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Nagoya Math. J.
Vol. 77 (1980), 47–60

MODULI SPACES OF THE STABLE VECTOR BUNDLES OVER ABELIAN SURFACES

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Let X be a projective non-singular variety and H an ample line bundle on X. The moduli space of H-stable vector bundles exists by Maruyama [4]. If X is a curve defined over C, the structure of the moduli space (or its compactification) $M(X, d, r)$ of stable vector bundles of degree d and rank r on X is studied in detail. It is known that the variety $M(X, d, r)$ is irreducible. Let L be a line bundle of degree d and let $M(X, L, r)$ denote the closed subvariety of $M(X, d, r)$ consisting of all the stable bundles E with $\det E = L$. We know the global Torelli theorem holds for the mapping $X \mapsto M(X, L, r)$ if the genus g of $X \geq 2$ and $(d, r) = 1$. Namely, let X' be a non-singular projective curve of genus g and L' be a line bundle of degree d on X'. Then the variety $M(X, L, r)$ is isomorphic to $M(X', L', r)$ if and only if X is isomorphic to X'. In higher dimensional case, very little is known about the moduli space of H-stable vector bundles. The moduli spaces have been studied only on two types of surfaces. When X is a hyperelliptic surface, we determined the moduli spaces of H-stable vector bundles with trivial Chern classes in Umemura [14]. In this case the Moduli spaces are not connected when we fixed the numerical Chern classes. Barth [1] proved the moduli space of H-stable vector bundles with $c_1 = 0$ of rank 2 over P_2 is irreducible and rational. In this paper, we work over abelian surfaces A and we study the moduli spaces of some H-stable vector bundles. In the first example, a component of the moduli spaces is isomorphic to $A \times A$ and in the second example, it is birationally isomorphic to the symmetric product $S^n(A)$. In both cases, the local Torelli theorem holds (see for precise statements, Theorem 5 and Theorem 21). We know nothing about the connectedness of the moduli spaces.

Let X be a non-singular algebraic surface defined over an algebraically closed field k and H an ample line bundle over X. We know that, if we

Received November 4, 1978.

fix numerical Chern classes c_1, c_2 and a number r, the coarse moduli space $M(c_1, c_2)$ of H-stable vector bundles of rank r exists. The moduli space $M(c_1, c_2)$ is a scheme of finite type over k and the Zariski tangent space T_x at a closed point $x \in M(c_1, c_2)$ is isomorphic to $H^1(X, \operatorname{End} E)$ where E is the H-stable vector bundle corresponding to the point x (see Maruyama [4]). The scheme consisting of all the irreducible components of $M(c_1, c_2)$ passing through x is called the moduli space of E and denoted by $X(E)$. From now on, we assume all schemes are defined over $\boldsymbol{C}$.

LEMMA 1. *Let Y be an irreducible non-singular algebraic variety and $\mathscr{E}$ be a family of H-stable vector bundles over X parametrized by Y, i.e. $\mathscr{E}$ is a locally free sheaf over $Y \times X$ and for any closed point $y \in Y$, the vector bundle E_y over $y \times X$ is H-stable. Let $f: Y \to M(c_1, c_2)$ denote the map defined by $\mathscr{E}$. We assume that, for any closed point $y \in Y$, $\dim H^1(X, \operatorname{End} E_y) = \dim Y$. If f is injective, then f is an open immersion. In particular Y is birationally equivalent to an irreducible component of $M(c_1, c_2)$.*

Proof. In fact, for any closed point $y \in Y$, $\dim H^1(X, \operatorname{End} E_y) = \dim Y \leq \dim X \leq \dim H^1(X, \operatorname{End} E_y)$. Hence $\dim f(Y) = \dim X$ and $f(X)$ is contained in the open subset of nonsingular points of $M(c_1, c_2)$. Therefore the analytic map $f^{an}: X^{an} \to M(c_1, c_2)^{an}$ is an open immersion. It follows from Mumford [9], f is étale. Now the lemma follows.

LEMMA 2. *Let A be an abelian surface (abelian variety of dimension 2), H an ample line bundle over A and E an H-stable vector bundle over A. If E is absolutely simple, of type M_0 and a model, then there is an injection $\check{A} \times A \to X(E)$.*

Proof. Let $\mathscr{L}$ be the Poincaré line bundle over $A \times \check{A}$, $p: A \times (\check{A} \times A) \to A$ be the map defined by

$$p((a, b, c)) = a + c\,,$$

and $p_{12}: A \times (A \times A) \to A \times A$ be the projection. Consider the vector bundle $\mathscr{E} = p^*E \otimes p_{12}^*\mathscr{L}$ over $A \times (\check{A} \times A)$. This is a family of H-stable vector bundles over A parametrized by $\check{A} \times A$. The vector bundle over a point $(L, a) \in \check{A} \times A$ is $T_a^*E \otimes L$. Let $f: \check{A} \times A \to X(E)$ be the map defined by $\mathscr{E}$. If $T_a^*E \otimes L \simeq T_{a'}^*E \otimes L'$, $T_a^*E \simeq T_{a'}^*E \otimes L' \otimes L^{-1}$ hence $T_{a-a'}^*E \simeq E \otimes L' \otimes L^{-1}$. In particular $T_{a-a'}^*P(E) \simeq P(E)$. The hypothesis that E is a model implies $H(P(E)) = 0$ where $H(P(E)) = \{x \in A \mid P(E) \simeq$

$T_x^* P(E)\}$ (Umemura [15]). Hence $a = a'$. We get $E \otimes L \simeq E \otimes L'$. It follows from the absolute simplicity of E that $L \simeq L'$. This proves f is injective. See Umemura [15].

Let F be an V-stable vector bundle with numerical Chern classes c_1, c_2 and of rank r over an abelian surface A. Let $y \in M(c_1, c_2)$ be the corresponding point. Let us put $\Delta(F) = (r-1)c_1^2 - 2rc_2$. $-\Delta(F)$ is equal to the second Chern class of $\operatorname{End} F$.

LEMMA 3. *The dimension of the Zariski tangent space at y is equal to* $-\Delta(F) + 2$.

Proof. It follows from the Riemann-Roch theorem, $\dim H^0(A, \operatorname{End} F) - \dim H^1(A, \operatorname{End} F) + \dim H^2(A, \operatorname{End} F) = -c_2(\operatorname{End} F) = \Delta(F)$. By the Serre duality $\dim H^0(A, \operatorname{End} F) = \dim H^2(A, \operatorname{End} F)$. Since a stable bundle is simple, $\dim H^0(A, \operatorname{End} F) = 1$. Hence $\dim H^1(A, \operatorname{End} F) = -\Delta(F) + 2$ which is equal to the dimension of the Zariski tangent space at y.

LEMMA 4. *Under the same hypothesis as in Lemma 2, moreover if $\Delta(E) = -2$, then $X(E)$ is irreducible and $\check{A} \times A$ is isomorphic to $X(E)$.*

Proof. Let $x' \in X(E)$ and E' the corresponding H-stable bundle. By Lemma 3 $\dim H^1(A, \operatorname{End} E') = -\Delta(E') + 2$. Since E' has the same numerical Chern classes as E, $-\Delta(E') + 2 = -\Delta(E) + 2 = 4$. Now it follows from Lemmas 1 and 2 that the map f constructed in the proof of Lemma 2 is an isomorphism.

EXAMPLE 1. Let C be a non-singular projective curve of genus 2, P a point of C and J the Jacobian variety of C. Let $C^{(n)}$ be the n-th symmetric product of C. We assume $n \geq 3$. There is a projection $\varphi\colon C^{(n)} \to J$ defined by

$$\varphi(Q_1 + \cdots + Q_n) = (Q_1 + \cdots + Q_n - nP)\,,$$

where $Q_i \in C$ for $1 \leq i \leq n$. We know that there exists a vector bundle E_{n-1} of rank $n-1$ over J such that $C^{(n)}$ is J-isomorphic to $P(E_{n-1})$. We proved in Umemura [4] that E is $\mathcal{O}(C)$-stable, of type M_0 and a model. The number $\Delta(E)$ is also calculated -2. Hence we can apply Lemma 4. The moduli space $X(E_{n-1})$ is isomorphic to $\check{J} \times J$. The determinant defines a map $\check{J} \times J \to \check{J}$ $(y \mapsto \det E_y)$. This map is surjective and all the fibers which we denote $X'(E_{n-1})$ are isomorphic to J. Hence we proved:

THEOREM 5. *The scheme $X'(E_{n-1})$ is isomorphic to J. The local Torelli theorem holds for $C \mapsto X'(E_{n-1})$.*

EXAMPLE 2. Let A be an abelian surface and L be a principal polarization of A, i.e., L is ample and $\dim H^0(A, L) = 1$. Let $n \geq 3$ be an integer and L_i be a line bundle algebraically equivalent to L and φ_i a non-zero section of L_i, $1 \leq i \leq n$. φ_i is uniquely determined up to the multiplication of a non-zero constant. Let $E(L_1, L_2, \cdots, L_n)$ be the coherent sheaf over A defined by the exact sequence:

$$\text{(i)} \qquad \begin{aligned} 0 \to \mathcal{O} \to L_1 \oplus L_2 \oplus \cdots \oplus L_n \to E(L_1, L_2, \cdots, L_n) \to 0\,, \\ 1 \mapsto (\varphi_1, \varphi_2, \cdots, \varphi_n)\,. \end{aligned}$$

The coherent sheaf $E(L_1, L_2, \cdots, L_n)$ does not depend on the choice of φ_i. For, let φ_i' be another non-zero section of L_i. There exists a non-zero constant c_i such that $\varphi_i' = c_i\varphi_i$. Hence the diagram

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{O} & \xrightarrow{\Phi} & L_1 \oplus L_2 \oplus \cdots \oplus L_n \\ & \text{Id}\,\Big\| & & \Big\downarrow \Psi \\ 0 \longrightarrow & \mathcal{O} & \xrightarrow{\Phi'} & L_1 \oplus L_2 \oplus \cdots \oplus L_n \end{array}$$

is commutative where Φ is the injection of the exact sequence (i), Φ' is the injection obtained from Φ by replacing φ_i by φ_i' and Ψ is the $\mathcal{O}_A$-linear map defined by the diagonal matrix

$$\begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}.$$

Let C be an irreducible non-singular projective curve of genus 2. Let P be a point of C and φ the map of C to J defined by $\varphi(Q) = (Q - P)$. We denote by C the image $\varphi(C)$. Let $P_1, P_2, \cdots, P_6$ be the points of C such that $\mathcal{O}(2P_i)$ is isomorphic to the canonical bundle K of C. Let C_i be the image of C in J defined by $\varphi_i(Q - P_i) = \mathcal{O}(Q - P_i) \in J$ for $Q \in C$.

LEMMA 5. *If x is a point of J such that $x \notin \bigcup_{i=1}^6 C_i$, then $C \cap (C + x) \cap (C + 2x) = \varnothing$.*

Proof. Suppose that $C \cap C + x \cap C + 2x$ is not empty. Then there exist three points Q, Q', Q'' of C such that $\mathcal{O}(Q - P) = \mathcal{O}(Q' - P) + x$, $\mathcal{O}(Q - P) = \mathcal{O}(Q'' - P) + 2x$. Hence $\mathcal{O}(Q - Q') = x$, $\mathcal{O}(Q - Q'') = 2x$. Therefore $\mathcal{O}(2Q - 2Q') = \mathcal{O}(Q - Q'')$. Finally we get $\mathcal{O}(Q + Q'' - 2Q') = \mathcal{O}$. We study two cases separately.

Case (i) $K \neq \mathcal{O}(2Q')$. In this case, by the Riemann-Roch theorem $\dim H^0(C, \mathcal{O}(2Q')) = 1$. Hence it follows from $\mathcal{O}(Q + Q'' - 2Q') = \mathcal{O}$ that $Q + Q'' = 2Q'$ as divisors. Consequently $Q = Q'$ and $Q'' = Q'$. This shows $x = 0$. Hence $x \notin C_i$ for every i. This is a contradiction.

Case (ii) $K = \mathcal{O}(2Q')$. Hence there exists an i such that $Q' = P_i$. Therefore $x = \mathcal{O}(Q - Q') = \mathcal{O}(Q - P_i)$ is in C_i. This is impossible.

Let D be the effective divisor on A such that $L \simeq \mathcal{O}(D)$. Such divisors are limited: (a) There exist a non-singular curve C of genus 2 and a point P of C such that the abelian variety A is isomorphic to the Jacobian variety J of C and D coincides with C. (b) There exist two elliptic curves C_1, C_2 such that the abelian variety A is isomorphic to $C_1 \times C_2$ and D is $C_1 \times 0 \cup 0 \times C_2$ (Weil [18]). Let us study first the case (a).

COROLLARY 6. *For any integer N_1, there exist an integer $N \geq N_1$ and a point $x \in J$ of order N such that $C \cap (C + x) \cap (C + 2x) = \varnothing$.*

This is an easy consequence of Lemma 5.

LEMMA 7. *Let x be a point of the Jacobian variety $J(= A)$ of order $N \geq 3$ and $L \simeq \mathcal{O}(C)$. If $C \cap (C + x) \cap \cdots \cap (C + (N - 1)x) = \varnothing$, then $E(L, T_x^*L, \cdots, T_{(N-1)x}^*L)$ is an L-stable locally free sheaf.*

Proof. The locally freeness of $E(L, T_x^*L, \cdots, T_{(N-1)x}^*)$ is evident. The cyclic group (x) operates on the exact sequence;

$$\text{(ii)} \qquad 0 \to \mathcal{O} \to L \oplus T_x^*L \oplus \cdots \oplus T_{(N-1)x}^*L \to E(L, T_x^*L, \cdots, T_{(N-1)x}^*L) \to 0 \,.$$

Hence there exists an exact sequence of vector bundles over $A' = A/(x)$;

$$\text{(iii)} \qquad 0 \to \mathcal{O} \to F \to E' \to 0$$

such that π^* (iii) is isomorphic to (ii) where π is the isogeny $A \to A/(x)$. F is nothing but the direct image $\pi_* L$ (Morikawa [7]). Hence by Takemoto [11], F is $\det F$-stable. The line bundle $\det F$ will be denoted by L'. Suppose that E' is not L'--stable. Then there exist a non-zero locally free sheaf G' of rank $< N - 1$ and a morphism g' of $\mathcal{O}_{A'}$-modules $E' \to G'$ such that g' is surjective on A' − (a subvariety of codimension ≥ 2) and such that

$$\text{(iv)} \qquad \frac{(c_1(E') \cdot L')}{r(E')} \geq \frac{(c_1(G') \cdot L')}{r(G')} \,.$$

Since F is L'-stable,

(v) $$\frac{(c_1(F')\cdot L')}{r(F')} < \frac{(c_1(G')\cdot L')}{r(G')} .$$

Let now M' be a line bundle over A'. Then $N(M'\cdot L') = \pi^*(M'\cdot L') = (\pi^*M'\cdot L^{\otimes N}) = N(\pi^*M'\cdot L)$. Hence we proved $(\pi^*M'\cdot L) = (M'\cdot L')$. Applying this rule to the inequalities (iv) and (v), we get

$$\frac{(L^{\otimes N}\cdot L)}{N} < \frac{(c_1(G)\cdot L)}{r(G)} \leq \frac{(L^{\otimes N}\cdot L)}{N-1}$$

where $G = \pi^*G'$. Consequently

$$2 < \frac{(c_1(G)\cdot L)}{r(G)} \leq 2 + \frac{2}{N-1} .$$

Since $r(G) < N - 1$, it follows that $(c_1(G)\cdot L) = 2r(G) + 1$. If we put $\tilde{G} = G \otimes L^{-1}$, then $(c_1(\tilde{G})\cdot L) = 1$. Let φ be the isogeny $y \mapsto Ny$ of A onto A itself. Then, since the morphism induced by g' $\varphi^*(L \oplus T_x^*L \oplus \cdots \oplus T_{(N-1)x}^*L) \otimes L^{-1} = \overbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}^{N} \to \varphi^*G$ is surjective on $A -$ (a subvariety of codimension $\geqslant 2$), $H^0(A, \varphi^*G \det \tilde{G}) \neq 0$. On the other hand, since the spectral sequence degenerates, we get $H^0(A, \varphi^* \det \tilde{G}) \simeq H^0(A, \varphi_*\varphi^* \det \tilde{G}) = \oplus_{\varphi^*\mathscr{L}=\mathcal{O}_A} H^0(A, \mathscr{L} \otimes \det \tilde{G})$. Therefore, there exists a line bundle $\mathscr{L}$ on A such that $\varphi^*\mathscr{L} \simeq \mathcal{O}_A$, $H^0(A, \mathscr{L} \otimes \det \tilde{G}) \neq 0$. Let D be an effective divisor on A such that $\varphi^*\mathscr{L} = \mathcal{O}_A$, $\mathcal{O}(D) = \mathscr{L} \otimes \det \tilde{G}$. Then $(D\cdot L) = 1$. This is impossible as we proved in Umemura [16].

LEMMA 8. *Using the same notation as in the preceding lemma, we assume moreover* $N \geq 4$ *and* $C \cap (C + x) \cap (C + 2x) = \varnothing$. *Then* $E(L, T_x^*L, \cdots, T_{mx}^*L)$ *is an* L-*stable locally free sheaf for* $3 \leq m \leq N$.

Proof. We put $E_r = E(L, T_x^*L, \cdots, T_{(r+1)x}^*L)$ for $2 \leq r \leq N-1$. The local freeness of E_r follows from the hypothesis $C \cap (C + x) \cap (C + 2x) = \varnothing$. Now we prove the L-stability of E by the descending induction on r. Lemma 7 shows that E_{N-1} is L-stable. Let us assume E_r is L-stable for an r, $3 \leq r \leq N-1$ and show E_{r-1} is L-stable. The diagram

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{O} & \xrightarrow{\Phi_r} & L \oplus T_x^*L \oplus \cdots \oplus T_{rx}^*L \\ & \text{Id}\Big\| & & \Big\downarrow \Psi_r \\ 0 \longrightarrow & \mathcal{O} & \xrightarrow{\Phi_{r-1}} & L \oplus T_x^*L \oplus \cdots \oplus T_{(r-1)x}^*L \end{array}$$

is commutative where $\Phi_r(1) = (\varphi, T_x^*\varphi, \cdots, T_{rx}^*\varphi)$, $0 \neq \varphi \in H^0(A, L)$, $\Phi_{r-1}(1) = (\varphi, T_x^*\varphi, \cdots, T_{(r-1)x}^*\varphi)$ and Ψ_r is the projection onto the first r factors. The projection Ψ_r induces a surjection: $\psi_r\colon E_r \to E_{r-1}$ and the $\operatorname{Ker}\psi_r = T_{rx}^*L$. Hence we get an exact sequence

$$0 \to L' \to E_r \to E_{r-1} \to 0\,,$$

where $L' = T_{rx}^*L$. Tensoring L'^{-1} with the exact sequence, we obtain a new exact sequence

$$0 \to \mathcal{O} \to E'_r \to E'_{r-1} \to 0\,.$$

Our induction hypothesis is that E'_r is L-stable and we have to show E'_{r-1} is L-stable. Let G be a non-zero locally free sheaf of rank $\leq r-1$ and $E'_{r-1} \to G$ be a morphism which is surjective on $X-$ (a subvariety of codimension ≥ 2). By the stability of E'_r, $(c_1(E'_r)\cdot L)/r(E'_r) < (c_1(F)\cdot L)/r(G)$. Since $c_1(E'_r)$ is algebraically equivalent to L, we get $2/r < (c_1(F)\cdot L)/r(G)$. If $(c_1(F)\cdot L) \geq 2$, then

$$\frac{(c_1(E'_{r-1})\cdot L)}{r(E'_{r-1})} = \frac{2}{r-1} < \frac{2}{r(G)} \leq \frac{(c_1(F)\cdot L)}{r(G)}\,.$$

Hence we may assume $(c_1(F)\cdot L) = 1$. Then there is a generically surjective homomorphism $(L \oplus T_x^*L \oplus \cdots \oplus T_{rx}^*L) \otimes T_{rx}^*L^{-1} \to G \otimes T_{rx}^*L^{-1}$. The argument of the proof of Lemma 7 shows this is impossible.

Let us now examine the case (b).

LEMMA 9. *Let C_1, C_2 be elliptic curves. $A = C_1 \times C_2$ and $L = \mathcal{O}(C_1 \times 0 + 0 \times C_2)$. Let r be an integer ≥ 2 and x_i be a point of order $r+1$ of C_i, $1 \leq i \leq 2$. If we put $x = (x_1, x_2)$, then $E(L, T_x^*L, \cdots, T_{rx}^*L)$ is an L-stable locally free sheaf.*

Proof. Let us put $E = E(L, T_x^*L, \cdots, T_{rx}^*L)$, $C_1 \times 0 \cup 0 \times C_2 = D$. Since $D \cap T_x^*D \cap T_{2x}^*D = \varnothing$, E is locally free. Let us show the restriction $E|_{C_1\times 0}$ (resp. $E|_{0\times C_2}$) on $C_1 \times 0$ (resp. $0 \times C_2$) of E is stable. We need

SUBLEMMA 10. *Let C be an elliptic curve and M a line bundle of degree 1 on C. Let s be an integer ≥ 2 and y be a point of C of order $s+1$. Let E be the coherent sheaf defined by the following exact sequence;*

$$\text{(vi)}\qquad \begin{array}{c} 0 \to \mathcal{O} \to L \oplus T_y^*L \oplus \cdots \oplus T_{sy}^*L \to E \to 0 \\ 1 \mapsto (\psi, T_y^*, \cdots, T_{sy}^*\psi) \end{array}$$

https://doi.org/10.1017/S0027763000018651 Published online by Cambridge University Press

where $0 \neq \psi \in H^0(C, L)$. *Then* E *is locally free and stable.*

Proof of the sublemma. For the same reason as in the proof of the lemma, E is locally free. As in the proof of Lemma 7 the cyclic group (y) operates on the exact sequence and there exists a exact sequence of vector bundles

$$0 \to \mathcal{O} \to F \to E' \to 0 \qquad \text{(vii)}$$

over $C' = C/(y)$ such that π^*(vii) is isomorphic to (vi) where π is the isogeny $C \to C/(y)$. For the same reason as before, F is stable. Let $E' \to G'$ be a non-trivial quotient vector bundle. Since F is stable, $1/(s+1) = d(F)/r(F) < d(G')/r(G')$. Hence $d(G') \geq 1$. Since $r(E') > r(G')$, $d(E')/r(E') = 1/r(E') < d(G')/r(G')$. This shows E' is stable. Since the degree of π is s, and s is relatively prime to $r(E) = s-1$, $\pi^*E' = E$ is stable.

Let us come back to the proof of Lemma 9. Let G be a non-zero locally free sheaf of rank r on A and $E(L, T_x^*L, \cdots, T_{rx}^*L) \to G$ be a morphism which is surjective on A – (a subvariety of codimension ≥ 2). Since the restrictions are stable,

$$\frac{(c_1(E)\cdot C_1 \times 0)}{r(E)} = \frac{d(E\,|\,C_1 \times 0)}{r(E)} < \frac{d(G\,|\,C_1 \times 0)}{r(G)} = \frac{(c_1(G)\cdot C_1 \times 0)}{r(G)}$$

and

$$\frac{(c_1(E)\cdot 0 \times C_2)}{r(E)} < \frac{(c_1(G)\cdot 0 \times C_2)}{r(G)} .$$

Therefore

$$\frac{(c_1(E)\cdot L)}{r(E)} = \frac{(c_1(E)\cdot C_1 \times 0)}{r(E)} + \frac{(c_1(E)\cdot 0 \times C_2)}{r(E)} < \frac{(c_1(G)\cdot C_1 \times 0)}{r(G)} + \frac{(c_1(G)\cdot 0 \times C_2)}{r(G)} = \frac{(c_1(G)\cdot L)}{r(G)} .$$

LEMMA 11. *Let us assume that* $E(L_1, L_2, \cdots, L_n) = E$ *is locally free. Then the number* $\Delta(E) = (r(E)-1)c_1(E)^2 - 2r(E)c_2(E)$ *is equal to* $-2n$.

Proof. The first Chern class $c_1(E)$ is numerically equivalent to $L^{\otimes n}$ and the second Chern class $c_2(E)$ is numerically equivalent to $(n(n-1)/2)L^2$. Hence,

$$\begin{aligned}\Delta(E) &= (n-2)n^2L^2 - 2(n-1)\frac{n(n-1)}{2}L^2 \\ &= 2(n-2)n^2 - 2(n-1)^2 n = -2n .\end{aligned}$$

Remark 12. We do not know whether all $E(L_1, L_2, \cdots, L_n) = E$ is L-stable. We can prove the following assertion which will be used in the sequel.

LEMMA 13. *Suppose that if* $i = j$, $L_i \neq L_j$. *If* $E(L_1, L_2, \cdots, L_n) = E$ *is locally free,* E *is simple* ($\dim H^0(A, \operatorname{End} E) = 1$).

SUBLEMMA 14. *For* $1 \leq i \leq n$, $H^0(A, \check{E} \times L_i) = 0$.

Proof. We have an exact sequence;

$$\text{(viii)} \qquad 0 \to \check{E} \to \check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n \to \mathcal{O} \to 0 .$$

Tensoring with L_i, we get

$$\text{(ix)} \qquad 0 \to \check{E} \oplus L_i \to (\check{L}_1 \oplus \cdots \check{L}_n) \otimes L_i \to L_i \to 0 .$$

The long exact sequence of cohomology group is;

$$0 \to H^0(\check{E} \otimes L_i) \to H^0((\check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n) \otimes L_i) \to H^0(L_i) \to \cdots .$$

From the hypothesis $H^0((\check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n) \otimes L_i) \simeq H^0(\mathcal{O})$ and the homomorphism $H^0((\check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n) \otimes L_i) \simeq H^0(\mathcal{O}) \to H^0(L_i)$ is not zero. Hence $H^0(A, \check{E} \otimes L_i) = 0$.

SUBLEMMA 15. $H^l(A, E \otimes L_i) \simeq H^l(\mathcal{O})$, *for* $l = 1, 2$.

Let us write the long exact sequence of cohomology of (ix) again;

$$\begin{aligned} 0 &\to H^0((\check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n) \otimes L_i) \to H^0(L_i) \to H^1(\check{E} \otimes L_i) \\ &\to H^1((\check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n) \otimes L_i) \to H^1(L_i) \to H^2(\check{E} \otimes L_i) \\ &\to H^2((\check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n) \otimes L_i) \to H^2(L_i) . \end{aligned}$$

Now the assertion follows from the following;

(1) $H^j((\check{L}_1 \oplus \check{L}_2 \oplus \cdots \oplus \check{L}_n) \otimes L_i) = H^j(\mathcal{O})$ for any j.

(2) $H^1(L_i) = 0$ for $1 = 1, 2$.

SUBLEMMA 16. $H^0(A, \check{E}) = 0$, $H^1(A, \check{E}) \simeq H^0(\mathcal{O})$, $\dim H^2(A, \check{E}) = n + 1$.

The first two assertions follow easily from the long exact sequence of cohomology groups of the exact sequence (viii). The last assertion follows from the Riemann-Roch theorem for $\check{E}$.

Proof of Lemma 13. Tensoring E with the exact sequence (viii), we get

$$\text{(x)} \qquad 0 \to \check{E} \otimes E \to \check{L}_1 \otimes E \oplus \check{L}_2 \otimes E \oplus \cdots \oplus \check{L}_n \otimes E \to E \to 0 .$$

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The last terms of the exact sequence are;

$$\cdots \to H^1(E) \to H^2(\check{E} \otimes E)$$
$$\to H^2(\check{L}_1 \otimes E \oplus \check{L}_2 \otimes E \oplus \cdots \oplus \check{L}_n \otimes E) \to H^2(E)\,.$$

By the Serre duality and Sublemmas 15 and 16,

$$\dim H^1(E) = \dim H^1(\check{E}) = 1\,,$$
$$\dim H^2(L_1 \otimes \check{E} \oplus L_2 \otimes \check{E} \oplus \cdots \oplus L_n \otimes \check{E})$$
$$= \dim H^0(\check{L}_1 \otimes E \oplus \check{L}_2 \otimes E \oplus \cdots \oplus \check{L}_n \otimes E) = 0\,.$$

Hence $\dim H^2(\check{E} \otimes E) \leq 1$. By the Serre duality $\dim H^0(\check{E} \otimes E) \leq 1$. But $H^0(\check{E} \otimes E)$ contains k as homothesies. Hence $\dim H^0(A, \check{E} \otimes E) = 1$.

LEMMA 17. *If $E(L_1, L_2, \cdots, L_n) \simeq E(L'_1, L'_2, \cdots, L'_n)$, the set $\{L_1, L_2, \cdots, L_n\}$ coincide with the set $\{L'_1, L'_2, \cdots, L'_n\}$ counted with multiplicity.*

In fact let M be a line bundle algebraically equivalent to L. Tensoring M^{-1} wth the exact sequence, we get

$$\text{(xi)} \qquad 0 \to M^{-1} \to (L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M^{-1}$$
$$\to E(L_1, L_2, \cdots, L_n) \otimes M^{-1} \to 0\,.$$

Since M is ample, $H^0(A, M^{-1}) = H^1(A, M^{-1}) = 0$. Hence $H^0((L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M^{-1}) \simeq H^0(E(L_1, L_2, \cdots, L_n) \otimes M^{-1})$. Since the dimension of $H^0((L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M^{-1})$ is the number of times that M appears is the set $L_1, L_2, \cdots, L_n$, the lemma follows.

LEMMA 18. *Let M, M' be line bundles algebraically equivalent to 0. If $E(L_1, L_2, \cdots, L_n) \otimes M \simeq E(L'_1, L'_2, \cdots, L'_n) \otimes M'$, then $M \simeq M'$ and $E(L_1, L_2, \cdots, L_n) \simeq E(L'_1, L'_2, \cdots, L'_n)$.*

Tensoring M^{-1}, we may assume $M' \simeq \mathcal{O}$. Suppose that M is not isomorphic to $\mathcal{O}$. Then tensoring M with the exact sequence (i), we get

$$0 \to M \to (L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M \to E(L_1, L_2, \cdots, L_n) \otimes M \to 0\,.$$

Since M is algebraically equivalent to 0, $H^i(A, M) = 0$ for any i and

$$H^0((L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M) \simeq H^0(E(L_1, L_2, \cdots, L_n) \otimes M)\,.$$

Hence $\dim H^0(E(L_1, L_2, \cdots, L_n) \otimes M) = n$. On the other hand, from the exact sequence (i),

$$0 \to H^0(\mathcal{O}) \to H^0(L'_1 \oplus L'_2 \oplus \cdots \oplus L'_n) \to H^0(E(L'_1, L'_2, \cdots, L'_n))$$
$$\to H^1(\mathcal{O}) \to H^1(L'_1 \oplus L'_2 \oplus \cdots \oplus L'_n) = 0\,.$$

https://doi.org/10.1017/S0027763000018651 Published online by Cambridge University Press

Hence $\dim H^0(E(L'_1, L'_2, \cdots, L'_n)) = n+1$ and $E(L_1, L_2, \cdots, L_n) \otimes M$ is not isomorphic to $E(L'_1, L'_2, \cdots, L'_n)$.

Let now $p_{in+1}\colon \overbrace{A \times \cdots \times A}^{n+1} \to A \times A$ $(1 \leq i \leq n)$ be the projection onto the product of i-th and $(n+1)$-th factors. Let $m\colon A \times A \to A$ be the group law of A. Let $\mathscr{L}_i$ be the inverse image $(m \circ p_{in+1})^* L$ and $\Theta_i = (m \circ p_{in+1})^*(\theta)$ where θ is a fixed non-zero section of L. The coherent sheaf $\mathscr{E}$ on $\overbrace{A \times \cdots \times A}^{n+1}$ is defined by the exact sequence;

(xii)
$$\begin{gathered} 0 \to \mathcal{O} \to \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \cdots \oplus \mathscr{L}_n \to \mathscr{E} \to 0 \\ 1 \mapsto (\Theta_1, \Theta_2, \cdots, \Theta_n) \end{gathered}.$$

The coherent sheaf $\mathscr{E}$ is considered as a family of coherent sheaves on the last A parametrized by the first $\overbrace{A \times \cdots \times A}^{n}$. Let $(x_1, x_2, \cdots, x_n)$ be a point of $\overbrace{A \times \cdots \times A}^{n}$. The restriction of the exact sequence (xii) to the fibre $(x_1, \cdots, x_n) \times A$ is

$$\begin{aligned} &0 \to \mathcal{O} \to T^*_{x_1}L + T^*_{x_2}L + \cdots + T^*_{x_n}L \to E(T^*_{x_1}L, T^*_{x_2}L, \cdots, T^*_{x_n}L) \to 0 \\ &\quad 1 \mapsto (T^*_{x_1}\Theta_1, T^*_{x_2}\Theta_2, \cdots, T^*_{x_n}\Theta_n) \end{aligned}.$$

The symmetric group $\mathfrak{S}_n$ operates on $\overbrace{A \times \cdots \times A}^{n}$ hence on $(\overbrace{A \times \cdots \times A}^{n}) \times A$. There is an operation of $\mathfrak{S}_n$ on $\mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \cdots \oplus \mathscr{L}_n$ covering its operation on $(\overbrace{A \times \cdots \times A}^{n}) \times A$. This operation is compatible with the injection $0 \to \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \cdots \oplus \mathscr{L}_n$. Hence $\mathfrak{S}_n$ acts on $\mathscr{E}$. It follows from the descent theory that there exists a coherent sheaf $\mathscr{E}$ over $S^n(A) \times A$ such that $\pi^*\mathscr{E}' \simeq \mathscr{E}$ where $S^n(A)$ is the n-th symmetric product of A and $\pi\colon \overbrace{A \times \cdots \times A}^{n} \to S^n(A)$ is the projection.

Let $\mathscr{P}$ be the Poincaré line bundle over $\check{A} \times A \simeq A \times A$ and $p_{23}\colon S^n(A) \times A \times A \to A \times A$ the projection. We put $p^*_{23}\mathscr{P} = \mathscr{P}'$ and $\mathscr{E}' \otimes \mathscr{P}' = \mathscr{E}''$. Then $\mathscr{E}''$ is a family of coherent sheaves on A parametrized by $S^n(A) \times A$. Let (x, y), $(x', y') \in S^n(A) \times A$. If $\mathscr{E}''|_{(x,y)\times A}$ is isomorphic to $\mathscr{E}''|_{(x,y)\times A}$, then $(x, y) = (x', y')$ by Lemma 17 and Lemma 18.

PROPOSITION 19. *There exists a non-empty open subset U of $S^n(A)$ such that* (i) $\mathscr{E}''|U \times A \times A = \mathscr{E}^{(3)}$ *is locally free,* (ii) *for any point $(x, y) \in U \times A$, $\mathscr{E}''|(x, y) \times A$ is L-stable.*

Proof. Let $X = \{(x, y, z) \in S^n(A) \times A \times A \mid \Theta_i(x, z) = 0$ for any $1 \leq i \leq n$. Then X is a closed subset of $S^n(A) \times A \times A$. The coherent sheaf $\mathscr{E}''$ is locally free outside X. Let $p_1 \colon S^n(A) \times A \times A \to S^n(A)$ be the projection onto the first factor. Since p_1 is proper, $p_1(X)$ is a proper closed subset of $S^n(A)$ by Corollary 6. Let $U' = S^n(A) - p_1(X)$. Then $\mathscr{E}'' \mid U' \times A \times A$ is locally free. By Corollary 6, Lemma 8 and Lemma 9, there exists a point $x \in U'$ such that $\mathscr{E}^{(3)} \mid (x, y) \times A$ is L-stable for any point $y \times A$. Since the stability is an open condition by Maruyama [4], there exists non-empty open subset U of U' satisfying the condition (i) and (ii) of the proposition.

THEOREM 20. *The algebraic variety $S^n(A) \times A$ is birationally isomorphic to a component of the moduli space of L-stable vector bundles.*

Proof. Since $U \times A$ parametrizes a family of L-stable bundles, we get a morphism from $U \times A$ to a component of the moduli space. This map is injective by Lemma 17 and Lemma 18. The theorem now follows from Lemma 1, Lemma 3 and Lemma 11.

This component will be denoted by $X_n(A; L)$. Let $g \colon X_n(A; L) \to \check{A} = A$ be the map such that $g(x) = (\det E) \otimes L^{-1}$. Since this map is birationally equivalent to $S^n(A) \times A \to A$, $((x_1, \cdots, x_r), y) \mapsto \sum x_j + (n-1)y$ for $x \in \check{A} = A$, the fibre $g^{-1}(z) = Y_n(A; L)$ is birational to $Z^n(A) = \{((x_1, \cdots, x_n), y) \in S^n(A) \times A \mid \sum x_i + (n-1)y = 0\}$.

THEOREM 21. *The map $(A, L) \mapsto Y_n(A, L)$ separates locally the moduli space of the principally polarized abelian surfaces.*

Proof. Let A, B be abelian varieties of any dimension. We put

$$Z^n(A) = \{((x_1, \cdots, x_n), y) \in S^n(A) \times A \mid \textstyle\sum x_i + (n-1)y = 0\} ,$$
$$Z^n(B) = \{((x_1, \cdots, x_n), y) \in S^n(B) \times B \mid \textstyle\sum x_i + (n-1)y = 0\} .$$

It is sufficient to show that, if $Z^n(A)$ is birational to $Z^n(B)$, then A is isomorphic to B. This will be proved in Proposition 24.

Let X be an algebraic variety and $n > 0$ an integer. The symmetric group $\mathfrak{S}_n$ acts on the product $\overbrace{X \times \cdots \times X}^{n}$. The quotient variety $\overbrace{X \times \cdots \times X}^{n}/\mathfrak{S}_n$ is the n-th symmetric product of X and denoted by $S^n(X)$.

LEMMA 22. *Let A be an abelian variety. The closed subvariety $\{(x_1, x_2, \cdots, x_n) \in A \times \cdots \times A \mid \sum_{i=1}^n x_i = 0\}$ of $S^n(A)$ will be denoted by $T^n(A)$.*

Let $\tilde{T}^n(A)$ be a non-singular model of $T^n(A)$. If $n \geq 2$, $H^0(\tilde{T}^n(A), \Omega^1) = 0$.

Let V be the universal covering space of A. Hence there exists a lattice $\Gamma \subset V$ such that $V/\Gamma \simeq A$. $H^1(A, \mathcal{O}) \simeq T$ where $T = \mathrm{Hom}_C(V, C)$ (see Mumford [10]). By the Künneth formula, $H^1(A \times \cdots \times A, \mathcal{O}) \simeq T \oplus \cdots \oplus T$. The symmetric group acts on $H^1(A \times \cdots \times A, \mathcal{O}) \simeq T \oplus \cdots \oplus T$ as permutation of factors. Let $U^n(A)$ denote the closed subvariety $\{(x_1, x_2, \cdots, x_n) \in A \times \cdots \times A \mid \sum_{i=1}^n x_i = 0\}$ of $\overbrace{A \times \cdots \times A}^{n}$ and i the inclusion $U^n(A) \hookrightarrow \overbrace{A \times \cdots \times A}^{n}$. $U^n(A)$ is isomorphic to $\overbrace{A \times \cdots \times A}^{n-1}$. Notice that the symmetric group $\mathfrak{S}_n$ operates on $U^n(A)$ and $U^n(A)/\mathfrak{S}_n = T^n(A)$. It is easy to check that the restriction of the map $i^*: H^1(A \times \cdots \times A, \mathcal{O}) = \overbrace{T \oplus \cdots \oplus T}^{n} \to H^1(U^n(A), \mathcal{O}) = \overbrace{T \oplus \cdots \oplus T}^{n-1}$ to the subspace $U = \{(t_1, t_2, \cdots, t_n) \in T + \cdots + T \mid \sum_{i=1}^n t_i = 0\}$ is an isomorphism. Hence $H^1(U^n(A), \mathcal{O})^{\mathfrak{S}_n} = U^{\mathfrak{S}_n} = 0$. On the other hand $H^1(U^n(A), \mathcal{O})^{\mathfrak{S}_n} = H^1(\tilde{T}^n(A), \mathcal{O})$ by Proposition 9. 24, Ueno [12] hence the lemma is proved.

LEMMA 23. *Any rational map f from $\tilde{T}^n(A)$ to B is trivial, i.e., $f(\tilde{T}_n))$ is a point.*

Proof. Blowing up the given $\tilde{T}^n(A)$ if necessary, we may assume f regular. Suppose that the dimension of $f(T^n(A))$ is positive. Then there exist a holomorphic 1-form ω on B such that $f^*\omega \neq 0$ which conducts Lemma 22.

PROPOSITION 24. *Let A, B be abelian varieties. If $Z^n(A)$ is birationally isomorphic to $Z^n(B)$, then A is isomorphic to B.*

Proof. If $n = 1$, then $Z^1(A) = A$, $Z^1(B) = B$ and the assertion is well known (see. Weil [17]). We assume $n \geqq 2$. Let $f: Z^n(A) \to Z^n(B)$ be a birational map. Let π_A (resp. π_B) be the map from $Z^n(A)$ to A(resp. from $Z^n(B)$ to B) defined by $\pi_A((x_1, \cdots, x_n), y) = \sum_{i=1}^n x_i$ (resp. $\pi_B((x_1, \cdots, x_n), y) = \sum_{i=1}^n x_i$). Then by Lemma 23, f induces a birational map $\bar{f}: A \to B$ such that $\pi_B \circ f = \bar{f} \circ \pi_A$. Hence A is biregularly isomorphic to B.

In our examples, the moduli spaces have irregularity 4. The abelian variety $A \times \hat{A}$ operates on the moduli spaces; $E \mapsto T_X^* E \otimes L$, $x \in A$, $L \in \hat{A}$. In most cases this operation is effective modulo finite group and hence by a theorem of Matsumura-Nishi [6], the irregularity of the moduli space ≥ 4.

QUESTION 25. When is the irregularity of the moduli space 4?

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