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THE DENSITY OF j-WISE RELATIVELY r-PRIME ALGEBRAIC INTEGERS

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Abstract

Let K be a number field with a ring of integers $\mathcal{O}$. We follow Ferraguti and Micheli ['On the Mertens–Cèsaro theorem for number fields', *Bull. Aust. Math. Soc.* **93**(2) (2016), 199–210] to define a density for subsets of $\mathcal{O}$ and use it to find the density of the set of j-wise relatively r-prime m-tuples of algebraic integers. This provides a generalisation and analogue for several results on natural densities of integers and ideals of algebraic integers.

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1. Introduction

A recurring theme in analytic number theory concerns the distribution of integers with certain special properties. Classical results in this field include Dirichlet's theorem for primes in an arithmetic progression and the prime number theorem. However, the following fascinating facts about the probabilities (or more precisely, natural densities) of integers are not as well-known.

- The probability that m integers are relatively prime is $1/\zeta(m)$.
- The probability that an integer is rth power-free (where $r \geq 2$) is $1/\zeta(r)$.

Mertens [7] proved the first fact when $m = 2$ in 1874 and Lehmer [6] subsequently proved it in full generality in 1900. Gegenbauer [4] proved the second result in 1885. It was not until 1976 that Benkoski [1] generalised the work of both Gegenbauer and Lehmer, showing that if $rm > 1$, then the probability that m positive integers are relatively r-prime (that is, these integers have no common rth power prime factor) is $1/\zeta(rm)$.

Another way to generalise these results is to consider refinements to the notion of relative primality of integers. In 2002, Tóth [9] established that the probability that m

positive integers are *pairwise* relatively prime equals

$$\prod_{p \text{ prime}} \Big(1 - \frac{1}{p}\Big)^{m-1} \Big(1 + \frac{m-1}{p}\Big).$$

In 2012, Hu [5] extended Tóth's result further by showing that the probability of m positive integers being j-wise relatively prime equals

$$\prod_{p \text{ prime}} \Big[\sum_{k=0}^{j-1} \binom{m}{k} \Big(1 - \frac{1}{p}\Big)^{m-k} \Big(\frac{1}{p}\Big)^{k} \Big].$$

We can extend the scope of these statements by considering a fixed ring of algebraic integers $\mathcal{O}$. Benkoski's statement has been extended to *ideals* in an algebraic integer ring in 2010 by Sittinger [8]. More precisely, if $rm > 1$, then the probability that m ideals in $\mathcal{O}$ are relatively r-prime is $1/\zeta_{\mathcal{O}}(rm)$. In 2016, DeMoss [2] adapted Hu's work, showing that the probability that m ideals in $\mathcal{O}$ are j-wise relatively prime equals

$$\prod_{\mathfrak{p}} \Big[\sum_{k=0}^{j-1} \binom{m}{k} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big)^{m-k} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p})}\Big)^{k} \Big],$$

where it is understood that the product is taken over all prime ideals of $\mathcal{O}$. These results in $\mathcal{O}$ were derived by taking the existing arguments for their analogues in $\mathbb{Z}$ and modifying them with ideals, taking full advantage of the uniqueness of factoring ideals in $\mathcal{O}$ into prime ideals.

In a departure from this line of inquiry, Ferraguti and Micheli [3] proposed an extension of natural densities to ordered tuples of algebraic integers (to be defined in the next section), and used this approach to show that the probability that *m elements* in $\mathcal{O}$ are relatively prime is $1/\zeta_{\mathcal{O}}(m)$. It should be remarked that this probability matches that from Sittinger [8] with ideals in the case $r = 1$.

In this article, we not only remove the restriction on r from Ferraguti and Micheli's work, but also use their techniques to provide generalisations of the results of Tóth and Hu (which were in $\mathbb{Z}$) to elements in $\mathcal{O}$. In fact, we prove a result that has all of these results as special cases. We accomplish this with the following notion.

Definition 1.1. Fix $r, j, m \in \mathbb{N}$ where $j \le m$. Given an algebraic number ring $\mathcal{O}$, we say that $\beta_1, \ldots, \beta_m \in \mathcal{O}$ are *j-wise relatively r-prime* if $\mathfrak{p}^r \nmid \langle \beta_{i_1}, \ldots, \beta_{i_j} \rangle$ for any prime ideal $\mathfrak{p} \subseteq \mathcal{O}$ and for any integers $1 \le i_1 < \cdots < i_j \le m$.

When $m = 1$, this definition reduces to an algebraic integer being rth power-free. When $r = 1$, the definition reduces to that of j-wise relative primality. In addition, if $j = m$ or $j = 2$, then we retrieve the definitions for relative primality and pairwise relative primality, respectively. We now state the main result of this article.

THEOREM 1.2. *Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let K be an algebraic number field over $\mathbb{Q}$ with a ring of integers $\mathcal{O}$. Then, the density of the set E of j-wise relatively r-prime ordered m-tuples of elements of $\mathcal{O}$ equals*

$$\prod_{\mathfrak{p}} \bigg[\sum_{k=0}^{j-1} \binom{m}{k} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{k} \bigg].$$

After setting up the pertinent notation in Section 2, we prove Theorem 1.2 in Section 3.

2. Notation

Let K be an algebraic number field of degree n over $\mathbb{Q}$ with $\mathcal{O}$ as its ring of integers having integral basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$. As a way to generalise the notion of all positive integers less than or equal to some positive constant M, we define

$$\mathcal{O}[M, \mathcal{B}] = \bigg\{ \sum_{i=1}^{n} c_i \alpha_i : c_i \in [-M, M) \cap \mathbb{Z} \bigg\}.$$

Wherever the basis is understood, we abbreviate this as $\mathcal{O}[M]$.

Following [3], we define a notion of density for a subset T of $\mathcal{O}^m$ as follows.

DEFINITION 2.1. Let $T \subseteq \mathcal{O}^m$ and fix an integral basis $\mathcal{B}$ of $\mathcal{O}$.

(1) *Upper density of T with respect to $\mathcal{B}$*:

$$\overline{\mathbb{D}}_{\mathcal{B}}(T) = \limsup_{M \to \infty} \frac{|\mathcal{O}[M, \mathcal{B}]^m \cap T|}{(2M)^{mn}}.$$

(2) *Lower density of T with respect to $\mathcal{B}$*:

$$\underline{\mathbb{D}}_{\mathcal{B}}(T) = \liminf_{M \to \infty} \frac{|\mathcal{O}[M, \mathcal{B}]^m \cap T|}{(2M)^{mn}}.$$

(3) If $\overline{\mathbb{D}}_{\mathcal{B}}(T) = \underline{\mathbb{D}}_{\mathcal{B}}(T)$, the common value is called the *density of T with respect to $\mathcal{B}$* and denoted by $\mathbb{D}_{\mathcal{B}}(T)$. Whenever this density is independent of the chosen integral basis $\mathcal{B}$, we denote it simply as $\mathbb{D}(T)$.

Observe that this definition reduces to the classic notion of density (or probability) over $\mathbb{Z}$. Although the manner in which we cover $\mathcal{O}$ could potentially depend on the choice of the given integral basis $\mathcal{B}$, a corollary of the main result of this paper shows that the density of the set of j-wise relatively r-prime elements in $\mathcal{O}$ is actually independent of the integral basis used.

Finally, for any fixed rational prime p, suppose that $\mathfrak{p}$ is a prime ideal in $\mathcal{O}$ that lies above p; that is, $\mathfrak{p} \mid \langle p \rangle$. Then, we write $D_p = \sum_{\mathfrak{p} \mid \langle p \rangle} f_{\mathfrak{p}}$, where $f_{\mathfrak{p}}$ denotes the inertial degree of $\mathfrak{p}$.

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3. Density of j-wise relatively r-prime elements in $\mathcal{O}$

Let S be a finite set of rational primes, and fix positive integers r, j, m such that $j \le m$. Define E_S to be the set of m-tuples $z = (z_1, \ldots, z_m)$ in $\mathcal{O}^m$ such that any ideal generated by j entries of z is relatively r-prime with each $p \in S$. That is, E_S consists of the m-tuples of algebraic integers from $\mathcal{O}$ which are j-wise relatively r-prime with respect to S.

For the following lemma and proposition, let

$$\pi : \mathcal{O}^m \to \Big(\prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} \mathcal{O}/\mathfrak{p}^r \Big)^m$$

be the surjective homomorphism induced by the family of natural projections

$$\pi_{\mathfrak{p}^r} : \mathcal{O} \to \mathcal{O}/\mathfrak{p}^r \quad \text{for all } \mathfrak{p} \mid \langle p \rangle \text{ where } p \in S.$$

The following lemma follows immediately from the definition of j-wise relative r-primality of algebraic integers.

Lemma 3.1. *For a given $\mathfrak{p} \mid \langle p \rangle$ where $p \in S$ and $k \in \{1, 2, \ldots, m\}$, let $A_k^{(\mathfrak{p})}$ denote the set of elements in $(\mathcal{O}/\mathfrak{p}^r)^m$ that have 0 in exactly k of their m components. Then,*

$$E_S = \pi^{-1}\Big(\prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} \bigcup_{k=0}^{j-1} A_k^{(\mathfrak{p})} \Big).$$

For the remainder of this section, set $N = \prod_{p \in S} p^r$, and let $\mathcal{O}[M]^m$ denote the set of m-tuples of elements of $\mathcal{O}[M]$.

Proposition 3.2. *Fix $q \in \mathbb{N}$. Then,*

$$|E_S \cap \mathcal{O}[qN]^m| = (2q)^{mn} \prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} \Big[p^{rm(n-D_p)} \sum_{k=0}^{j-1} \binom{m}{k} (\mathfrak{N}(\mathfrak{p}^r) - 1)^{m-k} \, \mathfrak{N}(\mathfrak{p}^r)^k \Big].$$

Proof. We first examine the map π. Let π_N denote the reduction modulo N homomorphism, and let $\psi = (\psi_p)_{p \in S}$, where $\psi_p : (\mathcal{O}/\langle p \rangle^r)^m \to (\prod_{\mathfrak{p} \mid \langle p \rangle} \mathcal{O}/\mathfrak{p}^r)^m$ is the homomorphism induced by the projection maps $\mathcal{O}/\langle p \rangle^r \to \prod_{\mathfrak{p} \mid \langle p \rangle} \mathcal{O}/\mathfrak{p}^r$. Finally, let $\overline{\psi}$ be the extension of ψ to $(\mathcal{O}/\langle N \rangle)^m$ (by applying the Chinese remainder theorem to the primes in S). Let

$$R_p = \prod_{\mathfrak{p} \mid \langle p \rangle} \mathcal{O}/\mathfrak{p}^r.$$

Then, we have the following diagram.

$$\begin{tikzcd}
\mathcal{O}^m \arrow[r, "\pi_N"] & (\mathcal{O}/\langle N \rangle)^m \arrow[r, "\overline{\psi}"] \arrow[d, "\cong"'] & (\prod_{p \in S} R_p)^m \arrow[d, "="] \\
& (\prod_{p \in S} \mathcal{O}/\langle p^r \rangle)^m \arrow[r, "\psi"] & (\prod_{p \in S} R_p)^m
\end{tikzcd}$$

With these maps, it follows that $\pi = \overline{\psi} \circ \pi_N$.

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To prove the proposition, we need to compute cardinalities of a few preimages. First, we consider ψ^{-1}. Since $\psi_p : (\mathcal{O}/\langle p^r\rangle)^m \to R_p^m$ is a surjective-free $\mathbb{Z}_{p^r}$-module homomorphism, for all $y_p \in R_p^m$,

$$|\psi_p^{-1}(y_p)| = |\ker(\psi_p)| = \frac{(p^r)^{mn}}{(p^r)^{mD_p}} = p^{rm(n-D_p)}.$$

Hence, for all $y \in (\mathcal{O}/\langle N\rangle)^m$, we conclude that

$$|\overline{\psi}^{-1}(y)| = \prod_{p\in S} |\psi_p^{-1}(y_p)| = \prod_{p\in S} p^{rm(n-D_p)}.$$

Next, we compute $|\pi_N^{-1}(z) \cap \mathcal{O}[qN]^m|$. To this end, let $\overline{z} = (\overline{z_1}, \ldots, \overline{z_m}) \in (\mathcal{O}/\langle N\rangle)^m$. Since $\mathcal{O}/\langle N\rangle$ is a free $\mathbb{Z}_N$-module with basis $\{\pi(\alpha_1), \ldots, \pi(\alpha_n)\}$, for each $j \in \{1, 2, \ldots, m\}$,

$$\overline{z_j} = \sum_{t=1}^{n} c_t^j \pi(\alpha_t),$$

for some unique $c_t^j \in [0, N) \cap \mathbb{Z}$. Then for $z = (z_1, \ldots, z_m) \in \mathcal{O}^m$, it follows that $\pi_N(z) = \overline{z}$ if and only if

$$z_j = \sum_{t=1}^{n} (c_t^j + l_t^j N)\alpha_t$$

for some $l_t^j \in \mathbb{Z}$. Therefore,

$$|\pi_N^{-1}(z) \cap \mathcal{O}[qN]^m| = (2q)^{mn},$$

since we need $l_t^j \in [-q, q) \cap \mathbb{Z}$ for each pair of indices j and t.

We are ready to compute $|E_S \cap \mathcal{O}[qN]^m|$. For notational convenience, let

$$H = \psi^{-1}\Big(\prod_{\substack{\mathfrak{p}|\langle p\rangle \\ p\in S}} \bigcup_{k=0}^{j-1} A_k^{(\mathfrak{p})}\Big),$$

so that $E_S = \pi^{-1}(H)$ by Lemma 3.1. Since $\pi = \overline{\psi} \circ \pi_N$,

$$E_S \cap \mathcal{O}[qN]^m = \pi_N^{-1}(H) \cap \mathcal{O}[qN]^m.$$

For any fixed k and $\mathfrak{p}$,

$$|A_k^{(\mathfrak{p})}| = \binom{m}{k}(\mathfrak{N}(\mathfrak{p}^r) - 1)^{m-k}\, \mathfrak{N}(\mathfrak{p}^r)^k,$$

and so our calculations with ψ^{-1} yield

$$|H| = \prod_{\substack{\mathfrak{p}|\langle p\rangle \\ p\in S}} \Big[p^{rm(n-D_p)} \sum_{k=0}^{j-1} \binom{m}{k}(\mathfrak{N}(\mathfrak{p}^r) - 1)^{m-k}\, \mathfrak{N}(\mathfrak{p}^r)^k\Big].$$

Therefore, we conclude that

$$\begin{aligned}|E_S \cap \mathcal{O}[qN]^m b| &= (2q)^{mn}|H| \\ &= (2q)^{mn} \prod_{\substack{\mathfrak{p}|\langle p\rangle \\ p\in S}} \Big[p^{rm(n-D_p)} \sum_{k=0}^{j-1} \binom{m}{k} (\mathfrak{N}(\mathfrak{p}^r)-1)^{m-k}\, \mathfrak{N}(\mathfrak{p}^r)^k\Big].\end{aligned}$$ □

We now compute the density of E_S.

Lemma 3.3. *With the previous notation, for any integral basis $\mathcal{B}$ of $\mathcal{O}$,*

$$\mathbb{D}(E_S) = \mathbb{D}_{\mathcal{B}}(E_S) = \prod_{\substack{\mathfrak{p}|\langle p\rangle \\ p\in S}} \Big[\sum_{k=0}^{j-1} \binom{m}{k}\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k}\Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^k\Big].$$

Proof. Let $a_j = |E_S \cap \mathcal{O}[j]^m|/(2j)^{mn}$ and let D denote the density in question.

First, we consider the subsequence $\{a_{qN}\}_{q\in\mathbb{N}}$, where $N = \prod_{p\in S} p^r$. We claim that this subsequence is constant. By the previous proposition along with the definitions for N and D_p,

$$\begin{aligned}a_{qN} &= \frac{1}{(2qN)^{mn}}\Big[(2q)^{mn}\cdot \prod_{\substack{\mathfrak{p}|\langle p\rangle \\ p\in S}} p^{rm(n-D_p)} \sum_{k=0}^{j-1}\binom{m}{k}(\mathfrak{N}(\mathfrak{p}^r)-1)^{m-k}\mathfrak{N}(\mathfrak{p}^r)^k\Big] \\ &= \prod_{\substack{\mathfrak{p}|\langle p\rangle \\ p\in S}} \Big[\sum_{k=0}^{j-1} \binom{m}{k}\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k}\Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^k\Big].\end{aligned}$$

Hence, $\{a_{qN}\}$ is a constant subsequence and converges to D.

Next, we show that $\{a_{c+qN}\}$ also converges to D for any $c \in \{1, 2, \ldots, N-1\}$. We first find bounds for a_{c+qN}. To this end, note that

$$\frac{a_{c+qN}}{a_{qN}} = \Big(\frac{2qN}{2c+2qN}\Big)^{mn} \cdot \frac{|\mathcal{O}[c+qN]^m \cap E_S|}{|\mathcal{O}[qN]^m \cap E_S|} \geq \Big(\frac{2qN}{2c+2qN}\Big)^{mn}.$$

Similarly,

$$\frac{a_{c+qN}}{a_{(q+1)N}} = \Big(\frac{2(q+1)N}{2c+2qN}\Big)^{mn} \cdot \frac{|\mathcal{O}[c+qN]^m \cap E_S|}{|\mathcal{O}[N+qN]^m \cap E_S|} \leq \Big(\frac{2(q+1)N}{2c+2qN}\Big)^{mn}.$$

Therefore, it follows that

$$a_{qN}\Big(\frac{2qN}{2c+2qN}\Big)^{mn} \leq a_{c+qN} \leq a_{(q+1)N}\Big(\frac{2(q+1)N}{2c+2qN}\Big)^{mn}.$$

By letting $q \to \infty$ and applying the squeeze theorem, we conclude that $\{a_{c+qN}\}$ converges to D for any $c \in \{1, 2, \ldots, N-1\}$. Finally, since $\{a_{c+qN}\}$ converges to D for any $c \in \{0, 1, 2, \ldots, N-1\}$, we conclude that $\{a_j\}$ converges to D. □

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Note that the density in Lemma 3.3 is independent of the basis used. Moreover, if we let S be the set of the first t rational primes, taking the limit as $|S| \to \infty$ yields

$$\lim_{|S|\to\infty} \mathbb{D}(E_S) = \prod_{\mathfrak{p}} \Big[\sum_{k=0}^{j-1} \binom{m}{k} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{k} \Big].$$

This motivates the main theorem of this section. For convenience, we restate it here before proving it.

THEOREM 3.4. *Fix $r, j, m \in \mathbb{N}$ such that $j \le m$ and $rm \ge 2$, and let K be an algebraic number field over $\mathbb{Q}$ with a ring of integers $\mathcal{O}$. Then, the density of the set E of j-wise relatively r-prime ordered m-tuples of elements of $\mathcal{O}$ equals*

$$\prod_{\mathfrak{p}} \Big[\sum_{k=0}^{j-1} \binom{m}{k} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{k} \Big].$$

PROOF. Fix $t \in \mathbb{N}$ and let S_t denote the set of the first t rational primes. For brevity, we write $E_t = E_{S_t}$. Since $E_t \supseteq E$,

$$\overline{\mathbb{D}}_{\mathcal{B}}(E) \le \overline{\mathbb{D}}_{\mathcal{B}}(E_t) = \mathbb{D}(E).$$

Observe that the last equality is due to the existence of $\mathbb{D}(E)$. Letting $t \to \infty$,

$$\mathbb{D}_{\mathcal{B}}(E) \le \prod_{\mathfrak{p}} \Big[\sum_{k=0}^{j-1} \binom{m}{k} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{k} \Big].$$

To show the opposite inequality, we first observe that $\mathbb{D}(E_t) - \overline{\mathbb{D}}(E_t \backslash E) \le \underline{\mathbb{D}}(E)$. Hence, it suffices to show that $\lim_{t\to\infty} \overline{\mathbb{D}}(E_t \backslash E) = 0$.

To this end, we introduce the following notation (following [3]). Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$, p_t be the tth rational prime and M be a positive integer.

(1) We say that $\mathfrak{p} \succ M$ if and only if $\mathfrak{p}$ lies over a rational prime greater than M.
(2) We say that $M \succ \mathfrak{p}$ if and only if the rational prime lying under $\mathfrak{p}$ is less than M.

Using this notation, we can write

$$E_t \backslash E \subseteq \bigcup_{\mathfrak{p} \succ p_t} \Big(\prod_{j=1}^{m} \mathfrak{p}^r \Big) \subseteq \mathcal{O}^m,$$

where it is understood that $\prod_{j=1}^{m} \mathfrak{p}^r$ is the subset of $\mathcal{O}^m$ such that each entry of the m-tuple is an element of $\mathfrak{p}^r$. Then, we see that

$$(E_t \backslash E) \cap \mathcal{O}[M]^m \subseteq \bigcup_{CM^n \succ \mathfrak{p} \succ p_t} \prod_{j=1}^{m} (\mathfrak{p}^r \cap \mathcal{O}[M])$$

for some constant $C > 0$ independent of M. It should be remarked that the upper bound $CM^n \succ \mathfrak{p}$ comes from noting that for a fixed integral basis for $\mathcal{O}$, the norm function is a polynomial of degree n in the coefficients (with respect to this basis) of the elements in $\mathcal{O}$. Thus, the norm of any element in $\mathcal{O}[M]$ is contained in $[-CM^n, CM^n]$ for some constant $C > 0$ depending only on the chosen basis. On the other hand, if an element in $\mathcal{O}[M]$ is in $\mathfrak{p}$, then its norm is divisible by the rational prime p lying under $\mathfrak{p}$. Hence, there do not exist prime ideals $\mathfrak{p} > CM^n$ containing a nonzero element of $\mathcal{O}[M]$. Therefore,

$$\overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) \leq \limsup_{M\to\infty} \sum_{CM^n > \mathfrak{p} > p_t} |(\mathfrak{p}^r \cap \mathcal{O}[M])^m| \cdot (2M)^{-mn}.$$

By [3, Proposition 13], there exist constants $c, d > 0$ independent of M and $\mathfrak{p}$ such that

$$|(\mathfrak{p}^r \cap \mathcal{O}[M])^m| \leq \frac{(2M)^{mn}}{\mathfrak{N}(\mathfrak{p}^r)^m} + c\Big(\frac{2M}{d\,\mathfrak{N}(\mathfrak{p}^r)^{1/n}} + 1\Big)^{mn-1}.$$

Using this proposition along with the facts that $\mathfrak{N}(\mathfrak{p}) \geq p$ for every $\mathfrak{p}$ lying above a fixed rational prime p and that at most n prime ideals lie above a fixed rational prime,

$$\begin{aligned}\overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) &\leq \limsup_{M\to\infty} \sum_{CM^n > \mathfrak{p} > p_t} \Big[\frac{1}{\mathfrak{N}(\mathfrak{p}^r)^m} + c\Big(\frac{2M}{d\,\mathfrak{N}(\mathfrak{p}^r)^{1/n}} + 1\Big)^{mn-1}(2M)^{-mn}\Big] \\ &\leq \limsup_{M\to\infty} \sum_{CM^n > p > p_t} \Big[\frac{n}{p^{rm}} + cn\Big(\frac{2M}{dp^{r/n}} + 1\Big)^{mn-1}(2M)^{-mn}\Big].\end{aligned}$$

It remains to show that the right side goes to 0 as $t \to \infty$. To this end, we first observe that for all sufficiently large M, we have $2M/dp^{r/n} > 1$ and thus

$$\Big(\frac{2M}{dp^{r/n}} + 1\Big)^{mn-1}(2M)^{-mn} < \Big(\frac{2}{d}\Big)^{mn} \cdot \frac{1}{p^{rm}}.$$

Therefore, writing $A = n + cn(2/d)^{mn}$ (which is a constant independent of M and p), for all sufficiently large M,

$$\overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) \leq \limsup_{M\to\infty} \sum_{CM^n > p > p_t} \frac{A}{p^{rm}}.$$

Finally, since $\sum_{k=1}^{\infty} 1/k^{rm}$ is convergent, we conclude that $\overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) = 0$ because

$$\limsup_{M\to\infty} \sum_{CM^n > p > p_t} \frac{A}{p^{rm}} \leq \sum_{k=p_t}^{\infty} \frac{A}{k^{rm}} \to 0 \quad \text{as } t \to \infty. \qquad \square$$

We now list two corollaries that show how our main result provides generalisations of many previous results. The first gives a generalisation of the work from [9] and [5].

Corollary 3.5. *Fix $j, m \in \mathbb{N}$ such that $j \leq m$, and let K be an algebraic number field over $\mathbb{Q}$ with a ring of integers $\mathcal{O}$. Then, the density of the set of j-wise relatively prime ordered m-tuples of elements of $\mathcal{O}$ equals*

$$\prod_{\mathfrak{p}} \Big[\sum_{k=0}^{j-1} \binom{m}{k}\Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big)^{m-k}\Big(\frac{1}{\mathfrak{N}(\mathfrak{p})}\Big)^{k}\Big].$$

Proof. Let $r = 1$ in Theorem 1.2. □

The next corollary gives an analogue of the results from [8] and [1].

Corollary 3.6. *Fix $r, j, m \in \mathbb{N}$ such that $rm \geq 2$, and let K be an algebraic number field over $\mathbb{Q}$ with a ring of integers $\mathcal{O}$. Then, the density of the set of relatively r-prime ordered m-tuples of elements of $\mathcal{O}$ equals*

$$\prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^{rm}}\right) = \frac{1}{\zeta_{\mathcal{O}}(rm)}.$$

Proof. Let $j = m$ in Theorem 1.2, and apply the binomial theorem. □

Note that letting $r = 1$ in Corollary 3.6 gives the main result of [3], and letting $m = 1$ gives the probability for an algebraic integer being rth power-free, generalising [4].

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