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LETTERS TO THE EDITOR

DISPERSIVE ORDERING IS THE SAME AS TAIL-ORDERING

J. V. DESHPANDE* AND
S. C. KOCHAR,* *Panjab University*

Recently Lewis and Thompson (1981) and Shaked (1982) have defined the following partial ordering on the space of probability distribution functions. Let F and G be two cumulative distribution functions with F^{-1} and G^{-1} as the respective left-continuous inverses. Then F is said to be dispersed with respect to $G(F \overset{\text{disp}}{<} G)$ if and only if

(1) $$F^{-1}(\beta)-F^{-1}(\alpha)\leqq G^{-1}(\beta)-G^{-1}(\alpha) \quad \text{whenever} \quad 0<\alpha<\beta<1.$$

In these papers many interesting and important properties of this ordering have been studied. We wish to point out some connections that this ordering has with some concepts extensively used in reliability theory and related statistical inference.

It is easily seen by putting $\alpha=F(x)$ and $\beta=F(y)$, where $x\leqq y$ that $F\overset{\text{disp}}{<} G$ if and only if

(2) $$G^{-1}F(x)-x \text{ is non-decreasing in } x.$$

In the literature (see Doksum (1969)) Condition (2) has already been used. F is said to be tail-ordered with respect to $G(F \overset{t}{<} G)$ if and only if (2) holds. Doksum (1969) has also studied some properties of this ordering. Yanagimoto and Sibuya (1976), (1980) call this partial ordering (introduced originally by Fraser (1957)), 'G is stochastically more spread than F,' and have discussed some properties of this ordering. *Thus we see that dispersive ordering is the same as tail-ordering.* Doksum (1969), Deshpande and Kochar (1982) and Deshpande and Mehta (1982) have used this concept of tail-ordering in some inferential problems to obtain bounds on efficiencies of tests, probabilities of correct selections, etc.

For distributions with $F(0)=G(0)=0$ another partial ordering is given in Shaked (1982). It is characterized by

(3) $$\frac{G^{-1}(\beta)}{G^{-1}(\alpha)}\geqq\frac{F^{-1}(\beta)}{F^{-1}(\alpha)}, \quad \text{whenever} \quad 0<\alpha<\beta<1.$$

Again substituting $x=F^{-1}(\alpha)$ and $y=F^{-1}(\beta)$, $x\leqq y$, we see that (3) is true if and only if

(4) $$\frac{G^{-1}F(x)}{x} \quad \text{is non-decreasing in } x,$$

as noted by Yanagimoto and Sibuya as early as 1976.

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* Postal address: Department of Statistics, Panjab University, Chandigarh, India.

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This is used in the literature to define the well-known concept of star-ordering. F is said to be star-ordered with respect to $G(F \overset{*}{<} G)$ if $G^{-1}\ F(x)$ is a star-shaped function on $(0, \infty)$ or in other words $G^{-1}F(x)/x$ is non-decreasing (see e.g. Barlow and Proschan (1975)). Thus the results of Saunders and Moran (1978) and Shaked (1982) regarding the gamma family parametrized by the shape parameter being ordered according to (3) are the same as the well-known result that it is ordered according to star-ordering. In fact, it is known to be ordered according to the stronger convex-ordering ($G^{-1}F(x)$ is a convex function of x) as well.

Some of the other results proved in Lewis and Thompson (1981) and Shaked (1982), when stated in terms of tail-ordering and star-ordering are perhaps known to the workers in the area. A connected result first proved in Kochar (1978) is given below.

Theorem. Let F and G be absolutely continuous such that $F(0) = G(0) = 0$ and let the corresponding density functions be such that $f(0) \geqq g(0) > 0$. Then $f \overset{*}{<} G$ implies $F \overset{t}{<} G$.

Proof.

$$F \overset{*}{<} G \Leftrightarrow \frac{G^{-1}F(x)}{x} \text{ is non-decreasing in } x$$

(5)

$$\Leftrightarrow \frac{f(x)}{g[G^{-1}F(x)]} \geqq \frac{G^{-1}F(x)}{x} \quad \text{for } x \geqq 0$$

by differentiation. Now since

$$\lim_{x \to 0+} \frac{G^{-1}F(x)}{x} = \frac{f(0)}{g(0)} \geqq 1,$$

and $G^{-1}F(x)/x$ is non-decreasing, (5) implies $f(x) \geqq g[G^{-1}F(x)]$, $x > 0$

$$\Leftrightarrow F \overset{t}{<} G.$$

The last step is observed also in Shaked (1982), Remark 2.3.

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