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GRADED π-RINGS

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1. Introduction. All rings considered will be commutative with identity. By a *graded ring* we will mean a ring graded by the non-negative integers.

A ring R is called a *π-ring* if every principal ideal of R is a product of prime ideals. A π-ring without divisors of zero is called a *π-domain*. A graded ring (domain) is called a *graded π-ring* (*-domain*) if every homogeneous principal ideal is a product of homogenous prime ideals. A ring R is called a *general* ZPI-*ring* if every ideal is a product of primes. A graded ring is called a *graded general* ZPI-*ring* if every homogenous ideal is a product of homogeneous prime ideals.

In Section 2 we review the known results about (ungraded) π-rings and general ZPI-rings. Eight characterizations of π-domains are given, several of which are new. The characterization to be used in Section 3 is that a domain D is a π-domain if and only if D is locally a UFD (D_M is a UFD for every maximal ideal M of D) and D is a Krull domain.

In Section 3 we investigate graded π-rings. We show that a graded π-ring is a finite direct product of special principal ideal rings, graded π-domains and a special type of graded π-ring which is not a π-ring. We show that a graded π-domain is actually a π-domain. We also show that a graded general ZPI-ring is a general ZPI-ring.

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Section 2. The ungraded case. Mori has completely characterized the structure of π-rings in a series of four papers [**12**]–[**15**]. We state this characterization as Theorem 1, the proof of which may also be found in [**7**].

THEOREM 1. *A ring R is a π-ring if and only if R is a finite direct product of π-domains and special principal ideal rings.*

Thus the study of π-rings is essentially reduced to the study of π-domains. Next we give eight characterizations of π-domains.

THEOREM 2. *For a domain D the following conditions are equivalent:*

(1) D is a π-domain, (2) every principal ideal is a product of invertible prime ideals, (3) every invertible ideal is a product of invertible prime ideals, (4) every nonzero prime ideal contains an invertible prime ideal, (5) D is locally a UFD *and the minimal primes are finitely generated, (6) D is locally a* UFD *and a Krull domain, (7) D is a Krull domain with the minimal primes being invertible, (8) $D(X)$ is a* UFD.

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Proof. (*1*) $\Rightarrow$ (*2*): Any factor of a principal ideal is invertible. (*2*) $\Rightarrow$ (*4*): Let P be a nonzero prime ideal and let $0 \neq x \in P$. Then $(x) = P_1 \ldots P_n$ a product of invertible prime ideals. Since P is prime, some $P_i \subset P$ and P_i is invertible. (*4*) $\Rightarrow$ (*3*): The proof then is similar to the proof of Theorem 5 [**8**] but using "generalized" multiplicatively closed sets. (Also see Theorem 4.6 [**2**]). As (*3*) $\Rightarrow$ (*1*) is trivial, we see that (*1*)–(*4*) are equivalent. (*1*) $\Rightarrow$ (*5*): A localization of a π-domain is a π-domain and in a quasi-local domain, invertible ideals are principal. (*5*) $\Rightarrow$ (*1*): Since D is locally a UFD, every nonzero prime contains a minimal prime P, which is by hypothesis finitely generated. Since P is finitely generated and locally principal, P is invertible. That (*1*) implies (*6*) is clear. (*6*) $\Rightarrow$ (*1*): Let $0 \neq x \in D$ be a nonunit. We show that xD is a product of prime ideals. Since D is a Krull domain, $xD = P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$ where $P_1, \ldots, P_s$ are the rank one primes containing x. We show that $xD = P_1^{n_1} \ldots P_s^{n_s}$ locally. Let M be a fixed maximal ideal of D. If $P_i \not\subseteq M$, then $P_{i_M}^{(n_i)} = D_M = P_{i_M}^{n_i}$. If $P_i \subseteq M$, then P_{i_M} is a rank one prime in the UFD D_M and hence is principal. Thus $P_{i_M}^{n_i}$ is primary and hence $P_{i_M}^{n_i} = P_{i_M}^{(n_i)}$. Since the P_{i_M}'s are principal,

$$xD_M = P_{1_M}^{(n_1)} \cap \ldots \cap P_{s_M}^{(n_s)} = P_{1_M}^{n_1} \cap \ldots \cap P_{s_M}^{n_s}$$
$$= P_{1_M}^{n_1} \ldots P_{s_M}^{n_s} = (P_1^{n_1} \ldots P_s^{n_s})_M.$$

Thus (*6*) $\Rightarrow$ (*1*). It is clear that (*1*)–(*6*) $\Rightarrow$ (*7*) and that (*7*) $\Rightarrow$ (*6*). If D is a π-domain, then $D[X]$ is also a π-domain as is easily seen from the equivalence of (*1*) and (*6*). Thus $D(X) = D[X]_S$ is a π-domain where $S = \{f \in D[X] | A_f = D\}$ and A_f is the content of f. Since every invertible ideal in $D(X)$ is principal (Theorem 2 [**4**]), $D(X)$ is a UFD. Hence (*1*) $\Rightarrow$ (*8*). Conversely, suppose that $D(X)$ is a UFD. By Proposition 6.10 [**6**], D is a Krull domain and every rank one prime ideal of D is invertible. Hence D is a π-domain.

Theorem 2 supports our philosophy that a π-domain is just a UFD where invertible ideals have taken the place of principal ideals. Thus π-domains are related to UFD's in a manner similar to the way that Dedekind domains are related to PID'S. One question of interest is: Given a π-domain D, does there exist a UFD D' such that D and D' have isomorphic lattices of ideals? (See [**1**] and [**3**] for a discussion of this question.)

The equivalence of (*1*), (*5*), and (*7*) appears as Theorem 46.7 [**7**, page 573].

The following theorem characterizes general ZPI-rings. The equivalence of (*1*) and (*2*) is due to Mori [**16**] and the equivalence of (*1*) and (*3*) to Levitz [**9**], [**10**]. Also see [**7**].

THEOREM 3. *For a ring R the following statements are equivalent:*

(*1*) *R is a general* ZPI*-ring,* (*2*) *every ideal of R generated by two elements is a product of prime ideals,* (*3*) *R is a finite direct product of Dedekind domains and special principal ideal rings.*

Section 3. The graded case. In this section we consider graded π-rings and graded $\rightarrow$ general ZPI-rings of the form $R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$. Our characterization of graded π-rings will be given by a number of lemmas. Our first lemma follows directly from Theorem 1.

LEMMA 1. *Suppose that $R = R_0 \oplus R_1 \oplus \ldots$ is a graded π-ring. Then R_0 is a π-ring. Moreover, R is a finite direct product of graded π-rings each of which has for its zero component a π-domain or a special principal ideal ring.*

The case where R_0 is a special principal ideal ring is easily handled.

LEMMA 2. *Suppose that $R = R_0 \oplus R_1 \oplus \ldots$ is a graded π-ring where R_0 is a special principal ideal ring. Then $0 = R_1 \oplus R_2 \oplus \ldots$.*

Proof. Let $0 \neq pR_0$ be the unique prime ideal of R_0 and suppose that $p^n = 0$. Let $a \in R_1$, then aR is a product of homogeneous prime ideals. Since the zero degree part of any homogeneous prime ideal must be pR_0, we see that $R_1 = pR_1$. Hence $R_1 = p^n R_1 = 0$. By induction $R_m = 0$ for $m > 0$.

Thus we are reduced to the case where R_0 is a π-domain.

LEMMA 3. *Let $R = R_0 \oplus R_1 \ldots$ be a graded π-ring. Then any rank zero prime P in R is a "homogeneous" multiplication ideal (i.e., $A \subseteq P$ with A homogeneous implies $A = BP$ for some homogeneous ideal B of R.) Furthermore, $P \cap R_0$ is a multiplication ideal of R_0.*

Proof. It is well-known that a rank zero prime in a graded ring is homogeneous. Let $A \subseteq P$ be a homogeneous ideal and let $A = (x_\alpha)$ where x_α is homogeneous. Then $x_\alpha R = P_{\alpha_1} \ldots P_{\alpha_t}$ is a product of homogeneous prime ideals. Now rank $P = 0$ implies some $P_{\alpha_i} = P$ so that $x_\alpha R = PB_\alpha$ for some homogeneous ideal B_α. Hence $A = (x_\alpha) = \sum PB_\alpha = P(\sum B_\alpha)$. It is easily seen that $P \cap R_0$ is a multiplication ideal in R_0.

LEMMA 4. *Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded π-ring where R_0 is a field. Then R is a domain or $R \approx R_0[X]/(X^n)$ for some $n > 1$ where X ix an indeterminate over R_0 assigned positive degree.*

Proof. Suppose that R is not a domain. Now $M = R_1 \oplus R_2 \oplus \ldots$ is the unique maximal homogeneous ideal of R. We show that rank $M = 0$. Now since (0) is a finite product of (homogeneous) primes, R has only a finite number of minimal primes $P_1, \ldots, P_n$, each of which is homogeneous. Assume that $P_i \subsetneq M$ for $i = 1, \ldots, n$. We set $A = P_1 \cap \ldots \cap P_n$ and $\bar{R} = R/A$. It is easy to see that $Z(\bar{R}) = P_1/A \cup \ldots \cup P_n/A$ (here $Z(\bar{R})$ denotes the zero-divisors of $\bar{R}$.) By Prop. 8 [**5**, p. 161] there exists a homogeneous element $m \in M - (P_1 \cup \ldots \cup P_n)$ and $\bar{m} = m + A$ is a regular element of $\bar{R}$. Let $(m) = Q_1 \ldots Q_t$ be a prime factorization of (m) into a product of homogeneous prime ideals. Then $(\bar{m}) = \bar{Q}_1 \ldots \bar{Q}_t$ is a prime factorization of $(\bar{m})$ in

$\bar{R}$. Since $\bar{m}$ is regular, the ideal $\bar{Q}_1$ is invertible and $\bar{Q}_1$ properly contains some $\bar{P}_i$. Therefore $\bar{P}_i = \bar{P}_i\bar{Q}_1$ and hence $\bar{P}_i = \bar{P}_i\bar{M}$. Suppose that $\bar{P}_i \neq 0$. Then there exists a nonzero homogeneous element $y \in \bar{P}_i$. By Lemma 3, $(y) = B\bar{P}_i$ for some homogeneous ideal B. Hence $(y) = B\bar{P}_i = B(\bar{P}_i\bar{M}) = (B\bar{P}_i)\bar{M} = (y)\bar{M}$. Thus $\bar{R} = \bar{M} + (\bar{0}:y)$. But since y is a nonzero homogeneous element, $(\bar{0}:y)$ is a proper homogeneous ideal and hence $(\bar{0}:y) \subseteq \bar{M}$, the unique maximal homogeneous ideal of $\bar{R}$. Thus $\bar{P}_i = \bar{0}$. Hence $P_i = A$ so R has a unique prime P of rank zero. Thus R/P is a graded π-domain, in fact since $(R/P)_0 = R_0$ is a field, R/P is a graded UFD and hence a UFD (Theorem 5). Choose a homogeneous non-zero prime element $q + P$ of R/P. If $(q) = Q_1 \ldots Q_t$ is a homogeneous prime factorization of (q) in R, then $(\bar{q}) = \bar{Q}_1 \ldots \bar{Q}_t$ is the prime factorization of $(\bar{q})$ in R/P. Consequently $t = 1$ and (q) is a homogeneous prime ideal of R with $P \subsetneq (q) \subseteq M$. Hence $P = P(q)$ and so $P = PM$. As before, this implies that $P = 0$. This contradiction shows that M is the unique minimal prime ideal of R and hence the unique homogeneous prime ideal of R. We show that M is principal. Let $M = (x_\alpha)$ where x_α is homogeneous. By Lemma 3, $(x_\alpha) = MB_\alpha$ where B_α is some homogeneous ideal. Hence $M = \sum(x_\alpha) = \sum MB_\alpha = M(\sum B_\alpha)$. If $\sum B_\alpha = R$, then some $B_{\alpha_0} = R$ so $M = (x_{\alpha_0})$ is principal. Otherwise $M = M^2$ and the argument used above shows that $M = 0$. Let X be an indeterminate over R_0 assigned the degree of x_{α_0}. Then the graded homomorphism $f : R_0[X] \to R$ given by $X \to x_0$ is clearly onto. Since M is the unique homogeneous prime of R, there exists an $n > 0$ such that $M^n = 0$, but $M^{n-1} \neq 0$. Thus $\ker f = (X^n)$ so $R \approx R_0[X]/(X^n)$.

LEMMA 5. *Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded π-ring where (R_0, M_0) is a quasi-local domain but not a field. Then R is either a domain or R_0 is a* DVR *and $R \approx R_0[X]/A$ where A is a homogeneous ideal with $\sqrt{A} = XM_0[X]$.*

Proof. First suppose that $\dim R_0 > 1$. Then R_0 is a quasi-local UFD with an infinite number of principal primes. Assume that R is not a domain, so that R has a finite number of minimal primes $P_1, \ldots, P_n$. By Lemma 3, $P_i \cap R_0$ is a multiplication ideal, so each $P_i \cap R_0$ is either 0 or a principal prime. Thus we can choose a homogeneous element in $M_0 \oplus R_1 \oplus R_2 \oplus \ldots$, but not in $P_1, \ldots, P_n$. Proceeding as in Lemma 4, we get that R must be a domain. Thus we may suppose that $\dim R_0 = 1$, so that R_0 must be a DVR. Since R_0 is a domain, $Q = R_1 \oplus R_2 \oplus \ldots$ is a prime ideal. We show that rank $Q = 0$. Let $S = R_0 - \{0\}$, then $R_S = R_{0_S} \oplus R_{1_S} \oplus \ldots$ is a graded π-ring with R_{0_S} a field. Hence by Lemma 4, R_S contains a unique minimal prime, and hence R must contain a unique minimal prime P with $P \cap R_0 = 0$. Let $M_0 = pR_0$. Now pR is a product of homogeneous primes and hence itself must be prime. Now pR must be minimal. For if $P' \subsetneq pR$ is a prime, then either $P' \cap R_0 = 0$ so $pR \supsetneq P' \supseteq P$ or $P' \cap R_0 = pR_0$ so $P' \supseteq pR$. If $pR \supset P$, then P would be the unique minimal prime of R. Passing to R/P we see that this would imply that $P = (0)$ and thus R would be a domain. Thus R has exactly two minimal primes: pR and P. As in Lemma 4, we see that P is principal. Suppose that

$Q \supsetneq P$. Then by Proposition 8 [**5**, page 161], there exists a homogeneous element $m \in pR_0 \oplus R_1 \oplus \ldots$, but not in pR or P. Proceeding as in Lemma 4, we see that $R/pR \cap P$ must have a unique minimal prime. This contradiction shows that $Q = P$. Thus $P = Q = R_1 \oplus R_2 \oplus \ldots$ is principal. The result now follows as in Lemma 4.

LEMMA 6. *Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded π-ring where R_0 is a domain but not a field. Then either R is a domain or $R \approx R_0[X]/A$ where A is a homogeneous ideal of $R_0[X]$ with $\sqrt{A} = XM_1 \ldots M_n[X]$ where $M_1, \ldots, M_n$ are invertible maximal ideals of R_0.*

Proof. Assume that R is not a domain. Let $S = R_0 - \{0\}$, then R_S is a graded π-ring with R_{0_S} a field, so that R_S is a domain or is isomorphic to $R_0[X]/(X^n)$ and hence contains a unique minimal prime. Hence R contains a unique minimal (necessarily homogeneous) prime P with $P \cap R_0 = 0$. Let M_0 be a maximal ideal of R_0 and put $S(M_0) = R_0 - M_0$. Then $R_{S(M_0)}$ is a graded π-ring so $R_{S(M_0)}$ is a domain or $P_{S(M_0)} = (R_1 \oplus R_2 \oplus \ldots)_{S(M_0)}$. In the latter case $P = R_1 \oplus R_2 \oplus \ldots$ (for both are prime ideals of R). Suppose that $P \neq R_1 \oplus R_2 \oplus \ldots$. Then we may assume that $R_{S(M_0)}$ is a domain for every maximal ideal M_0 of R_0. Thus $P_{S(M_0)} = 0_{S(M_0)}$ for every maximal ideal M_0 of R_0, so that $P_M = 0_M$ for every homogeneous maximal ideal of R. Hence $P = 0$ and R is a domain. This contradiction shows that $P = R_1 \oplus R_2 \oplus \ldots$ is the unique minimal prime ideal of R contracting to 0 in R_0.

Suppose that $P, P_1, \ldots, P_n$ are the minimal prime ideals of R ($n > 0$ since R is not a domain). Then $P_i' = P_i \cap R_0 \neq 0$ is a multiplication ideal in the domain R_0. Thus P_i' is invertible [**7**, page 77]. Let M be a maximal ideal of R_0 containing P_i' and put $S = R_0 - M$. Then P_{iS} and P_S are distinct minimal primes in R_S. By Lemma 5, R_{0_S} must be a DVR and hence we see that each P_i' is also a maximal ideal in R_0. Also, $P_i'R$ and P_i are homogeneous ideals that are equally locally at the maximal homogeneous ideals of R. Thus $P_i'R = P_i$. We next show that $P = R_1 \oplus R_2 \oplus \ldots$ is principal. Let M be a maximal homogeneous ideal containing P. Let $M_0 = M \cap R_0$ and $S = R_0 - M_0$. If $P_i \subseteq M$ for some i, then R_S contains two minimal prime ideals. By Lemma 5, $M = P_i' \oplus R_1 \oplus R_2 \oplus \ldots$. If $P_i \not\subseteq M$ for all $i = 1, \ldots, n$, then P_S is the unique minimal prime ideal of R_S and hence R_S is a domain. Then $P_M = 0_M$. Thus $P_M = 0_M$ for almost all maximal homogeneous ideals M of R. An easy modification of Theorem 2 [**3**] shows that P is principal. Thus $R \approx R_0[X]/A$ where A is a homogeneous ideal of $R_0[X]$. Since $\sqrt{0} = P \cap P_1 \cap \ldots \cap P_n$ in R, we have $\sqrt{A} = (X) \cap P_1'[X] \cap \ldots \cap P_n'[X] = XP_1' \ldots P_n'[X]$ in $R_0[X]$.

LEMMA 7. *Let R_0 be a π-domain that is not a field. Suppose that A is a homogeneous ideal of $R_0[X]$ with $\sqrt{A} = XM_1 \ldots M_n[X]$ where $\{M_1, \ldots, M_n\}$ is a (possibly empty) set of invertible maximal ideals of R_0. Then $R = R_0[X]/A$ is a graded π-ring if and only if $A = X^s M_1^{s_1} \ldots M_n^{s_n}[X]B$ where $s, s_1, \ldots, s_n$ are* *positive integers, B is a (possibly vacuous) product of* $M_i[X] + (X)$*-primary ideals and* $s = 1$ *unless* $\{M_1, \ldots, M_n\}$ *is the set of all maximal ideals of R. R is a* π*-ring if and only if* $A = (X)$.

Proof. Suppose that $A = X^s M^{s_1} \ldots M^{s_n}[X]B$. Then the ideals $\bar{X}R$, $M_1 R$, $\ldots$, $M_n R$ are prime ideals in R. If N is another invertible prime ideal in R_0, then $N[X]$ and $M_1^{s_1} \ldots M_n^{s_n}[X]B$ are comaximal. Thus

$$N[X] + M_1^{s_1} \ldots M_n^{s_n}[X]B = R[X] \quad \text{so}$$
$$XN[X] + XM_1^{s_1} \ldots M_n^{s_n}[X]B = (X).$$

Since in this case $s = 1$, $N[X] + A = N[X] + (X)$ so NR is also a prime ideal in R. Since every homogeneous element of R has the form $r\bar{X}^m$ where $r \in R_0$ and $\bar{X} = X + A$, R is a graded π-ring.

Conversely, suppose that R is a graded π-ring. Now A has a homogeneous primary decomposition with minimal primes (X), $M_1[X], \ldots, M_n[X]$. Since each of these primes is invertible, the primary ideals belonging to these minimal primes are prime powers. From Lemma 5 we see that $M_i[X] + (X)$, $i = 1, \ldots, n$ are the only possible embedded prime ideals. Thus

$$A = (X)^s \cap M_1^{s_1}[X] \cap \ldots \cap M_n^{s_n}[X] \cap Q_1 \cap \ldots \cap Q_n$$

where Q_i is either $M_i[X] + (X)$-primary or $R_0[X]$. Since $(X)^s$, $M_1^{s_1}[X], \ldots, M_n^{s_n}[X]$ are invertible primary ideals, we have

$$(X)^s \cap M_1^{s_1}[X] \cap \ldots \cap M_n^{s_n}[X] = (X)^s M_1^{s_1}[X] \ldots M_n^{s_n}[X].$$

Hence

$$A = (X)^s M_1^{s_1}[X] \ldots M_n^{s_n}[X] \cap Q_1 \cap \ldots \cap Q_n$$
$$= (X)^s M_1^{s_1}[X] \ldots M_n^{s_n}[X](Q_1 \cap \ldots \cap Q_n : (X)^s M_1^{s_1}[X] \ldots M_n^{s_n}[X]).$$

But

$$(Q_1 \cap \ldots \cap Q_n : (X)^s M_1^{s_1}[X] \ldots M_n^{s_n}[X])$$
$$= \cap_{i=1}^n (Q_i : (X)^s M_1^{s_1}[X] \ldots M_n^{s_n}[X] \quad \text{and}$$
$$Q_i' = Q_i : (X)^s M_1^{s_1}[X] \ldots M_n^{s_n}[X]$$

is either $M_i[X] + (X)$-primary or $R_0[X]$. Since $Q_1', \ldots, Q_n'$ are comaximal, $Q_1' \cap \ldots \cap Q_n' = Q_1' \ldots Q_n'$. Suppose that M is a maximal ideal of R_0 other than $M_1, \ldots, M_n$. Then $R_{(R_0 - M)} = R_{0_M}[X]/(X)^s R_{0_M}$ is a graded π-ring. By Lemma 5 this is not possible unless $R_{(R_0 - M)}$ is a domain, that is, $s = 1$.

Clearly if $A = (X)$, $R = R_0[X]/A$ is a π-domain. If $A \neq (X)$, then R is not a domain. Since R is indecomposable, R cannot be a π-ring.

Thus we have established

THEOREM 4. *Let* $R = R_0 \oplus R_1 \oplus \ldots$ *be a graded* π*-ring. Then R is a finite direct product of graded* π*-domains and special graded* π*-rings of the following types:*

(1) special principal ideal rings (ungraded), (2) $k[X]/(X^n)$, k a field, X an indeterminate assigned positive degree, (3) $D[X]/A$ where D is a π-domain, X is an indeterminate over D assigned positive degree and A is a homogeneous ideal of $D[X]$ with

$$A = X^s M_1^{s_1}[X] \ldots M_n^{s_n}[X]B$$

where $s, \ldots, s_n$ are positive integers, $\{M_1, \ldots, M_n\}$ is a (possibly empty) set of invertible maximal ideals of D and B is a (possibly vacuous) product of $M_i[X] + (X)$-primary ideals. If $M_1, \ldots, M_n$ are not all the invertible prime ideals of D, then $s = 1$.

We are now reduced to the case where $R = R_0 \oplus R_1 \oplus \ldots$ is a graded π-domain.

THEOREM 5. *Let $R = R_0 \oplus R_1 \oplus \ldots$. If R is a graded* UFD, *then R is a* UFD. *If R is a graded π-domain where R_0 is quasi-local, then R is a graded* UFD *and hence a* UFD.

Proof. We may assume that $R \neq R_0$. Let S be the set of homogeneous non-zero elements of R. Now S is a multiplicatively closed set in R generated by the non-zero homogeneous principal primes. By Lemma 1.2 [**11**], R_S is isomorphic to $K[u, u^{-1}]$ where K is a field and u is transcendental over K. Thus R_S is a UFD. By Nagata's Lemma to show that R is a UFD it is sufficient to show that R satisfies ACC on principal ideals. Let $(f_1) \subseteq (f_2) \subseteq (f_3) \subseteq \ldots$ be an ascending chain of principal ideals in R. Surely R satisfies ACC on principal homogeneous ideals. It is easily verified that $R[X]$ satisfies ACC on homogeneous principal ideals when X is an indeterminate assigned degree 1. We homogenize the chain of principal ideals to $R[X]$ and then de-homogenize them back into R (for the process of homogenization see [**11**] or [**17**, p. 179]). Thus $(f_1)^h \subseteq (f_2)^h \subseteq (f_3)^h \subseteq \ldots$ is an ascending chain of homogeneous principal ideals in $R[X]$. Hence the chain becomes stable, say $(f_n)^h = (f_{n+1})^h = \ldots$. De-homogenizing the chain we get that $(f_n)^{ha} = (f_{n+1})^{ha} = \ldots$ in R. But since for any ideal I in R, $I^{ha} = I$, we have $(f_n) = (f_{n+1}) = \ldots$. Thus R satisfies the ascending chain condition on principal ideals. We remark that this same proof also applies to Z-graded UFD's.

Suppose that R is a graded π-domain where R_0 is quasi-local. Then every homogeneous invertible ideal of R is principal. Hence R is a graded UFD and hence a UFD.

THEOREM 6. *A graded π-domain $R = R_0 \oplus R_1 \oplus \ldots$ is a π-domain.*

Proof. Let M be a maximal ideal of R and let $M_0 = M \cap R_0$. Then $R_{(R_0 - M_0)}$ is a π-domain with $R_{(R_0 - M_0)}$ quasi-local. By Theorem 5, $R_{(R_0 - M_0)}$ is a UFD and hence R_M is a UFD. Thus R is locally a UFD. We show that R is a Krull domain. Since R is locally a UFD, R_P is a DVR for every rank one prime P in R and $R = \cap R_P$ where the intersection runs over all rank one primes of

R. Let $0 \neq x \in R$ be a nonunit. We must show that x is contained in only finitely many rank one primes of R. If x is homogeneous, the result is clear, so suppose that x is not homogeneous. Since a homogeneous component of x can be contained in only finitely many rank one homogeneous prime ideals, x can be contained in only finitely many rank one homogeneous prime ideals of R. Now any rank one non-homogeneous prime ideal Q containing x must satisfy $Q \cap R_0 = 0$ (since rank $Q = 1$, Q^*, the prime ideal generated by the homogeneous elements of Q, must be 0). Putting $S = R_0 - \{0\}$, $R_S = R_{0_S} \oplus T_{1_S} \oplus \ldots$ is a graded π-domain with R_{0_S} a field, so R_S is a graded UFD and hence a UFD. Thus xR_S is contained in only finitely many rank one primes and hence the same is true of xR.

THEOREM 7. *Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded ring in which every ideal generated by two homogeneous elements is a product of homogeneous prime ideals. Then R is a general* ZPI *ring. Further, R is a finite direct product of the following types of (graded) general* ZPI *rings: (1) R_0 a special principal ideal ring and $0 = R_1 \oplus R_2 \oplus \ldots$, (2) R_0 a Dedekind domain and $0 = R_1 \oplus R_2 \oplus \ldots$, (3) R_0 a field (a) $0 = R_1 \oplus R_2 \oplus \ldots$, (b) $R \approx R_0[X]$, (c) $R \approx R_0[X]/(X^n)$.*

Proof. It is easily seen that in R_0 every ideal generated by two elements is a product of prime ideals. Hence R_0 is a general ZPI-ring and hence by Theorem 3 is a finite direct product of special principal ideal rings and Dedekind domains. Thus we see that R is a finite direct product of graded rings where the zero coordinate is either a special principal ideal ring or a Dedekind domain. If R_0 is a special principal ideal ring, then $0 = R_1 \oplus R_2 \oplus \ldots$ by Lemma 2. Thus we may assume that R_0 is a field or a Dedekind domain. If R_0 is a field, but R is not a domain, then $R \approx R_0[X]/(X^n)$ by Lemma 4. So suppose that R is a domain and $0 \neq R_1 \oplus R_2 \oplus \ldots$. By Lemma 4.8 [**2**], we see that $R_1 \oplus R_2 \oplus \ldots$ is a principal prime ideal and hence $R \approx R_0[X]$. We are reduced to the case where R_0 is a Dedekind domain. It is easily seen that the rings occurring in case (3) of Theorem 4 do not satisfy the hypothesis of the Theorem. Thus R must be a domain. By Theorem 4.9 [**2**] we see that every homogeneous non-zero prime ideal in R is maximal. Thus since $0 \subseteq R_1 \oplus R_2 \oplus \ldots \subsetneq M \oplus R_1 \oplus R_2 \ldots$ for any maximal ideal M of R_0, we must have $0 = R_1 \oplus R_2 \oplus \ldots$.

REFERENCES

1. D. D. Anderson, *A remark on the lattice of ideals of a Prüfer domain*, Pacific J. Math. *57* (1975), 323–324.
2. ——— *Abstract commutative ideal theory without chain condition*, Algebra Universalis *6* (1976), 131–145.
3. ——— *Multiplication ideals, multiplication rings, and the ring R(X)*, Can. J. Math. *28* (1976), 760–768.
4. ——— *Some remarks on the ring R(X)*, Comment. Math. Univ. St. Pauli *26* (1977), 137–140.
5. N. Bourbaki, *Commutative algebra* (Addison-Wesley, Reading, Mass., 1972).
6. R. Fossum, *The divisor class group of a Krull domain* (Springer-Verlag, New York, 1973).
7. R. Gilmer, *Multiplicative ideal theory* (Marcel Dekker, Inc., New York, 1972).

8. I. Kaplansky, *Commutative rings* (Allyn and Bacon, Boston, 1969).
9. K. Levitz, *A characterization of general* ZPI-*rings*, Proc. Amer. Math. Soc. *32* (1972), 376–380.
10. ——— *A characterization of general* ZPI-*rings II*, Pacific J. Math. *42* (1972), 147–151.
11. J. Matijevic, *Three local conditions on a graded ring*, Trans. Amer. Math. Soc. *205* (1975), 275–284.
12. S. Mori, *Über die produktzerlegung der hauptideale I*, J. Sci. Hiroshima Univ. (A), *8* (1938), 7–13.
13. ——— *Über die produktzerlegung der hauptideale II*, J. Sci. Hiroshima Univ. (A), *9* (1939), 145–155.
14. ——— *Über die produktzerlegung der hauptideale III*, J. Sci. Hiroshima Univ. (A), *10* (1940), 85–94.
15. ——— *Über die produktzerlegung der hauptideale IV*, J. Sci. Hiroshima Univ. (A), *11* (1941), 7–14.
16. ——— *Allegemeine* Z.P.I.-*ringe*, J. Sci. Hiroshima Univ. (A), *10* (1941), 117–136.
17. O. Zariski and P. Samuel, *Commutative algebra*, Vol. II (Van Nostrand, New York, 1960).

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