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ON THE STABILITY OF PSEUDOCONVEXITY FOR CERTAIN COVERING SPACES

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Abstract. It is proved that, if $X \xrightarrow{\pi} T$ is a proper holomorphic map with one-dimensional fibers and $\tilde{X} \xrightarrow{\varpi} X$ a covering map, a point $t \in T$ has a neighbourhood U such that $\varpi^{-1}(\pi^{-1}(U))$ is holomorphically convex if and only if $\varpi^{-1}(\pi^{-1}(t))$ is holomorphically convex.

§**1.** It has long been known that the parameter space of a complex analytic family of complex manifolds often admits a significant geometric structure that deserves intensive studies. To describe such a structure, theory is extremely useful when the fibers are Riemann surfaces, because any family is then transformed into that of quasifuchsian groups in $\mathrm{PSL}(2, \mathbf{C})$. In higher dimensions, however, the uniformization theory has not been developed enough to capture important aspects of the variation of complex structures. Nevertheless it seems natural to expect that covering spaces somehow carry information about the deformation. Thus we would like to continue the previous work [O-2] on the holomorphic convexity of the covering spaces of a family of compact Riemann surfaces over the unit disc.

§**2.** First we ask for the possibility of constructing a plurisubharmonic function on the covering of the family by extending one from the special fiber. We shall give an answer to this question which is almost trivial (see Theorem 1). However, as we shall see after Theorem 2, this poses an interesting question on the extension of bounded plurisubharmonic functions. Next we would like to extend the result in [O-2] to the case of families with singular fibers. It will turn out that the absence of infinite chain of compact complex curves is sufficient for the holomorphic convexity at the germ level (cf. Theorem 3).

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§3. Let T be any contractible complex space, let X be a complex space with a proper and surjective holomorphic map $\pi : X \to T$, and let t_0 be any point of T. We put $X_t = \pi^{-1}(t)$ for $t \in T$ and $X_U = \pi^{-1}(U)$ for $U \subset T$. Let $\varpi : \tilde{X} \to X$ be any covering space. For simplicity we put $\tilde{X}_t = \varpi^{-1}(X_t)$ and $\tilde{X}_U = \varpi^{-1}(X_U)$.

§4. First we study the case where X and T are nonsingular and π is everywhere of maximal rank. It is remarkable that $\tilde{X}_t$ are not necessarily Stein even if $\tilde{X}_{t_0}$ is biholomorphically equivalent to $\mathbf{C}^n$ (cf. [N]). In contrast to this phenomenon we are going to give a criterion for the stability of Steinness. Under the above situation, we fix a neighbourhood $U_1 \ni t_0$ and a C^∞ retraction $r : X_{U_1} \to X_{t_0}$. Furthermore let $\tilde{r} : \tilde{X}_{U_1} \to \tilde{X}_{t_0}$ be the lift of r.

THEOREM 1. *Let ψ be a C^∞ plurisubharmonic function on $\tilde{X}_{t_0}$. Suppose that there exists a Hermitian metric g on X_{t_0} such that the complex Hessian of ψ is larger than $\varpi^* g$ and that the covariant derivatives of ψ with respect to $\varpi^* g$ are bounded up to the second order. Then, for any strictly plurisubharmonic function φ on T there exists a neighbourhood $U \ni t_0$ such that $\psi \circ \tilde{r} + \varphi \circ \pi \circ \varpi$ is strictly plurisubharmonic on $\tilde{X}_U$.*

Proof. In terms of a local expression $(r_1, \ldots, r_n)$ of $\tilde{r}$ with respect to a holomorphic local coordinate $(z_1, \ldots, z_n)$ of $\tilde{X}_{t_0}$,

$$\begin{aligned}\partial\bar{\partial}(\psi \circ r) &= \partial\bar{\partial}\psi(r_1, \ldots, r_n) = \partial\left(\sum_k \frac{\partial\psi}{\partial z_k}\bar{\partial} r_k \sum_k \frac{\partial\psi}{\partial \bar{z}_k}\bar{\partial}\bar{r}_k\right) \\ &= \sum_{j,k} \frac{\partial^2\psi}{\partial z_j \partial z_k}\partial r_j \bar{\partial} r_k + \sum_{j,k} \frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k}\partial \bar{r}_j \bar{\partial} r_k + \sum_k \frac{\partial\psi}{\partial z_k}\partial\bar{\partial} r_k \\ &\quad + \sum_{j,k} \frac{\partial^2\psi}{\partial \bar{z}_j \partial z_k}\partial \bar{r}_j \bar{\partial} r_k + \sum_{j,k} \frac{\partial^2\psi}{\partial \bar{z}_j \partial \bar{z}_k}\partial \bar{r}_j \bar{\partial} \bar{r}_k + \sum_k \frac{\partial\psi}{\partial z_k}\partial\bar{\partial} \bar{r}_k.\end{aligned}$$

The desired conclusion follows from the boundedness of the derivatives of ψ and that $(\partial^2\psi/\partial z_j \partial \bar{z}_k) > 0$, $\bar{\partial} r_k = \partial \bar{r}_j = \partial\bar{\partial} r_k = \partial\bar{\partial}\bar{r}_k = 0$ on $\tilde{X}_{t_0}$. □

This shows that the pseudoconvexity property of covering spaces is stable under some hyperbolicity condition, which is certainly not satisfied by $\mathbf{C}^n$.

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COROLLARY. *Under the hypothesis of Theorem 1, suppose moreover that ψ is an exhaustion function, i.e. all the sublevel sets of ψ are relatively compact in $\tilde{X}_{t_0}$. Then there exists a neighbourhood $U \ni t_0$ and a C^∞ plurisubhamonic functions φ on U such that $\psi \circ r + \varphi \circ \pi \circ \varpi$ is a strictly plurisubharmonic exhaustion function on $\tilde{X}_U$. In particular, $\tilde{X}_U$ is a Stein manifold for such a neighbourhood $U \ni t_0$.*

§**5.** The following is also an immediate consequence of Theorem 1, but we want to state it separately as a theorem because of an interesting question it poses.

THEOREM 2. *Let $\pi : X \to T$ be everywhere of maximal rank. If there is a biholomorphic map*

$$\sigma : \{z \in \mathbf{C}\,;\, |z| < 1\} \to \tilde{X}_{t_0}$$

then for any $\eta \in (0,1)$ there exists a neighbourhood $U \ni t_0$ and a negative plurisubharmonic function Φ_η on X_U such that $\Phi_\eta(z) = -(1-|z|^2)^\eta$ and $\pi \circ \varpi \,|\, \{x\,;\, \Phi_\eta(x) \le c\}$ is proper for all $c < 0$.

Proof. After extending the function $(\sigma^{-1})^*(-\log(1-|z|^2)$ to a saturated neighbourhood $\tilde{X}_U$ by Theorem 1, compose $-\exp(-\eta t)$ to it.

We note that $-\log(1-|z|^2)$ satisfies the condition of Theorem 1 because it is a potential of the Bergman metric whose covariant derivatives (up to order two) are all bounded.

§**6.** From now on we allow X_{t_0} to have singular points. The main achievement of the present note is the following.

THEOREM 3. *If $\dim X_{t_0} = 1$ and $\tilde{X}_{t_0}$ is holomorphically convex, then there exists a neighbourhood $U \ni t_0$ such that $\tilde{X}_U$ is holomorphically convex.*

Proof. We may assume that $\tilde{X}_{t_0}$ is noncompact since the conclusion is trivial otherwise. Moreover we may assume that X_{t_0} is connected since it suffices to show the holomorphic convexity for each connected component of $\tilde{X}_U$ for some neighbourhood $U \ni t_0$. Since $\tilde{X}_{t_0}$ is holomorphically convex, the union of compact irreducible components of $\tilde{X}_{t_0}$ consists of compact connected components, say $\{C_\alpha\}_{\alpha\in A}$. Since any of these components intersects with noncompact irreducible components, there exists a complex space $\hat{X}$ and a proper holomorphic map $f : \tilde{X} \to \hat{X}$ such that for each $\alpha \in A$ $f(C_\alpha)$ consists of a single point and $f \,|\, \tilde{X}_{t_0} \setminus \cup_{\alpha\in A} C_\alpha$ is a biholomorphism. Therefore it suffices to show that $f(\tilde{X}_U)$ is a Stein space for some neighbourhood $U \ni t_0$. We shall prove it by extending an increasing family of open subsets $\{R_j\}_{j\in\mathbf{N}}$ of $f(\tilde{X}_{t_0})$ to its neighbourhood of the form $f(\tilde{X}_U)$ so that the extension $\{R_j^*\}_{j\in\mathbf{N}}$ satisfies

1) $$(R_j^*, R_{j+1}^*) \text{ is a Runge pair for every } j$$

and

2) $$f(\tilde{X}_U) = \bigcup_{j=1}^{\infty} R_j^*.$$

For that, let us take a C^∞ strictly subharmonic exhaustion function on $f(\tilde{X}_{t_0})$. Such a function exists since $f(\tilde{X}_{t_0})$ contains no compact irreducible components. In fact, it is proved in [O-1] in a more general setting that given any discrete subset containing all the singular points of $f(\tilde{X}_{t_0})$, say Γ, and any strictly subharmonic function p defined on a neighbourhood $W \supset \Gamma$ such that $p|\Gamma$ is exhaustive, one can find a C^∞ function $\tilde{p} : f(\tilde{X}_{t_0}) \to \mathbf{R}$, a neighbourhood $W' \supset \Gamma$ with $W' \subset W$, and a C^∞ convex increasing function $\lambda : \mathbf{R} \to \mathbf{R}$ such that $p = \tilde{p}$ on W' and $\lambda \circ \tilde{p}$ is a strictly subharmonic exhaustion function. On the other hand, since $\dim X_{t_0} = 1$, there exist a finite set $\{x_1, \ldots, x_m\} \subset X_{t_0}$ and arbitrarily small disjoint neighbourhoods $V_j \ni x_j$ in X such that one can find a Stein neighbourhood $U \ni t_0$ and a holomorphic retraction from $X_U \setminus \cup_{j=1}^m V_j$ onto $X_{t=0} \setminus \cup_{j=1}^m V_j$, say κ. The lift of κ to $\tilde{X}_U \setminus \varpi^{-1}(\cup_{j=1}^m V_j)$ will be denoted by $\tilde{\kappa}$. Let Y be the union of noncompact components of $\varpi^{-1}(X_{t_0} \setminus \cup_{j=1}^m V_j)$ and let $Y_U = \tilde{\kappa}^{-1}(Y)$. Note that $f|Y_U$ is biholomorphic, so that we may identify Y_U with $f(Y_U)$. Shrinking V_j if necessary, one has an exhaustive strictly subharmonic function q on $f(\tilde{X}_{t_0})$ such that the ranges of q on the connected components Z_k, $k \in \mathbf{N}$, of $f(\tilde{X}_{t_0}) \setminus Y$, are mutually disjoint, and that $q|\partial Z_k = \text{const}$ and $dq|\partial Z_k$ has no zeroes for every k. Let $c_k = \sup_{Z_k} q$ and $d_k = \inf_{Z_k} q$. Then we put

$$\begin{aligned} R_k' &= \{x\,;\, q(x) < d_k\} \\ R_k'' &= R_k' \cup Z_k \end{aligned}$$

$$\begin{aligned} R'''_k &= \{x\,;\,q(x) < c_k\} \\ R_{3j-2} &= R'_j \\ R_{dj-1} &= R''_j \\ R_{3j} &= R'''_j \end{aligned}$$

and let $R*_j$ be an open subset of $f(\tilde{X}_U)$ satisfying $R^*_j \cap f(\tilde{X}_{t_0}) = R_j$ and $\partial R^*_j = \tilde{\kappa}^{-1}(\partial R_j)$. By a criterion of Grauert and Narasimhan (cf. [G], [Nr]), it is readily seen that R^*_j are all Stein spaces. Furthermore (R^*_{3j-1}, R^*_{3j}) are Runge pairs, since the open sets R^*_s defined by $R^*_s \cap f(\tilde{X}_{t_0}) = \{x\,;\,q(x) < s\}$ and $\partial R^*_s = \tilde{\kappa}^{-1}(\{x\,;\,q(x) = s\})$, for $c_k < s < d_k$, provides a semicontinuous holmorphic extension (halbstetige holomorphe Ausdehnung) $\{R^*_s \cup R^*_{3j-1}\}$ from R^*_{3j-1} to R^*_{3j} in the sense of Docquier-Grauert [D-G].

We recall that an open subset D of a complex manifold X admits a semicontinuous holomorphic extension to X if there exists an increasing family of open subsets $\{D_t\}_{0\le t\le 1}$ of X such that

1) D_t is Stein for almost all t.
2) $D_0 = D$, $\bigcup_{t<1} D_t = X$.
3) $\bigcup_{0\le t<t_0} D_t = D_{t_0}$ for all t_0.
4) $\left(\bigcap_{t_0<t} D_t\right)^o = D_{t_0}$ for all t_0.

Here A^o denotes the set of interior points of A.

Similarly (R^*_{3j}, R^*_{3j+1}) are Runge pairs for all j. That (R^*_{3j-2}, R^*_{3j-1}) is a Runge pair is trivial. Thus $f(\tilde{X}_U)$ is a Stein space, so that $\tilde{X}_U$ is holomorphically convex.

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