A note on inverse central factorial series

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1. The purpose of this brief note is to draw attention to a type of inverse factorial series which, so far as the writer can judge, has not been intensively studied. The central difference formulae of interpolation and the corresponding infinite series in central factorial polynomials have in the past received much attention, and so also has the ordinary inverse factorial series

$$\frac{a_0}{x} + \frac{a_1\, 1!}{x(x+1)} + \frac{a_2\, 2!}{x(x+1)(x+2)} + \dots \quad (1)$$

which from the time of Stirling (*Methodus Differentialis*, 1730) has been known to be of value in transforming slowly convergent series into series of more rapid convergence.

We shall first give a few examples of representation by inverse central factorial series, all capable of fairly simple formal proof, and we shall set out for comparison some parallel results in ordinary inverse factorial series. Thus, corresponding to

$$\frac{1}{x-a} = \frac{1}{x} + \frac{a}{x(x+1)} + \frac{a(a+1)}{x(x+1)(x+2)} + \dots \quad (2)$$

there is

$$\frac{1}{x^2-a^2} = \frac{1}{x^2-\frac{1}{4}} + \frac{a^2-\frac{1}{4}}{(x^2-\frac{1}{4})(x^2-\frac{9}{4})} + \frac{(a^2-\frac{1}{4})(a^2-\frac{9}{4})}{(x^2-\frac{1}{4})(x^2-\frac{9}{4})(x^2-\frac{25}{4})} + \dots \quad (3)$$

In this case an exact form for the remainder can be given. It is simply the next following term, with the last polynomial factor in the denominator changed to $x^2 - a^2$.

Again, corresponding to

$$\sum_{x=n}^{\infty} \frac{1}{x^2} = \frac{1}{n} + \frac{1}{2}\cdot\frac{1!}{n(n+1)} + \frac{1}{3}\cdot\frac{2!}{n(n+1)(n+2)} + \dots, \quad (4)$$

an example known to Stirling (*loc. cit.*), there is

$$\sum_{x=n+a}^{\infty} \frac{1}{x^2-a^2} = \frac{1}{\nu} + \frac{1}{3}\cdot\frac{a^2-\frac{1}{4}}{\nu(\nu^2-1)} + \frac{1}{5}\cdot\frac{(a^2-\frac{1}{4})(a^2-\frac{9}{4})}{\nu(\nu^2-1)(\nu^2-4)} + \dots,$$
$$\nu = n + a - \tfrac{1}{2}, \quad (5)$$

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to which we may also add a further useful result, namely

$$\sum_{x=n+a}^{\infty} \frac{1}{x(x^2-a^2)} = \frac{1}{2}\cdot\frac{1}{\nu^2-\frac{1}{4}} + \frac{1}{4}\cdot\frac{a^2-1}{(\nu^2-\frac{1}{4})(\nu^2-\frac{9}{4})} + \frac{1}{6}\cdot\frac{(a^2-1)(a^2-4)}{(\nu^2-\frac{1}{4})(\nu^2-\frac{9}{4})(\nu^2-\frac{25}{4})} + \cdots\cdots \quad (6)$$

The above are but a few examples of many cases where an inverse central factorial series represents a function in a useful and possibly elegant form.

The individual terms of such a series involve inverse central factorial polynomials, and the formal rules that these obey, under central differencing and central summing operations, are at once evident.

2. We proceed to illustrate the use of these series in transforming a slow convergence into a rapid one by taking three examples well known in this domain, (i) Brouncker's series for $\log_e 2$, (ii) the series Σx^{-2}, (iii) Σx^{-3}.

(i) The sum of the first ten terms of Brouncker's series is found by addition to be 0·6687714032, to ten decimals. Next, by applying § 1 (3) with $a = \frac{1}{4}$ we have for the remaining terms

$$\frac{1}{21.22} + \frac{1}{23.24} + \frac{1}{25.26} + \cdots\cdots = \frac{1}{41} - \frac{1}{3}\cdot\frac{1.3}{37.41.45} + \frac{1}{5}\cdot\frac{1.3.5.7}{33.37.41.45.49} - \cdots\cdots \quad (1)$$

The fifth term of the series on the right is already less than 10^{-9}, and the sum of five terms yields 0·0243757775. We thus derive 0·6931471807, while $\log_e 2 = 0{\cdot}693147180559\ldots\ldots$

It is interesting to compare the series (1) above with the equivalent one in Stirling's procedure (*cf.* Whittaker and Robinson, *Calculus of Observations*, p. 369)

$$\frac{1}{2}\left(\frac{1}{21} + \frac{1}{2}\cdot\frac{1}{21.22} + \frac{1}{3}\cdot\frac{1.3}{21.22.23} + \frac{1}{4}\cdot\frac{1.3.5}{21.22.23.24} + \cdots\cdots\right). \quad (2)$$

In both cases convergence begins strongly in the early terms, but becomes less as the terms proceed. In fact in the inverse central factorial series we soon reach a minimal term; after this the terms increase again in numerical value.

(ii) The sum of the first ten terms is 1·5497677312. By applying § 1 (5) with $a = 0$, we have for the remaining terms

$$\frac{2}{21}\left(1 - \frac{1}{3} \cdot \frac{1}{437} + \frac{1}{5} \cdot \frac{1.9}{437.425} - \frac{1}{7} \cdot \frac{1.9.25}{437.425.405} + \ldots\right), \quad (3)$$

and five of these yield 0·0951663362. Thus we have 1·6449340674, for comparison with $\pi^2/6$, namely 1·6449340668 .....

(iii) The sum of the first ten terms gives 1·197531986. By § 1 (6), again with $a = 0$, we have for the remaining terms

$$\frac{1}{2}\left(\frac{1}{10.11} - \frac{1}{2} \cdot \frac{1}{9.10.11.12} + \frac{1}{3} \cdot \frac{1.4}{8.9.10.11.12.13} - \ldots\right), \quad (4)$$

and five of these give 0·004524918. So we have 1·202056904, for comparison with the true value 1·202056903 .....

These examples are sufficient to show the power of the transformation. An interesting point in (iii) is that the eleventh and all further terms of the series (4) are infinite, yet this does not affect the approximating value of the earlier terms.

3. From the theoretical point of view, interesting questions arise in regard to the representation of a given function $f(x)$ by an inverse factorial series, the generating function of the coefficients and the domain of convergence. The writer owes to Dr A. Erdélyi the remark that just as the coefficients a_n in an ordinary inverse factorial series for a Laplace transform,

$$f(x) = \int_0^\infty e^{-xt} g(t)\, dt \quad (1)$$

depend on the coefficients of $g(t)$ when expanded in powers of $1 - e^{-t}$, so here, since

$$\frac{(2n)!\,\Gamma(x-n)}{\Gamma(x+n+1)} = \int_0^\infty e^{-xt} (2 \sinh \tfrac{1}{2} t)^{2n}\, dt, \quad (2)$$

the inverse central factorial series

$$f(x) = \Sigma\, a_n (2n)!\,\Gamma(x-n) / \Gamma(x+n+1) \quad (3)$$

will bear an equivalence with the expansion of $g(t)$ in powers of $\sinh^2 \frac{1}{2} t$. There is here scope for considerable investigation.

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