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ABSTRACT

A celebrated theorem in two-dimensional dynamics due to John Franks asserts that every area-preserving homeomorphism of the sphere has either two or infinitely many periodic points. In this work we re-prove Franks' theorem under the additional assumption that the map is smooth. Our proof uses only tools from symplectic topology and thus differs significantly from previous proofs. A crucial role is played by the results of Ginzburg and Kerman concerning resonance relations for Hamiltonian diffeomorphisms.

1. Introduction

Consider the unit sphere $S^2 \subset \mathbb{R}^3$ equipped with the standard area form ω inherited from \mathbb{R}^3 . Let ϕ_α be the rotation of the sphere by $2\pi\alpha$ radians about the vertical axis. Each ϕ_α is an area-preserving diffeomorphism, and there are two simple alternatives for the number of its periodic points: either α is irrational and ϕ_α has exactly two periodic points, the poles; or α is rational, in which case some iterate of ϕ_α is the identity and hence ϕ_α has infinitely many periodic points. The following remarkable theorem due to Franks [Fra92, Fra96] states that these alternatives for the number of periodic points of area-preserving maps of S^2 are universal.

THEOREM 1.1 [Fra92, Fra96]. *Every area-preserving homeomorphism of S^2 has either two or infinitely many periodic points.*

In the case of smooth maps, this theorem was strengthened by Franks and Handel in [FH03], with the addition of new information on the growth rate of periodic points. The smoothness condition in [FH03] was then relaxed by Le Calvez in [Le06a]. As stated and proved, these results all belong to the world of two-dimensional dynamical systems. In particular, all the previous proofs known to the present authors utilize results, such as Brouwer's translation theorem, that capture phenomena unique to dimension two. On the other hand, Franks' theorem (in the smooth category) and the results in [FH03] can be recast as statements about Hamiltonian diffeomorphisms of S^2 . From this perspective, they can be viewed as the two-dimensional models of a more general class of results that are expected to hold for Hamiltonian diffeomorphisms of large families of symplectic manifolds (see below).

A first step in the process of absorbing Franks' theorem into symplectic topology is to re-prove it using only the tools from this field. This is the goal of the present paper. Other *symplectic*

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approaches to a similar set of results concerning area-preserving disc maps have been developed by Bramham (see [BH12]) and by Ghrist *et al.* in [vdBGVW09].

Here we use some well-known symplectic tools, as well as the results on resonance relations for Hamiltonian diffeomorphisms from [GK10], to prove the following.

THEOREM 1.2. *Every Hamiltonian diffeomorphism ϕ of (S^2, ω) has either two or infinitely many periodic points. If ϕ has exactly two periodic points, say P and Q , then both are nondegenerate. In particular, both are elliptic fixed points of ϕ , and their mean indices $\Delta(P), \Delta(Q) \in \mathbb{R}/4\mathbb{Z}$ are irrational and satisfy*

$$\Delta(P) + \Delta(Q) = 0 \pmod{4}.$$

We defer a discussion of the mean index to §2.2.

Theorem 1.2 implies Theorem 1.1 in the smooth category. For the case of Hamiltonian diffeomorphisms with exactly two periodic points, the restrictions on these points included in the statement of Theorem 1.2 are not new. As pointed out to us by Bramham, nondegeneracy follows from the results on area-preserving homeomorphisms of annuli in [Fra92, Fra96], and the restrictions on the mean indices can be derived from the Lefschetz fixed point theorem and the Poincaré–Birkhoff theorem. However, we establish these restrictions by other means, which we hope will lead to analogous restrictions in some of the generalizations of Franks’ theorem that are expected to hold in higher-dimensional settings. Perhaps the best known (and most approachable) of these conjectured generalizations is the assertion that every Hamiltonian diffeomorphism of $\mathbb{C}\mathbb{P}^n$ must have either $n + 1$ or infinitely many periodic points (see, for example, [HZ94, p. 263]). Applications of the ideas developed here to such problems will be considered elsewhere.

Remark 1. Much is known about the set of Hamiltonian diffeomorphisms (in fact homeomorphisms) of S^2 with exactly two periodic points. In [Le06b] it is shown that, up to conjugacy, every such Hamiltonian diffeomorphism (in fact homeomorphism) is the compactification of an irrational pseudo-rotation, an area-preserving map of the closed annulus such that every positively recurrent point has the same irrational rotation number. On the other hand, such maps are known to exhibit a variety of different behaviors: from simple irrational rotations to the smooth examples from [AK70, FH77, FK04], which have only three ergodic invariant measures.

1.1 On the proof of Theorem 1.2

The crucial first step is to prove that if a Hamiltonian diffeomorphism ϕ of S^2 has finitely many periodic points, then at least two of these points, say P and Q , must have irrational mean indices. This is established as an essentially immediate implication of the theory of resonance relations for Hamiltonian diffeomorphisms developed in [GK10]. It is important to note that these results from [GK10] are themselves implied by the ideas inherent in the recent proofs of the Conley conjecture by Hingston [Hin09] and Ginzburg [Gin10], as well as the applications and refinements of these ideas developed by Ginzburg and Gürel in [GG09a, GG10]. With P and Q in hand, it is then easy to show that in order to prove Theorem 1.2, it suffices to show that ϕ cannot have another periodic point, say R , with an integer mean index. Assuming the existence of such an R , in two distinct cases, we blow up a suitable iteration of ϕ at two points and glue the resulting map to itself to obtain an area-preserving diffeomorphism of the torus (following Arnold). Using index relations and the Floer theory of symplectic diffeomorphisms, we then prove that the resulting maps cannot exist.

2. Background material, definitions and conventions

2.1 Symplectic isotopies and Hamiltonian diffeomorphisms

Let (M, ω) be a closed symplectic manifold of dimension $2n$ and minimal Chern number N . Our basic object of study will be a smooth isotopy ψ_t of symplectic diffeomorphisms of (M, ω) , where t takes values in $[0, 1]$ and ψ_0 is the identity map. In particular, we will be interested in the periodic points of ψ_1 . Denoting the set of fixed points of ψ_1 by $\text{Fix}(\psi_1)$, the set of periodic points of ψ_1 is defined as

$$\text{Per}(\psi_1) = \bigcup_{k \in \mathbb{N}} \text{Fix}((\psi_1)^k).$$

The period of a point $X \in \text{Per}(\psi_1)$ is defined to be the smallest positive integer k for which $X \in \text{Fix}((\psi_1)^k)$. We will also associate to each periodic point X of ψ_1 with period k the unique element of $\pi_1(M)$ or $H_1(M; \mathbb{Z})$ represented by the closed loop $t \mapsto (\psi_t)^k(X)$.

To facilitate our study of periodic points, we will assume from now on that the time-dependent vector field X_t generating our symplectic isotopy ψ_t extends to a smooth time-periodic vector field of period one. This imposes no new restrictions as any symplectic isotopy is homotopic, relative its endpoints, to one with this property. (In particular, ψ_t is homotopic to $\psi_{\zeta(t)}$ where $\zeta : [0, 1] \rightarrow [0, 1]$ is smooth, nondecreasing, onto and constant near 0 and 1.) This assumption allows us to define ψ_t for all $t \in \mathbb{R}$ and to identify ψ_k with $(\psi_1)^k$.

The subset of symplectic isotopies that we are most interested in consists of those corresponding to Hamiltonian flows. A Hamiltonian on (M, ω) is a function $H : \mathbb{R}/\mathbb{Z} \times M \rightarrow \mathbb{R}$ or, equivalently, a smooth one-periodic family of functions $H_t(\cdot) = H(t, \cdot)$. Each Hamiltonian determines a one-periodic vector field X_H on M via the equation $i_{X_H}\omega = -dH_t$. The time- t flow of X_H , denoted by ϕ_H^t , is defined for all $t \in \mathbb{R}$. For $t \in [0, 1]$, ϕ_H^t is a smooth isotopy of symplectic diffeomorphisms. The set of Hamiltonian diffeomorphisms of (M, ω) consists of all the time-one maps $\phi = \phi_H^1$ of Hamiltonian flows.

2.2 The Conley–Zehnder and mean indices

Let $A : [0, 1] \rightarrow \text{Sp}(n)$ be a continuous path in the group $\text{Sp}(n)$ of $2n \times 2n$ symplectic matrices such that $A(0)$ is the identity matrix. One can associate to A its Conley–Zehnder index $\mu(A) \in \mathbb{Z}$, as defined in [CZ84], and its mean index $\Delta(A) \in \mathbb{R}$, as defined in [SZ92]. As shown in [SZ92], these indices satisfy the inequality

$$|\mu(A) - \Delta(A)| \leq n, \tag{1}$$

where the strict form of the inequality holds if $A(1)$ has at least one eigenvalue different from 1.

Consider a smooth isotopy ψ_t of symplectic diffeomorphisms as above. Let X be a fixed point of ψ_1 and let $x : [0, 1] \rightarrow M$ be the closed curve $\psi_t(X)$. Given a symplectic trivialization ξ of x^*TM , the linearized flow of ψ_t along $x(t)$ yields a smooth path $A_\xi : [0, 1] \rightarrow \text{Sp}(n)$ starting at the identity matrix. The quantities $\mu(A_\xi)$ and $\Delta(A_\xi)$ depend only on the homotopy class of the symplectic trivialization ξ . We denote this class by $[\xi]$ and define the Conley–Zehnder index and the mean index of X with respect to this choice as, respectively,

$$\mu(X; \psi_t, [\xi]) = \mu(A_\xi)$$

and¹

$$\Delta(X; \psi_t, [\xi]) = \Delta(A_\xi).$$

¹The symplectic isotopy is included in this notation because we will need to consider fixed points shared by different symplectic diffeomorphisms.

In this context, inequality (1) becomes

$$|\mu(X; \psi_t, [\xi]) - \Delta(X; \psi_t, [\xi])| \leq n, \tag{2}$$

where the strict form of the inequality holds if the linearization of ψ_1 at X , $D(\psi_1)_X$, has at least one eigenvalue different from 1.

2.2.1 Iteration formula. Each fixed point X of ψ_1 is also a fixed point of the k th iteration $(\psi_1)^k = \psi_k$. As shown in [SZ92], the mean index grows linearly under iteration; that is,

$$\Delta(X; \psi_{tk}, [\xi^k]) = k\Delta(X; \psi_t, [\xi]), \tag{3}$$

where ξ^k is the trivialization of TM along $\psi_{tk}(X)$ induced by ξ .

2.2.2 Continuity. We now recall a continuity property of the mean index established in [SZ92]. To do so, we first note that if two fixed points X and X' , of possibly different maps ψ_1 and ψ'_1 , represent the same homotopy class $c \in \pi_1(M)$, then we can specify a unique class of symplectic trivializations along both their trajectories by choosing a homotopy class of symplectic trivializations of $z^*(TM)$ where $z : S^1 \rightarrow M$ is any smooth representative of c (in [BH01] such a choice is referred to as a c -structure). In particular, a choice of $[\xi]$ for X determines a unique class of symplectic trivializations for X' , which we still denote by $[\xi]$. When we compare indices of fixed points in the same homology class, we will always assume that the classes of trivializations being used are coupled in this manner.

Now let $\tilde{\psi}_t$ be a symplectic isotopy that is C^1 -close to ψ_t . Under this perturbation, each fixed point X of ψ_1 splits into a collection of fixed points of $\tilde{\psi}_1$ which are close to X (and hence in the same homotopy class as X). If \tilde{X} is one of these fixed points of $\tilde{\psi}_1$, then

$$|\Delta(X; \psi_t, [\xi]) - \Delta(\tilde{X}; \tilde{\psi}_t, [\xi])|$$

is small.

2.2.3 A useful fact in dimension two. The following result is a simple consequence of the definition of the indices. It can be derived, for example, from [Lon02, ch. 8, Theorem 7].

LEMMA 2.1. *Let (M, ω) be a two-dimensional symplectic manifold, and suppose that ψ_t is an isotopy of symplectic diffeomorphisms of (M, ω) starting at the identity. If X is a fixed point of ψ_1 and $\Delta(X; \psi_t, [\xi])$ is not an integer, then $\mu(X; \psi_t, [\xi])$ is the odd integer closest to $\Delta(X; \psi_t, [\xi])$.*

2.2.4 Indices of contractible fixed points modulo $2N$. When X is a contractible fixed point of ψ_t , that is, $x(t) = \psi_t(X)$ is contractible, it is often useful to restrict attention to trivializations of x^*TM determined by a choice of smooth spanning disc $u : \mathbb{D}^2 \rightarrow M$ with $u(e^{2\pi it}) = x(t)$. For such choices of trivializations, the corresponding indices are well-defined modulo $2N$, twice the minimal Chern number. In fact, the corresponding elements of $\mathbb{R}/2N\mathbb{Z}$ depend only on the time-one map ψ_1 and hence will be denoted by $\mu(X)$ and $\Delta(X)$. The quantities $\Delta(P)$ and $\Delta(Q)$ appearing in the statement of Theorem 1.2 are meant to be understood in this way.

2.3 Floer homology for symplectic diffeomorphisms of the torus

Finally, we recall the properties of the Floer homology of symplectic diffeomorphisms required for the proof of Theorem 1.2. We will only need to consider the special case where (M, ω) is a two-dimensional symplectic torus (\mathbb{T}^2, Ω) and the symplectic diffeomorphism is isotopic to the

identity. Consider then a smooth isotopy ψ_t of symplectic diffeomorphisms of (\mathbb{T}^2, Ω) starting at the identity such that the fixed points of ψ_1 are all nondegenerate. The Floer homology of ψ_1 , $\text{HF}(\psi_1)$, is then well-defined and has the properties described below. The reader is referred to [DS94, LO95, Sei97a, Sei97b, Sei02a] for more details on the general construction of this Floer homology, and to [Cot09, Cot10] for more thorough reviews of the Floer theory of symplectic diffeomorphisms of surfaces.

Invariance under Hamiltonian isotopy. If ϕ is a Hamiltonian diffeomorphism of (\mathbb{T}^2, Ω) , then

$$\text{HF}(\psi_1\phi) = \text{HF}(\psi_1).$$

Splitting. The Floer homology $\text{HF}(\psi_1)$ admits a decomposition of the form

$$\text{HF}(\psi_1) = \bigoplus_{c \in H_1(\mathbb{T}^2; \mathbb{Z})} \text{HF}(\psi_1; c).$$

Here, each summand $\text{HF}(\psi_1; c)$ is the homology of a chain complex $(\text{CF}(\psi_1; c), \partial_J)$, where the chain group $\text{CF}(\psi_1; c)$ is a torsion-free module over a suitable Novikov ring, and the rank of this module is the number of fixed points of ψ_1 which represent the class c . The group $\text{HF}(\psi_1; 0)$ coincides with the Floer–Novikov homology constructed by Lê and Ono in [LO95, Sei02a]. Moreover, if ψ_t is a Hamiltonian isotopy, then $\text{HF}(\psi_1) = \text{HF}(\psi_1; 0)$, which is further equal to the usual Hamiltonian Floer homology of (\mathbb{T}^2, Ω) (see [Flo88]); in particular, it is canonically isomorphic to $H(\mathbb{T}^2; \mathbb{Z})$.

Grading. Each chain complex $(\text{CF}(\psi_1; c), \partial_J)$ above has a relative \mathbb{Z} -grading, and the boundary operator decreases degrees by one. For the $c = 0$ case, the grading can be set by using the usual Conley–Zehnder index of contractible fixed points (which is well-defined since $c_1(\mathbb{T}^2) = 0$). In particular, if ψ_t is a Hamiltonian isotopy, we have

$$\text{HF}_*(\psi_1; 0) = H_{*+1}(\mathbb{T}^2; \mathbb{Z}). \tag{4}$$

For a general class $c \in H_1(\mathbb{T}^2; \mathbb{Z})$, the (relative) grading of $(\text{CF}_*(\psi_1; c), \partial_J)$ is again determined by the Conley–Zehnder index, and the overall shift can be fixed by choosing a homotopy class of symplectic trivializations of $z^*(T\mathbb{T}^2)$ where $z : S^1 \rightarrow \mathbb{T}^2$ is a smooth representative of c .

Extension to all smooth isotopies. The property of invariance under Hamiltonian isotopy allows one to also define the Floer homology for any smooth symplectic isotopy $\tilde{\psi}_t$ of (\mathbb{T}^2, Ω) . One simply sets

$$\text{HF}_*(\tilde{\psi}_1) = \text{HF}_*(\tilde{\psi}_1 \circ \phi)$$

where ϕ is a Hamiltonian diffeomorphism for which the fixed points of $\tilde{\psi}_1 \circ \phi$ are nondegenerate. For example, if $\tilde{\psi}_t = \text{id}$ for all $t \in [0, 1]$, we can perturb by the Hamiltonian flow of a C^2 -small Morse function to obtain

$$\text{HF}(\text{id}) = \text{HF}(\text{id}; 0) = H(\mathbb{T}^2; \mathbb{Z}). \tag{5}$$

Dichotomy. Finally, we recall the following well-known result on alternatives for the Floer homology, whose proof we include for the sake of completeness.

PROPOSITION 2.1. *Either ψ_1 is a Hamiltonian diffeomorphism, in which case $\text{HF}(\psi_1) = \text{HF}(\psi_1; 0)$ and $\text{HF}_*(\psi_1; 0) = H_{*+1}(\mathbb{T}^2; \mathbb{Z})$, or $\text{HF}(\psi_1)$ is trivial.*

Proof. This result can be derived easily by using the flux homomorphism. Let $\widetilde{\text{Symp}}_0(\mathbb{T}^2, \Omega)$ denote the universal cover of $\text{Symp}_0(\mathbb{T}^2, \Omega)$, the identity component of the space of symplectic diffeomorphisms of (\mathbb{T}^2, Ω) . The points of $\widetilde{\text{Symp}}_0(\mathbb{T}^2, \Omega)$ are of the form $[\psi_t]$ where ψ_t is a symplectic isotopy starting at the identity and $[\psi_t]$ denotes the homotopy class of ψ_t relative its endpoints. The *flux homomorphism*

$$\mathcal{F} : \widetilde{\text{Symp}}_0(\mathbb{T}^2, \Omega) \rightarrow H^1(\mathbb{T}^2; \mathbb{R})$$

is then defined by

$$\mathcal{F}([\psi_t]) = \int_0^1 [\vartheta_t] dt,$$

where $\vartheta_t = -i_{X_t}\Omega$ and X_t is the vector field generating ψ_t . Besides the fact that it is indeed a homomorphism (where the target $H^1(\mathbb{T}^2; \mathbb{R})$ is identified with $\text{Hom}(\pi_1(\mathbb{T}^2), \mathbb{R})$), the other crucial property of \mathcal{F} is that its kernel consists of the classes in $\widetilde{\text{Symp}}_0(\mathbb{T}^2, \Omega)$ which can be represented by Hamiltonian isotopies; see [MS95].

Without loss of generality we may assume that $\Omega = d\theta_1 \wedge d\theta_2$ where $\theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}$ are global angular coordinates on \mathbb{T}^2 . Let a_1 and a_2 be the standard generators of $\pi_1(\mathbb{T}^2)$ corresponding to these coordinates. The flux $\mathcal{F}([\psi_t])$ is then specified by the two numbers $A_1 = \mathcal{F}([\psi_t])(a_1)$ and $A_2 = \mathcal{F}([\psi_t])(a_2)$.

Consider now the symplectic isotopy

$$\mathbf{S}_t(\theta_1, \theta_2) = (\theta_1 + tA_1, \theta_2 + tA_2).$$

The flux of $[\mathbf{S}_t]$ is equal to that of $[\psi_t]$. Since \mathcal{F} is a homomorphism, we have

$$\mathcal{F}([\psi_t]^{-1} \circ [\mathbf{S}_t]) = 0 \in H^1(\mathbb{T}^2; \mathbb{R}).$$

The characterization of the kernel of \mathcal{F} then implies that $[(\psi_t)^{-1} \circ \mathbf{S}_t] = [\phi_G^t]$ for some Hamiltonian flow ϕ_G^t . In particular, we have

$$\mathbf{S}_1 = \psi_1 \circ \phi_G^1.$$

By the invariance of Floer homology under Hamiltonian isotopies, this yields

$$\text{HF}(\psi_1) = \text{HF}(\mathbf{S}_1).$$

The Floer homology of \mathbf{S}_1 is now easy to compute. If $A_1 = A_2 = 0 \pmod 1$, then \mathbf{S}_1 is the identity map, and by (4) and (5) we have

$$\text{HF}(\mathbf{S}_1) = \text{HF}(\mathbf{S}_1; 0)$$

and

$$\text{HF}_*(\mathbf{S}_1; 0) = H_{*+1}(\mathbb{T}^2; \mathbb{Z}).$$

Otherwise, \mathbf{S}_1 has no fixed points and hence $\text{HF}(\mathbf{S}_1)$ is trivial. The result follows. □

3. Proof of Theorem 1.2

Let ϕ be a Hamiltonian diffeomorphism of S^2 with finitely many periodic points. It suffices to prove Theorem 1.2 for any iteration of ϕ . Using the freedom to choose this iteration, we may assume that the periodic points of ϕ are all fixed points (i.e. have period one). The iteration formula (3) implies that we may also assume that the mean index of any fixed point of ϕ is either

irrational or equal to zero modulo 4. In particular, we have

$$\text{Per}(\phi) = \text{Fix}(\phi) = \{p_1, \dots, p_l, r_1, \dots, r_m\},$$

where for $j = 1, \dots, l$ the mean indices $\Delta(p_j)$ are irrational, and for $j = 1, \dots, m$ we have $\Delta(r_j) = 0 \pmod 4$.

3.1 Resonance and periodic points with irrational mean indices

The starting point for the proof of Theorem 1.2 is to show that the number of fixed points of ϕ with irrational mean indices is at least two. To prove this, we require the theory of resonance relations for Hamiltonian diffeomorphisms developed in [GK10]. We now present a condensed version of these results, keeping only those features which are relevant to the task at hand.

As before, let (M, ω) be a closed symplectic manifold of dimension $2n$ with minimal Chern number N . Suppose also that (M, ω) is both weakly monotone (see, for example, [MS04]) and rational. A Hamiltonian diffeomorphism is said to be *perfect* if it has finitely many contractible periodic points which are all fixed points. Let φ be a perfect Hamiltonian diffeomorphism of (M, ω) , and let $\Delta_1, \dots, \Delta_m$ be the collection of irrational mean indices of the contractible fixed points of φ (as described in §2.2.4, these are defined modulo $2N$). A *resonance relation* for φ is a vector $\vec{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$ such that

$$a_1\Delta_1 + \dots + a_m\Delta_m = 0 \pmod{2N}.$$

The set of resonance relations of φ forms a free abelian group $\mathcal{R} = \mathcal{R}(\varphi) \subset \mathbb{Z}^m$.

THEOREM 3.1 [GK10]. *Assume that $n + 1 \leq N < \infty$.*

(i) *Then $\mathcal{R} \neq 0$, i.e. the irrational mean indices Δ_i satisfy at least one nontrivial resonance relation.*

(ii) *If the rank of \mathcal{R} is equal to one, then it has a generator of the form $r\vec{a} = (ra_1, \dots, ra_m)$, where $a_i \geq 0$ for all i ,*

$$\sum a_i \leq \frac{N}{N - n},$$

and r is the smallest natural number such that the mean index² of each fixed point of ϕ^r is either irrational or equal to zero modulo $2N$.

Returning to the proof of Theorem 1.2, we derive the following consequence for our Hamiltonian diffeomorphism ϕ .

COROLLARY 3.1. *At least two of the fixed points of ϕ , say P and Q , have irrational mean indices, i.e. are strongly nondegenerate elliptic fixed points. Moreover, if P and Q are the only fixed points of ϕ , then their irrational mean indices $\Delta(P)$ and $\Delta(Q)$ satisfy*

$$\Delta(P) + \Delta(Q) = 0 \pmod 4.$$

Proof. The sphere is weakly monotone and rational, and its minimal Chern number is two ($N = 2 = n + 1$). Hence, Theorem 3.1 applies and statement (i) implies the first assertion of this corollary, since there must be at least two fixed points of ϕ with irrational mean indices in order for a single nontrivial resonance relation to exist.

² In [GK10], assertion (ii) is stated for the collection all nonzero mean indices, in which case one can take $r = 1$. The formulation here is as described in [GK10, Remark 2.1].

Suppose that ϕ has exactly two fixed points, P and Q ; then, by the first assertion of this corollary, both P and Q have irrational mean indices. These indices cannot satisfy two independent resonance relations, because otherwise they would be the unique solutions (modulo 4) of a linear system with integer coefficients and hence would be rational. So, in this case, the rank of \mathcal{R} is equal to one and the conclusion of Theorem 3.1(ii) holds, where the natural number r can be taken to be 1. This immediately implies the second assertion of the corollary. \square

3.2 An assumption and two paths to a contradiction

By Corollary 3.1 and our previous choices, we now have

$$\text{Per}(\phi) = \text{Fix}(\phi) = \{P, Q, p_3, \dots, p_l, r_1, \dots, r_m\}.$$

LEMMA 3.1. *If there is no fixed point of ϕ of type r_j (with $\Delta(r_j) = 0 \pmod{4}$), then the points P and Q are the only fixed points of ϕ .*

Proof. Arguing by contradiction, assume that $\text{Fix}(\phi) = \{P, Q, p_3, \dots, p_l\}$ where $l > 2$ and the $\Delta(p_j)$ are all irrational. The fixed points of ϕ are then all nondegenerate and elliptic and so their topological indices are even. By the Lefschetz fixed point theorem, we would then have the Euler characteristic of S^2 being equal to $l > 2$. \square

By Corollary 3.1 and Lemma 3.1, we will be done if we can show that it is impossible for ϕ to have even one fixed point, r_1 , with $\Delta(r_1) = 0 \pmod{4}$. *Arguing by contradiction, we assume that such a point exists.* At this point, the route to a contradiction splits into two paths, with the first path corresponding to the case where at least one of the r_j is degenerate, and the second to the case where all the r_j are nondegenerate.

3.3 Path 1: one of the r_j is degenerate

Assume that ϕ as above has a fixed point, say $R = r_1$, which is degenerate (and satisfies $\Delta(R) = 0 \pmod{4}$).

3.3.1 *A useful generating Hamiltonian.* We now choose a generating Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times S^2 \rightarrow \mathbb{R}$ for ϕ such that P and R are both static fixed points of the flow of H , that is, $\phi = \phi_H^1$ and both P and R are fixed points of ϕ_H^t for all $t \in \mathbb{R}$. We begin with any Hamiltonian G that generates ϕ . Let $u_R : \mathbb{D}^2 \rightarrow S^2$ be a smooth spanning disc for $\phi_G^t(R)$. As described in [SZ92, § 9] (see also [Gin10, § 5.1]), one can use this disc to construct a contractible loop of Hamiltonian diffeomorphisms, γ_1^t , such that $\gamma_1^t \circ \phi_G^t(R) = R$ for all $t \in \mathbb{R}$ and γ_1^t is supported in an arbitrarily small neighborhood of the image of u_R (which could be all of S^2). The curve $\gamma_1^t \circ \phi_G^t(P)$ does not pass through R and is contractible in its complement. Hence, we can choose a spanning disc u_P for $\gamma_1^t \circ \phi_G^t(P)$ whose image does not contain R . Using it we can then construct, as above, a contractible loop of Hamiltonian diffeomorphisms, γ_2^t , which is trivial in a neighborhood of R and satisfies $\gamma_2^t \circ \gamma_1^t \circ \phi_G^t(P) = P$ for all $t \in \mathbb{R}$. Let H be the unique generating Hamiltonian of the Hamiltonian path $\gamma_2^t \circ \gamma_1^t \circ \phi_G^t$ such that $H(t, R) = 0$ for all $t \in \mathbb{R}/\mathbb{Z}$. By reparameterizing the path $\gamma_2^t \circ \gamma_1^t \circ \phi_G^t$, we may also assume that H vanishes when t is within some small fixed distance, say $0 < \delta_H \ll 1$, of $0 \in \mathbb{R}/\mathbb{Z}$.

3.3.2 *A generic perturbation of ϕ^k .* For a $k \in \mathbb{N}$, the Hamiltonian diffeomorphism ϕ^k is generated by the Hamiltonian

$$H_k(t, p) = kH(kt, p).$$

More precisely, we have $\phi_{H_k}^t = \phi_H^{kt}$ for all $t \in \mathbb{R}$. Note that P and R are still static fixed points of the flow of H_k , and by (3) we have

$$\Delta(P; \phi_{H_k}^t, [\xi^k]) = k\Delta(P; \phi_H^t, [\xi]) \tag{6}$$

and, similarly,

$$\Delta(R; \phi_{H_k}^t, [\xi^k]) = k\Delta(R; \phi_H^t, [\xi]) \tag{7}$$

for any choice of the class $[\xi]$.

LEMMA 3.2. *For each $k \in \mathbb{N}$ there is a neighborhood U_k of R and a Hamiltonian flow $\tilde{\phi}_{k,t}$ which is arbitrarily C^∞ -close to $\phi_{H_k}^t$, equal to $\phi_{H_k}^t$ outside of U_k , and such that its fixed point set has the form*

$$\text{Fix}(\tilde{\phi}_{k,1}) = \{P, Q, p_3, \dots, p_l, R, R_1, \dots, R_d, r_2, \dots, r_m\}$$

where:

- (i) R is a fixed point of $\tilde{\phi}_{k,t}$ for all t , an elliptic fixed point of $\tilde{\phi}_{k,1}$, and such that

$$\Delta(R; \tilde{\phi}_{k,t}, [\xi^k]) = k\Delta(R; \phi_H^t, [\xi]) + \lambda/\pi$$

where $[\xi]$ is any class of symplectic trivializations and $\lambda > 0$ is arbitrarily close to 0;

- (ii) the R_j are all contained in U_k , nondegenerate, and such that $\Delta(R_j; \tilde{\phi}_{k,t}, [\xi^k])$ is arbitrarily close to $k\Delta(R; \phi_H^t, [\xi])$ for $j = 1, \dots, d$ and any choice of $[\xi]$;
- (iii) none of the $\tilde{\phi}_{k,t}$ trajectories of the remaining fixed points of $\tilde{\phi}_{k,1}$ enter U_k .

Proof. Besides some simple manipulations, we will require only the following generic transversality result for Hamiltonian diffeomorphisms. Let ϕ_F^1 be a Hamiltonian diffeomorphism of a symplectic manifold (M, ω) , and let U be an open subset of M whose boundary is smooth and contains no fixed point of ϕ_F^1 . Then there is a Hamiltonian \tilde{F} arbitrarily C^∞ -close to F which equals F in the complement of $S^1 \times U$ and whose fixed points in U are all nondegenerate.

Now, choose a Darboux ball U_k around R such that none of the one-periodic trajectories of the Hamiltonian flow of H_k , other than the trajectory through R , enter U_k . Let (x, y) be the Darboux coordinates in U_k . By our choice of H , it follows from the definition of H_k that it vanishes for $t \in [1 - \delta_H/k, 1]$. Let G be a small function supported in U_k which equals $(\lambda_0/2\pi)(x^2 + y^2)$ near R for a λ_0 which is (arbitrarily) small and positive. Let $\kappa : \mathbb{R} \rightarrow [0, 1]$ be a smooth bump function such that $\kappa(t) = 1$ for $t \in [1 - 3\delta_H/4k, 1 - \delta_H/4k]$ and κ vanishes outside $(1 - \delta_H/k, 1)$. Viewing κ as a one-periodic function, we set $G'(t, p) = \kappa(t)G(p)$ and let ϕ_t^G be the Hamiltonian flow of $G' + H_k$. Clearly, R is still a static fixed point of ϕ_t^G and, since the flows of G' and H_k are supported in disjoint time domains, we have

$$\Delta(R; \phi_{G'+H_k}^t, [\xi^k]) = \Delta(R; \phi_{H_k}^t, [\xi^k]) + \Delta(R; \phi_{G'}^t, [\xi^k]) = k\Delta(R; \phi_H^t, [\xi]) + \lambda/\pi,$$

where $\lambda = \lambda_0 \int_0^1 \kappa(t) dt$. This settles assertion (i) of the lemma.

In appropriate coordinates, the linearization of ϕ_1^G at R is rotation by λ radians. Hence there are no fixed points of ϕ_1^G in some Darboux ball V around R in U_k . Using the fact above, we can then perturb $G' + H_k$ in $S^1 \times (U_k \setminus V)$ to obtain a Hamiltonian \tilde{F} whose fixed points R_1, \dots, R_d in $U_k \setminus V$ are all nondegenerate. Upon setting $\tilde{\phi}_{k,t} = \phi_{\tilde{F}}^t$ we are done. In particular, the continuity property of the mean index described in § 2.2.2 implies that each $\Delta(R_j; \tilde{\phi}_{k,t}, [\xi^k])$ is arbitrarily close to $\Delta(R; \phi_{H_k}^t, [\xi^k])$ and hence $k\Delta(R; \phi_H^t, [\xi])$. Thus condition (ii) is satisfied. Our choice of U_k ensures that condition (iii) is also satisfied. \square

3.3.3 *Completing the restriction of $\tilde{\phi}_{k,1}$ to $S^2 \setminus \{P, R\}$.* The symplectic manifold $(S^2 \setminus \{P, R\}, \omega)$ is symplectomorphic to the open cylinder $(-1, 1) \times \mathbb{R}/2\pi\mathbb{Z}$ equipped with the symplectic form $dz \wedge d\theta$. We now show that the map $\tilde{\phi}_{k,1}$ can be completed to an area-preserving diffeomorphism $\bar{\phi}_k$ of the closed cylinder $[-1, 1] \times \mathbb{R}/2\pi\mathbb{Z}$, where $\bar{\phi}_k$ acts on the boundary circles $\Gamma_P = \{1\} \times \mathbb{R}/2\pi\mathbb{Z}$ and $\Gamma_R = \{-1\} \times \mathbb{R}/2\pi\mathbb{Z}$ as rotations by $\pi\Delta(P; \tilde{\phi}_{k,t}, [\xi])$ and λ , respectively, for any choice of the class $[\xi]$.

In general, if X is an elliptic fixed point of a symplectic diffeomorphism ψ_1 of S^2 which is isotopic to the identity, then the eigenvalues of $D(\psi_1)_X$ are $e^{\pm i\pi\Delta(X, \psi_t, [\xi])}$, which are independent of the choice of the class $[\xi]$. Hence, the eigenvalues of $D(\phi_{k,1})_R$ are $e^{\pm i\pi\lambda}$. If $B(\epsilon)$ is the open ball in \mathbb{R}^2 of radius $\epsilon > 0$ centered at the origin, then for sufficiently small ϵ there is a symplectic embedding $A_R : B(\epsilon) \rightarrow (S^2, \omega)$ such that $A_R(0) = R$ and

$$D(A_R^{-1} \circ \tilde{\phi}_{k,1} \circ A_R)_0 = \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix}.$$

Now consider the map $(r, \theta) \mapsto (\rho = r^2/2, \theta)$ which takes $(B(\epsilon) \setminus 0, r dr \wedge d\theta)$ to $((0, \epsilon^2) \times \mathbb{R}/2\pi\mathbb{Z}, d\rho \wedge d\theta)$. It follows from the linearization above that in (ρ, θ) coordinates, the map $A_R^{-1} \circ \tilde{\phi}_{k,1} \circ A_R$ extends to the boundary circle $\{0\} \times \mathbb{R}/2\pi\mathbb{Z}$ as the map

$$(\rho, \theta) \mapsto (\rho, \theta + \lambda).$$

Applying the same procedure near P gives the desired map $\bar{\phi}_k$.

Since R and P are static fixed points of the flow $\tilde{\phi}_{k,t}$, we can complete each of the maps $\tilde{\phi}_{k,t}$ to an area-preserving diffeomorphism $\bar{\phi}_{k,t}$ of the same closed cylinder. (The restriction of $\bar{\phi}_{k,t}$ to the boundary circle Γ_P (respectively, Γ_R) will only be a rotation when P (respectively, R) is an elliptic fixed point of $\tilde{\phi}_{k,t}$, but this is inconsequential since the boundary circles are always invariant.) In this way we obtain a smooth isotopy of area-preserving diffeomorphisms $\bar{\phi}_{k,t}$ starting at the identity and ending at $\bar{\phi}_k$.

3.3.4 *Transfer of dynamics to the torus.* Following Arnold’s famous argument from [Arn89, Appendix 9] in support of his conjectured lower bound on the number of fixed points of Hamiltonian diffeomorphisms, we now extend the map $\bar{\phi}_k$ to the torus formed by gluing two copies of the domain cylinder $[-1, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ along their common boundaries. In fact, as in [Arn89], we first insert two narrow *connecting cylinders* along the boundary circles to obtain a symplectic torus (\mathbb{T}^2, Ω) of total symplectic area $(8 + 4\tau)\pi$, where each connecting cylinder is symplectomorphic to $[0, \tau] \times \mathbb{R}/2\pi\mathbb{Z}$. This allows us to extend the map $\bar{\phi}_k$ to an area-preserving map ψ_k of (\mathbb{T}^2, Ω) which agrees with $\bar{\phi}_k$ on the two large cylinders and is defined on the connecting cylinders so that the overall map is smooth. Since $\bar{\phi}_k$ has no fixed points on the boundary of its domain, we may also assume (again as in [Arn89]) that no new fixed points are introduced into the connecting cylinders. Hence, $\text{Fix}(\psi_k)$ consists of two copies of $\text{Fix}(\bar{\phi}_k)$, which we denote by

$$\text{Fix}(\bar{\phi}_k^\pm) = \{Q^\pm, p_3^\pm, \dots, p_l^\pm, R_1^\pm, \dots, R_d^\pm, r_2^\pm, \dots, r_m^\pm\}.$$

3.3.5 *The contradiction at the end of path 1.* The isotopy $\bar{\phi}_{k,t}$ induces a smooth isotopy $\psi_{k,t}$ from the identity to ψ_k . Hence, the Floer homology of ψ_k is well-defined. Now, there are two fixed points of ψ_k , Q^\pm , corresponding to the fixed point Q of ϕ . As described below, the following result concerning the role of Q^+ in $\text{HF}(\psi_k)$ contradicts Proposition 2.1.

PROPOSITION 3.1. *If k is sufficiently large, then Q^+ represents a nontrivial class in $\text{HF}(\psi_k)$; and if Q^+ is contractible, then the degree of the class $[Q^+]$ is greater than one in absolute value.*

Proof. Fix a class $[\xi]$ of symplectic trivializations of TS^2 along $\phi_H^t(Q)$. The resulting class $[\xi^k]$ of symplectic trivializations of TS^2 along $\phi_{H_k}^t(Q)$ then determines an equivalence class $[\xi_+^k]$ of symplectic trivializations of $T\mathbb{T}^2$ along $\psi_{k,t}(Q^+)$. Let X^\pm be any fixed point of ψ_k in the same homotopy class as Q^+ , where X is the corresponding fixed point of $\tilde{\phi}_{k,1}$. Since the Floer boundary operator decreases degrees by one, to prove the first assertion of Proposition 3.1 it suffices to show that for k large enough we have either

$$\Delta(X^\pm; \psi_{k,t}, [\xi_+^k]) = \Delta(Q^+; \psi_{k,t}, [\xi_+^k])$$

or

$$|\Delta(X^\pm; \psi_{k,t}, [\xi_+^k]) - \Delta(Q^+; \psi_{k,t}, [\xi_+^k])| > 3.$$

More precisely, by Lemma 2.1 the first equality implies

$$\mu(X^\pm; \psi_{k,t}, [\xi_+^k]) = \mu(Q^+; \psi_{k,t}, [\xi_+^k]),$$

whereas, if the alternative inequality holds and \tilde{X}^\pm is a nondegenerate fixed point obtained from X by perturbation, then it follows from the continuity property of the mean index and (2) that

$$|\mu(\tilde{X}^\pm; \psi_{k,t}, [\xi_+^k]) - \mu(Q^+; \psi_{k,t}, [\xi_+^k])| > 1.$$

Since there are only finitely many points X^\pm , for any k , it will follow that for large enough k the point Q^+ is in the kernel of the corresponding Floer boundary map and is not in its image, thus establishing the first assertion of Proposition 3.1.

Case 1: X corresponds to a fixed point of ϕ^k , i.e. $X \neq R_j^\pm$. By construction, we have

$$\Delta(Q^+; \psi_{k,t}, [\xi_+^k]) = \Delta(Q; \phi_{H_k}^t, [\xi^k]),$$

and the iteration formula (3) then implies that

$$\Delta(Q^+; \psi_{k,t}, [\xi_+^k]) = k\Delta(Q; \phi_H^t, [\xi]). \tag{8}$$

Similarly,

$$\Delta(X^\pm; \psi_{k,t}, [\xi_+^k]) = k\Delta(X; \phi_H^t, [\xi]). \tag{9}$$

If $\Delta(X; \phi_H^t, [\xi]) = \Delta(Q; \phi_H^t, [\xi])$, then we are done, as this would imply

$$\Delta(X^\pm; \psi_{k,t}, [\xi_+^k]) = \Delta(Q^+; \psi_{k,t}, [\xi_+^k]).$$

If $\Delta(X; \phi_H^t, [\xi]) \neq \Delta(Q; \phi_H^t, [\xi])$, then equations (8) and (9) yield

$$|\Delta(X^\pm; \psi_{k,t}, [\xi_+^k]) - \Delta(Q^+; \psi_{k,t}, [\xi_+^k])| = k|\Delta(X; \phi_H^t, [\xi]) - \Delta(Q; \phi_H^t, [\xi])|,$$

and for sufficiently large k we have

$$|\Delta(X^\pm; \psi_{k,t}, [\xi_+^k]) - \Delta(Q^+; \psi_{k,t}, [\xi_+^k])| > 3,$$

as desired.

Case 2: $X = R_j^\pm$. Since $\Delta(R_j^\pm; \psi_{k,t}, [\xi_+^k]) = \Delta(R_j; \tilde{\phi}_{k,t}, [\xi])$, it follows from Lemma 3.2 that $\Delta(R_j^\pm; \psi_{k,t}, [\xi_+^k])$ is arbitrarily close to $k\Delta(R; \phi_H^t, [\xi])$. By assumption, $\Delta(R; \phi_H^t, [\xi])$ is an integer (multiple of 4) and hence not equal to the irrational number $\Delta(Q; \phi_H^t, [\xi_Q])$. Arguing as in Case 1, we see that for sufficiently large k we have

$$|\Delta(R_j^\pm; \psi_{k,t}, [\xi_+^k]) - \Delta(Q^+; \psi_{k,t}, [\xi_+^k])| > 3.$$

Finally, we settle the second assertion of Proposition 3.1. If Q^+ is a contractible fixed point of ψ_k , then $\phi_H^t(Q)$ is contractible in $S^2 \setminus \{P, R\}$. We choose $[\xi]$ in this case so that it is determined by a spanning disc for $\phi_H^t(Q)$. Then the induced class $[\xi_+^k]$ determines the canonical grading of $\text{HF}_*(\psi_k)$. As established above, we have

$$\Delta(Q^+; \psi_{k,t}, [\xi_+^k]) = k\Delta(Q; \phi_H^t, [\xi]).$$

Since $\Delta(Q; \phi_H^t, [\xi])$ is irrational and hence nonzero, we therefore have

$$|\Delta(Q^+; \psi_{k,t}, [\xi_+^k])| > 2$$

for large enough $k \in \mathbb{N}$. For such k , inequality (2) then yields

$$|\mu(Q^+; \psi_{k,t}, [\xi_+^k])| > 1. \quad \square$$

Now, Propositions 2.1 and 3.1 cannot both be true. Proposition 2.1 and the first assertion of Proposition 3.1 together imply that ψ_k must be a Hamiltonian diffeomorphism, in which case $\text{HF}_d(\psi_k; 0)$ must be trivial when $|d| > 1$. But this contradicts the second assertion of Proposition 3.1. Thus, ϕ cannot have a degenerate fixed point.

3.4 Path 2: all the r_j are nondegenerate

To begin, we choose as in § 3.3.1 a generating Hamiltonian H for ϕ such that, this time, P and Q are static fixed points of ϕ_H^t . Fix (arbitrarily large) $k \in \mathbb{N}$ such that the fixed points of ϕ^k are also all nondegenerate. Following § 3.3.3, we can then complete the restriction of ϕ^k to $S^2 \setminus \{P, Q\}$ to obtain a smooth area-preserving map $\overline{\phi^k}$ of the closed cylinder $[-1, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ which acts on the boundary circles $\Gamma_P = \{1\} \times \mathbb{R}/2\pi\mathbb{Z}$ and $\Gamma_Q = \{-1\} \times \mathbb{R}/2\pi\mathbb{Z}$ as (irrational) rotations by $\pi\Delta(P; \phi_{H_k}^t, [\xi])$ and $\pi\Delta(Q; \phi_{H_k}^t, [\xi])$, respectively, for any choice of the class $[\xi]$. Moreover, the flow $\phi_{H_k}^t$ again induces an isotopy $\overline{\phi_t^k}$ from the identity to $\overline{\phi^k}$.

Proceeding as in § 3.3.4, we extend the map $\overline{\phi^k}$ to the torus formed by gluing two copies of the domain cylinder $[-1, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ to one another with two narrow connecting cylinders in between. In this way we obtain an area-preserving map Ψ_k of the symplectic torus (\mathbb{T}^2, Ω) which agrees with $\overline{\phi^k}$ on the two large cylinders and is defined on the connecting cylinders so that the overall map is smooth and has no new fixed points. In particular, $\text{Fix}(\Psi_k)$ consists of two copies of $\text{Fix}(\overline{\phi^k})$, which we denote by

$$\text{Fix}^\pm(\overline{\phi^k}) = \{p_3^\pm, \dots, p_l^\pm, r_1^\pm, \dots, r_m^\pm\}.$$

The isotopy $\overline{\phi_t^k}$ induces a smooth isotopy $\Psi_{k,t}$ from the identity to Ψ_k , so we can again consider the Floer homology $\text{HF}(\Psi_k)$. The following result again contradicts Proposition 2.1.

PROPOSITION 3.2. *If k is sufficiently large, then no contractible fixed point of Ψ_k has Conley–Zehnder index equal to one, and r_1^+ represents a nontrivial class in $\text{HF}(\Psi_k)$.*

Proof. Let X^k be a contractible fixed point of Ψ_k , where X denotes the corresponding fixed point of ϕ . Since $\pi_2(\mathbb{T}^2)$ is trivial, all classes of symplectic trivializations determined by spanning discs for $\Psi_{k,t}(X^k)$ yield the same values of the mean index and Conley–Zehnder index of X^k . So, in what follows, we denote these simply by $\Delta(X^k; \Psi_{k,t})$ and $\mu(X^k; \Psi_{k,t})$. Since X^k is contractible, X must admit a spanning disc with image in $S^2 \setminus \{P, Q\}$. Let $\Delta(X; \phi_H^t)$ denote the mean index computed with respect to the corresponding class of trivializations along $\phi_H^t(X)$. By (3) we have

$$\Delta(X^k; \Psi_{k,t}) = k\Delta(X; \phi_H^t). \quad (10)$$

Case 1: $X^k = p_j^\pm$. Since $\Delta(p_j; \phi_H^t)$ is irrational, it follows from (10) that for large enough k we have

$$|\Delta(p_j^\pm; \Psi_{k,t})| = k|\Delta(p_j; \phi_H^t)| > 2.$$

By (2) it then follows that for sufficiently large k we have

$$|\mu(p_j^\pm; \Psi_{k,t})| > 1.$$

Case 2: $X^k = r_j^\pm$. In this case, $\Delta(r_j; \phi_H^t) = 0 \pmod 4$. If $\Delta(r_j; \phi_H^t) \neq 0$, then we can argue as in the previous case to show that for sufficiently large k we have

$$|\mu(r_j^\pm; \Psi_{k,t})| > 1.$$

Otherwise, it follows from (10) that

$$\Delta(r_j^\pm; \Psi_{k,t}) = 0.$$

Since r_j^\pm is nondegenerate, the strong form of (2) applies and implies that

$$\mu(r_j^\pm; \Psi_{k,t}) = 0.$$

This settles the first assertion of Proposition 3.2.

To approach the second assertion, we first fix a class $[\xi]$ of symplectic trivializations of TS^2 along $\phi_H^t(r_1)$. This determines an equivalence class $[\xi_+^k]$ of symplectic trivializations of $T\mathbb{T}^2$ along $\Psi_{k,t}(r_1^+)$. Let X^k be any fixed point of Ψ_k in the same homotopy class as r_1^+ . Arguing as in Proposition 3.1, to prove the second assertion of Proposition 3.2 it suffices to show that³ for k sufficiently large we have either

$$\Delta(X^k; \Psi_{k,t}, [\xi_+^k]) = \Delta(r_1^+; \Psi_{k,t}, [\xi_+^k])$$

or

$$|\Delta(X^k; \Psi_{k,t}, [\xi_+^k]) - \Delta(r_1^+; \Psi_{k,t}, [\xi_+^k])| > 3.$$

Case 1: $X^k = p_j^\pm$. By our construction of Ψ_k and (3), we have

$$\Delta(r_1^+; \Psi_{k,t}, [\xi_+^k]) = k\Delta(r_1; \phi_H^t, [\xi])$$

and

$$\Delta(p_j^\pm; \Psi_{k,t}, [\xi_+^k]) = k\Delta(p_j; \phi_H^t, [\xi]).$$

Now $\Delta(r_1; \phi_H^t, [\xi]) = 0 \pmod 4$ and $\Delta(p_j; \phi_H^t, [\xi])$ is irrational, so for k sufficiently large we have

$$|\Delta(r_1^+; \Psi_{k,t}, [\xi_+^k]) - \Delta(p_j^\pm; \Psi_{k,t}, [\xi_+^k])| > 3.$$

Case 2: $X^k = r_j^\pm$. In this case,

$$\Delta(r_1^+; \Psi_{k,t}, [\xi_+^k]) - \Delta(r_j^\pm; \Psi_{k,t}, [\xi_+^k]) = 0 \pmod 4.$$

If the mean indices are equal, we are done. If not, we can argue as in the previous case to show that for large k the indices differ by more than 3. \square

This leads to the desired contradiction at the end of path 2, as Proposition 3.2 contradicts Proposition 2.1. In particular, the first assertion of Proposition 3.2 implies that Ψ_k cannot be Hamiltonian, and the second assertion of Proposition 3.2 implies that the Floer homology $\text{HF}(\Psi_k)$ is nontrivial. With this, the proof of Theorem 1.2 is complete.

³ A very similar argument to the one that follows appears in [GG09b].

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