$ is a linear monad module:

$$m_\otimes \otimes m_\otimes \leq m_\otimes, \quad m_\otimes \leq m_\oplus \oplus m_\otimes \quad \text{and} \quad m_\otimes \leq m_\otimes \oplus m_\oplus$$

by the definition of linear $\mathcal{Q}$-categories. Alternatively, we can simply note that $f \otimes g$ and $m_\otimes$ being $\otimes$-monad morphisms implies they are $\oplus$-comonad morphisms, being the remark made earlier. The quantaloid structure is directly inherited from $(\mathcal{Q}, \otimes, \top)$. Similarly, $(\mathsf{Mon}\mathcal{Q}^{co}, \oplus, \perp\!\!\perp_{(a,m)})$ is a quantaloid. $\square$

Then we get this result:

Theorem 6.17. $\mathsf{Mon}\mathcal{Q}$ *is a linear quantaloid if and only if $\mathcal{Q}$ is a linear quantaloid.*

Proof. Suppose $\mathcal{Q}$ is a linear quantaloid, then it is immediate that $\mathsf{Mon}\mathcal{Q}$ is a linear quantaloid as compositions $\otimes$ and $\oplus$ are inherited directly.

Suppose $\mathsf{Mon}\mathcal{Q}$ is a linear quantaloid, consider the mapping of each object $a \in \mathcal{Q}$ to the trivial linear monad $(a, \top_a, \perp_a)$ and each morphism $f\colon a \to b$ to itself viewed as the trivial linear monad module $f\colon (a, \top_a, \perp_a) \nrightarrow (b, \top_b, \perp_b)$. Then, $\forall f\colon a \to b, g\colon b \to c, h\colon c \to d$, we know $f \otimes (g \oplus h) \leq (f \otimes g) \oplus h$ and $(f \oplus g) \otimes h \leq f \oplus (g \otimes h)$, since the inequalities hold when they are viewed as 1-cells in $\mathsf{Mon}\mathcal{Q}$. $\square$

We now return to the proof of Theorem 6.8:

Proof. Suppose $\mathscr{Q}$ is a linear quantaloid, then $\mathsf{Matr}\mathscr{Q}$ is a linear quantaloid, and therefore, we can perform the linear monad construction and see that $\mathsf{MonMatr}\mathscr{Q}$ is a linear quantaloid. By looking at the definitions, it is immediate that $(\mathsf{MonMatr}\mathscr{Q}, \otimes, \top\!\!\top_{(X,\gamma),m})$ is biequivalent to $(\mathscr{Q}\text{-}\mathsf{Mod}, \otimes, \perp\!\!\perp_M)$ and $(\mathsf{MonMatr}\mathscr{Q}^{co}, \oplus, \perp\!\!\perp_{(X,\gamma),m)})$ is biequivalent to $(\mathscr{Q}\text{-}\mathsf{Mod}^{co}, \oplus, \perp\!\!\perp_M)$. Thus, $\mathscr{Q}\text{-}\mathsf{Mod}$ and $\mathscr{Q}\text{-}\mathsf{Mod}^{co}$ are quantaloids.

Moreover, given linear $\mathscr{Q}$-modules $\Theta\colon M \nrightarrow N$, $\Pi\colon N \nrightarrow P$ and $\Sigma\colon P \nrightarrow R$,

$$\Theta \otimes (\Pi \oplus \Sigma) \leq (\Theta \otimes \Pi) \oplus \Sigma \quad \text{and} \quad (\Theta \oplus \Pi) \otimes \Sigma \leq \Theta \oplus (\Pi \otimes \Sigma)$$

since the inequalities hold when they are viewed as linear $(M, N), (N, P)$ and (P, R)-monad modules in $\mathsf{MonMatr}\mathscr{Q}$.

Suppose $\mathscr{Q}\text{-}\mathsf{Mod}$ is a linear quantaloid, then consider the mapping of each object $a \in \mathscr{Q}$ to the trivial linear $\mathscr{Q}$-category $M_a = (1, \rho_a)$ where $1 = \{*\}$ is the singleton set, $\rho_a(\,*\,) = a$, singleton set, $\rho_a(\,*\,) = a$, $M_{a\otimes} = \top_a$ and $M_{a\oplus} = \perp_a$, each morphism to $f\colon a \to b$ to $\mathscr{Q}$-module $\Theta_f\colon M_a \nrightarrow M_b$, where $\Theta_f(*, *) = f$. This mapping ensures the linear distributivities hold in $\mathscr{Q}$ as they hold in $\mathscr{Q}\text{-}\mathsf{Mod}$. □

Remark 6.18. *As expected, the constructions are related: let $\mathscr{Q}$ be a linear quantaloid, then* $\mathscr{Q}\text{-}\mathsf{Mod} \cong \mathsf{MonMatr}\mathscr{Q}$. *Therefore, given any LD-quantale Q, we can then consider linear* $\mathsf{Mon}(Q\text{-}\mathsf{Rel})$, *which is isomorphic to* $\mathsf{MonMatr}\mathscr{B}(Q)$ *and* $\mathscr{B}(Q)\text{-}\mathsf{Mod}$.

Example 6.19. *Let* Rel *be the Girard quantaloid of sets and relations. The linear monads in Rel are preordered sets $(X, \leq)$ additionally endowed with the relation $\leq^{\perp}$ defined by*

$$x \leq^{\perp} y \iff y \nleq x$$

The linear monad modules are order ideals $R\colon (X, \leq_X) \to (Y, \leq_Y)$. In other words, the linear quantaloid of linear monads in ***Rel*** *is isomorphic to the quantaloid of ordered sets and ideals* $\mathsf{Mon}(\mathsf{Rel}) \cong \mathsf{Ord}$.

The above example follows from a more general result about Girard quantaloids (and even further cyclic $*$-autonomous bicategories). Given a linear monad $(a, m_{\otimes}, m_{\oplus})$ in a quantaloid $\mathscr{Q}$, the monad maps $m_{\otimes}\colon a \to a$ and $m_{\oplus}\colon a \to a$ are cyclic linear adjoints. Recall that these are unique (up to isomorphism, which is equality in the posetal context) and, as a cyclic $*$-autonomous bicategory, cyclic linear adjoints are canonically given by the $(-)^{\perp}$. Thus, $m_{\oplus} = m_{\otimes}^{\perp}$. Furthermore, monad modules $f\colon (a, m) \to (b, n)$ automatically become linear monad modules $f\colon (a, m, m^{\perp}) \to (b, n, n^{\perp})$.

Therefore, the quantaloid of linear monads and linear monad modules of a Girard quantaloid is isomorphic to the standard quantaloid of monad and monad modules. In other words, considering Girard quantaloids does not give us any truly new quantaloids.

Example 6.20. *Let* $\mathsf{P}_{\mathsf{max}}\text{-}\mathsf{Rel}$ *be the linear quantaloid of sets and "extended" distance relations. Then consider Mon* $(\mathsf{P}_{\mathsf{max}}\text{-}\mathsf{Rel})$, *a linear quantaloid of sets X endowed with a relation $m_{\otimes}\colon X \times X \times X \to [0, \infty]$ satisfying*

$$m_{\otimes}(x, x) \leq 0 \quad \textit{and} \quad m_{\otimes}(x, x'') \leq \max(m_{\otimes}(x, x'), m_{\otimes}(x', x''))$$

and relation $m_{\oplus}\colon X \times X \to [0, \infty]$ satisfying

$$\infty \leq m_{\oplus}(x, x) \quad \textit{and} \quad \min(m_{\oplus}(x, x'), m_{\oplus}(x', x'')) \leq m_{\oplus}(x, x'')$$

such that

$$\min(m_{\oplus}(x, x'), m_{\otimes}(x', x'')) \leq m_{\otimes}(x, x''), \qquad \min(m_{\otimes}(x, x'), m_{\oplus}(x', x'')) \leq m_{\otimes}(x', x'')$$

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$$m_\oplus(x,x'') \le \max(m_\otimes(x,x'), m_\oplus(x',x'')) \quad \textit{and} \quad m_\oplus(x,x'') \le \max(m_\oplus(x,x'), m_\otimes(x',x''))$$

These are Lawvere ultrametric spaces $(X, m_\otimes)$ with an additional distance relation $m_\oplus$, which interact coherently with $m_\otimes$; in particular, they are cyclic linear adjoints. The arrows $(X, m) \nrightarrow (Y, n)$ are real-valued functions $f\colon X \times Y \to [0, \infty]$ such that

$$f(x,y') \le \max(f(x,y), n_\otimes(y,y')) \quad \textit{and} \quad f(x,y) \le \max(m_\otimes(x,x'), f(x',y))$$

$$\min(f(x,y'), n_\oplus(y',y)) \le f(x,y) \quad \textit{and} \quad \min(m_\otimes(x,x'), f(x',y)) \le f(x,y)$$

Note that the top two inequalities imply the bottom two and vice versa. We can of course do the same construction with P_+-Rel and $[0,1]$-Rel and get similar examples of "linearized" metric spaces.

7. Non-Locally Posetal Examples, $\mathscr{L}$oc, $\mathscr{Q}$uant, and $\mathscr{Q}$tld

In this section, we present examples of linear bicategories that are not locally posetal.

While the bicategory of locales being a linear bicategory will be a consequence of Theorem 7.8, we start with the case of locales, as it is the setting in which these greater results were first investigated.

Definition 7.1. *(Joyal and Tierney 1984) Let W, X and Y be locales.*

- *An* (X, Y)-module $A\colon X \nrightarrow Y$ *is a suplattice A which is a left X-module and a right Y-module satisfying $x(ay) = (xa)y$.*
- *If B is a (W, Y)-module, then the suplattice of right Y-module homomorphisms, denoted by $B \multimapinv A$, becomes a (W, X)-module via $(wf)(a) = wf(a)$ and $(fx)(a) = f(xa)$. Dually, if C is a left (X, Z)-module, $A \multimap C$ is a (Y, Z)-module.*
- *A function $f\colon A \to A'$ is a* module homomorphism *if it is a left X-module and right Y-module homomorphism.*
- *Suppose A is an (X, Y)-module. Then the opposite lattice A° becomes a (Y, X)-module as follows. Given $x \in X$, the function $x \cdot -\colon A \to A$ is sup-preserving map and hence has a right adjoint $-/x$. Thus, A° becomes a right X-module via $(a, x) \mapsto a/x$ and a left Y-module via $(y, a) \mapsto r\backslash a$, where $y\backslash -$ is right adjoint to $- \cdot y$, and $(A^\circ)^\circ \cong A$.*

While not explicitly stated, Joyal and Tierney (1984) develop the required results to say that:

Theorem 7.2. *There is a bicategory whose objects are locales, 1-cells are modules, and 2-cells are module homomorphisms. Composition of $A\colon X \nrightarrow Y$ and $B\colon Y \nrightarrow Z$ is given by*

$$A \otimes B = A \otimes_Y B \cong (B^\circ \multimapinv A)^\circ \cong (B \multimap A^\circ)^\circ$$

with identity 1-cells $X\colon X \nrightarrow X$. We denote this bicategory by $\mathcal{L}$oc.

Now, we note that $\mathcal{L}$oc admits another composition of 1-cells

$$A \oplus B = (B^\circ \otimes A^\circ)^\circ = (B^\circ \otimes_Y A^\circ)^\circ \cong A^\circ \multimap B$$

with identity 1-cells $X^\circ\colon X \nrightarrow X$, since $X^\circ \oplus A \cong A \cong A \oplus Y^\circ$ and is associative as follows

$$A \oplus (B \oplus C) = ((C^\circ \otimes B^\circ) \otimes A^\circ)^\circ \cong (C^\circ \otimes (B^\circ \otimes A^\circ))^\circ = (A \oplus B) \oplus C$$

To see that $\mathcal{L}$oc is a linear bicategory, we will define the linear distributivity

$$A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

Since $A \otimes (B \oplus C) \cong A \otimes_Y (B^\circ \multimap C)$ and $(A \otimes B) \oplus C \cong (B^\circ \multimapinv A) \multimap C$, it suffices to define a 2-cell $A \otimes_Y (B^\circ \multimap C) \to (B^\circ \multimapinv A) \multimap C$ or equivalently,

$$(B^\circ \multimapinv A) \otimes_X (A \otimes_Y (B^\circ \multimap C)) \to C$$

For this, we can use the evaluation maps

$$(B^\circ \multimapinv A) \otimes_X (A \otimes_Y (B^\circ \multimap C)) \cong ((B^\circ \multimapinv A) \otimes_X A) \otimes_Y (B^\circ \multimap C) \xrightarrow{\epsilon_{A,Y}} B^\circ \otimes_Y (B^\circ \multimap C) \xrightarrow{\epsilon_{Z,B^\circ}} C$$

Similarly, we get a 2-cell $(A \oplus B) \otimes C \to A \oplus (B \otimes C)$, as desired.

Theorem 7.3. *Under the above operations, $\mathcal{L}oc$ is a linear bicategory.*

7.1 $\mathscr{V}$-Mat, Mon$\mathscr{B}$, and $\mathscr{V}$-Prof

Recall $\mathscr{V}$-Matr, the bicategory of sets, $\mathscr{V}$-matrices, and $\mathscr{V}$-matrix, which is biclosed when $\mathscr{V}$ is a symmetric monoidal closed category with set-indexed products and coproduct by Lemma 2.17. If $\mathscr{V}$ is further $*$-autonomous, $\mathscr{V}$-Matr becomes a linear bicategory.

Proposition 7.4. *Consider* $\mathscr{V}$-Matr, *where $\mathscr{V}$ is a $*$-autonomous category with set-indexed products and coproducts. Given a set X, define $\perp\!\!\!\perp_X : X \nrightarrow X$ by*

$$\perp\!\!\!\perp_X(x, x') = \begin{cases} \perp & x = x' \\ \mathbf{1} & \textit{otherwise} \end{cases}$$

where $\perp$ is the dualizing object and $\mathbf{1}$ is the terminal object of $\mathscr{V}$. For $\mathscr{V}$-matrix $A: X \nrightarrow Y$, defining $A^\perp : Y \nrightarrow X$ by $A^\perp(y, x) = A(x, y)^\perp = A(x, y) \multimap \perp$, one can show that $(A \multimap \perp\!\!\!\perp_Y) \cong A^\perp \cong (\perp\!\!\!\perp_X \multimapinv A)$ and $A \cong (A^\perp)^\perp$, and so $\mathscr{V}$-Matr *is a cyclic $*$-autonomous bicategory with cyclic dualizing family $\mathcal{D} = \{\perp\!\!\!\perp_X : X \nrightarrow X\}$.*

Now consider Mon$\mathscr{B}$, the bicategory of monads in $\mathscr{B}$, modules, and module morphisms, which is biclosed if $\mathscr{B}$ is biclosed with local equalizers and coequalizers stable under composition by Lemma 2.19. Then Mon$\mathscr{B}$ can be a linear bicategory:

Proposition 7.5. *Suppose $\mathscr{B}$ is a cyclic $*$-autonomous bicategory with local equalizers and coequalizers stable under composition, then* Mon$\mathscr{B}$ *is a cyclic $*$-autonomous bicategory.*

Proof. Suppose $\mathcal{D} = \{\perp_X : X \to X \mid X \in \mathscr{B}\}$ is a cyclic dualizing family for $\mathscr{B}$. Define the family of modules

$$\mathcal{D}^\perp = \{Q^\perp : (X, Q) \nrightarrow (X, Q) \mid (X, Q) \in \mathsf{Mon}\mathscr{B}\}$$

This family is well-defined since the 1-cell $Q^\perp = Q \multimap \perp_X : X \to X$ is a module $(X, Q) \nrightarrow (X, Q)$ with action $\lambda : Q \otimes Q^\perp \to Q^\perp$ given by currying

$$Q \otimes (Q \otimes Q^\perp) \xrightarrow{\sim} (Q \otimes Q) \otimes Q^\perp \xrightarrow{m \otimes Q^\perp} Q \otimes Q^\perp \xrightarrow{\epsilon_{X,Q}} \perp_X$$

and action ρ defined as $Q^\perp \otimes Q \xrightarrow{\sim} (\perp_X \multimapinv Q) \otimes Q \xrightarrow{\rho'} \perp_X \multimapinv Q \xrightarrow{\sim} Q^\perp$, where ρ' is obtained by currying

$$((\perp_X \multimapinv Q) \otimes Q) \otimes Q \xrightarrow{\sim} (\perp_X \multimapinv Q) \otimes (Q \otimes Q) \xrightarrow{(\perp_X \multimapinv Q) \otimes m} (\perp_X \multimapinv Q) \otimes Q \xrightarrow{\epsilon_{Q,X}} \perp_X$$

Given a monad module $A : (X, Q) \nrightarrow (Y, R)$, applying Proposition 2.14, we see that

$$A \multimap_Q Q^\perp = A \multimap_Q (Q \multimap \perp_X) = A \multimap_Q (Q \multimap_{\top_X} \perp_X) \cong A \multimap_{\top_X} \perp_X = A^\perp$$

and

$$R^{\perp} \multimap_R A \cong (\perp_Y \multimap R) \multimap_R A = (\perp_Y \multimap_{\top_Y} R) \multimap_R A \cong \perp_Y \multimap_{\top_Y} A = \perp_Y \multimap A \cong A^{\perp}$$

Since all these 2-cells are natural in A, $\mathcal{D}^{\perp}$ is a cyclic dualizing family for $\mathsf{Mon}\mathscr{B}$. □

Finally, recalling that $\mathscr{V}$-Prof can be constructed as $\mathsf{Mon}(\mathscr{V}\text{-Matr})$ Proposition 2.20, and by the above two results:

Proposition 7.6. *If $\mathscr{V}$ is a complete and cocomplete $*$-autonomous category, then $\mathscr{V}$-Prof is a cyclic $*$-autonomous bicategory.*

7.2 Linear bicategories $\mathscr{Q}$uant and $\mathscr{Q}$tld

Rosenthal (1994b) observed that, for a fixed commutative unital quantale Q, the category of left Q-modules and module homomorphisms is a $*$-autonomous category. Rosenthal (1996) expanded this result and showed that, given a small quantaloid $\mathscr{Q}$, the category of $(\mathscr{Q}, \mathscr{Q})$-modules is cyclic $*$-autonomous.

Now that we have access to the notion of cyclic $*$-autonomous bicategories and linear bicategories, Q and $\mathscr{Q}$ no longer need to be fixed, and we can consider the following examples.

Definition 7.7.

- *Given unital quantales Q and R, a (Q, R)-module $Q \nrightarrow R$ is a suplattice A which is a left Q-module and a right R-module, that is there are suplattice homomorphisms $\star\colon Q \times A \to A$ and $\cdot\colon A \times R \to A$ such that*

$$(q \otimes q') \star a = q \star (q' \star a), \quad a \cdot (r \otimes r') = (a \cdot r) \cdot r'$$

$$\top \star a = a, \quad a \cdot \top = a \quad \text{and} \quad (q \star a) \cdot r = q \star (a \cdot r)$$

- *Given (Q, R)-modules A and B, a module homomorphism is a suplattice homomorphism $f\colon A \to B$ such that*

$$f(q \star a) = q \star f(a) \quad \text{and} \quad f(a \cdot r) = f(a) \cdot r$$

- *Let $\mathscr{Q}$uant denote the bicategory of unital quantales, modules and module morphisms. Then, $\mathscr{Q}$uant is $\mathsf{Mon}\mathcal{B}(\mathrm{Sup})$, the bicategory of monoids, monoid modules and module morphisms in* Sup.

If $\mathscr{V}$ is $*$-autonomous category with equalizers and coequalizers, its suspension $\mathscr{B}(V)$ is a cyclic $*$-autonomous bicategory with local equalizers and co-equalizers stable under composition. Then, by Proposition 7.5, the bicategory $\mathsf{Mon}(\mathscr{B}(V))$ is a cyclic $*$-autonomous bicategory. Taking $\mathscr{V} = \mathrm{Sup}$, we get:

Theorem 7.8. *The bicategory $\mathscr{Q}$uant is a cyclic $*$-autonomous bicategory.*

Definition 7.9.

- *Given small quantaloids $\mathscr{Q}$ and $\mathcal{R}$, a $(\mathscr{Q}, \mathcal{R})$-module $A\colon \mathscr{Q} \nrightarrow \mathcal{R}$ consists of, for each $q, q' \in$ ob$\mathscr{Q}$, $r, r' \in$ ob$\mathcal{R}$:*
  - *a suplattice $A(q, r)$,*

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– *a left action suplattice homomorphism* $\star\colon \mathcal{Q}(q,q')\times A(q',r)\to A(q,r)$ *such that given* $a\in A(q,r)$

$$\top_q\star a=a \quad \text{for} \quad q=q' \qquad (f\otimes g)\star a=f\star(g\star a) \quad \text{for} \quad f\colon q\to q', g\colon q'\to q''$$

– *a right action suplattice homomorphism* $\cdot\colon A(q,r)\times\mathcal{R}(r,r')\to A(q,r')$ *such that given* $a\in A(q,r)$

$$a\cdot\top_r=a \quad \text{for} \quad r=r' \qquad a\cdot(h\otimes k)=(a\cdot h)\cdot k \quad \text{for} \quad h\colon r\to r', k\colon r'\to r''$$

satisfying $\forall a\in A(q,r), f\colon q\to q'\in\mathcal{Q}, h\colon r\to r'\in\mathcal{R}$,

$$(f\star a)\cdot h=f\star(a\cdot h)$$

- *Given* $(\mathcal{Q},\mathcal{R})$*-modules* A *and* B, *a module homomorphism* $f\colon A\to B$ *is a family of suplattice homomorphisms* $f_{q,r}\colon A(q,r)\to B(q,r)$ *satisfying, for* $f\colon q\to q'\in\mathcal{Q}, a'\in A(q',r), a\in A(q,r), h\colon r\to r'\in\mathcal{R}$,

$$f_{q,r}(f\star a')=q\star f_{q',r}(a') \quad \textit{and} \quad f_{q,r'}(a\cdot h)=f_{q,r}(a)\cdot h$$

- *Let* $\mathcal{Q}$tld *denote the bicategory of small quantaloids, modules, and module homomorphisms. Then,* $\mathcal{Q}$tld *is* Sup-Prof, *the bicategory of* Sup*-categories,* Sup*-profunctors, and* Sup*-transformations.*

By Proposition 7.6 and taking $\mathscr{V}=$ Sup, we get:

Theorem 7.10. *The bicategory* $\mathcal{Q}$tld *is a cyclic* $*$*-autonomous bicategory.*

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