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IDEALS AND HIGHER DERIVATIONS IN COMMUTATIVE RINGS

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Introduction. In this paper, we wish to generalize the following lemma first proven by O. Zariski [**5**, Lemma 4]. Let O be a complete local ring containing the rational numbers and let m denote the maximal ideal of O. Assume there exists a derivation δ of O such that $\delta(x)$ is a unit in O for some x in m. Then O contains a ring O_1 of representatives of the (complete local) ring O/Ox having the following properties: (a) δ is zero on O_1; (b) x is analytically independent over O_1; (c) O is the power series ring $O_1[[x]]$. In [**4**], A. Seidenberg used Zariski's lemma extensively to study conditions under which an affine algebraic variety V over a base field of characteristic zero is analytically a product along a given subvariety W of V. We should like to generalize Zariski's lemma by removing the condition that O contain the rationals. We could then get some conditions under which an arbitrary affine variety V would be analytically a product along a subvariety W.

Unfortunately, Zariski's lemma is false in the characteristic $q \neq 0$ case. That is, if O is a complete local ring containing a field k of characteristic $q \neq 0$, and if δ is a derivation of O such that $\delta(x)$ is a unit for some x in m, then there may be no subring O_1 of O such that properties (a), (b), and (c) hold. An example is given in § II of this paper. Thus there is no hope of a straightforward generalization of the lemma. It is well known that in certain problems involving fields of characteristic q, higher derivations, i.e., derivations of infinite rank, often yield results which are unobtainable for ordinary derivations. So we are led naturally to study higher derivations and their relationships to ideals in commutative rings.

If we replace δ in Zariski's lemma with a higher derivation $\{\delta_i\}$, then we do get an appropriate generalization of the lemma. This result is given in § II of this paper. In § I, we present some general information on the relationship between ideals and higher derivations. This investigation is carried out mainly in the setting of commutative rings of finitely generated type, i.e., those rings which naturally arise in Algebraic Geometry. We have basically tried to reproduce the results known for ordinary derivations [**4**] in the more general setting of higher derivations.

Preliminaries. Throughout this paper, all rings will be assumed to be commutative and to contain an identity. A ring O_1 will be called a subring

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of a ring O if $O_1 \subset O$ and the identity of O_1 is the same as that of O. We shall also assume that all ring homomorphisms take the identity to the identity. We shall let k denote a field and $X_1, \ldots, X_n$ indeterminates over k. We shall say that a ring O is a finitely generated extension of the field k if O is a homomorphic image of a polynomial ring $k[X_1, \ldots, X_n]$. We shall often write $O = k[x_1, \ldots, x_n]$ if O is not the null ring and is finitely generated extension of k.

Let O be a ring. Then a derivation on O is an abelian group homomorphism $\delta: O \to O$ such that for all a, b in O

$$\delta(ab) = a\delta(b) + b\delta(a).$$

A higher derivation on O is an infinite sequence of abelian group homomorphisms $\delta_i: O \to O$, $i = 0, 1, \ldots$ such that:

(1) δ_0 is the identity map.

(2) $\delta_i(ab) = \Sigma_{j+k=i}\, \delta_j(a)\delta_k(b)$ for all i and all a, b in O.

We shall call (2) Leibnitz's rule. We shall let $\mathrm{Der}(O)$ denote the collection of all derivations of O into O and $H(O, O)$ denote the collection of all higher derivations of O into O. Note that if $D = \{\delta_i\} \in H(O, O)$, then $\delta_1 \in \mathrm{Der}(O)$.

Let A be an ideal of O. We shall say that A is differential under $\mathrm{Der}(O)$ if $\delta(A) \subset A$ for all $\delta \in \mathrm{Der}(O)$. Similarly, A is differential under $H(O, O)$ if for all $D = \{\delta_i\} \in H(O, O)$, $\delta_i(A) \subset A$ for all $i = 0, 1, \ldots$. We note that these two ideas are somewhat independent of each other. We shall give an example in § I which shows that a derivation $\delta \in \mathrm{Der}(O)$ need not be the term of degree one in any higher derivation $D = \{\delta_i\} \in H(O, O)$. Thus, if A is differential under $\mathrm{Der}(O)$, it does not necessarily follow that A is differential under $H(O, O)$ and vice versa. We shall say more about this relationship in § I.

Let V be an affine algebraic variety over a field k and let O denote the coordinate ring of V. Let p be a prime ideal in O. Then we say that V is analytically a product along the subvariety $W = \mathscr{V}(p)$ if the completion $\hat{O}_p$ of the local ring O_p is of the form $O_1[[t]]$ with O_1 a complete local ring and t analytically independent over O_1.

Finally, we shall assume the reader is familiar with the main results in [**4**].

I. Ideals and higher derivations. Let O be a ring and $D = \{\delta_i\} \in H(O, O)$. Let $O^* = O[[t]]$, t an indeterminate. Thus

$$O^* = \left\{\sum_{i=0}^{\infty} \alpha_i t^i \middle| \alpha_i \in O\right\}$$

is the ring of all formal power series in t with coefficients in O. If O is Noetherian and A an ideal of O, then

$$AO^* = \left\{\sum_{i=0}^{\infty} \alpha_i t^i \middle| \alpha_i \in A\right\}.$$

For each $j = 0, 1, \ldots$ we may extend the definition of δ_j to O^* as follows:

$$\delta_j\left(\sum_{i=0}^{\infty} \alpha_i t^i\right) = \sum_{i=0}^{\infty} \delta_j(\alpha_i)t^i.$$

The δ_j's thus extended are clearly a collection of abelian group homomorphisms of $O^* \to O^*$. A routine computation shows that $\{\delta_i\}$ actually forms a higher derivation on O^*, i.e. , $\{\delta_i\}$ satisfies (1) and (2) on O^*. Thus given $D = \{\delta_i\} \in H(O, O)$ we can construct a higher derivation which we also call D in $H(O^*, O^*)$.

If $D = \{\delta_i\} \in H(O^*, O^*)$, then we may define a ring homomorphism $\tau_D: O^* \to O^*$ as follows:

$$\tau_D(\alpha) = \sum_{i=0}^{\infty} \delta_i(\alpha)t^i,$$

for $\alpha \in O^*$.

Note that $\delta_i(1) = 0$ for all $i \geqq 1$ and thus $\tau_D(1) = 1$. Thus if $D = \{\delta_i\} \in H(O, O)$, we can associate with D a ring endomorphism τ_D of O^*. It is well known [**1**, Theorem 1] that $H(O, O)$ forms a group and that τ_D is an automorphism of O^*. We need the following lemma.

LEMMA 1. *Let O be a Noetherian ring and A an ideal in O. Let $D = \{\delta_i\} \in H(O, O)$. Then A is differential under D (i.e., $\delta_i(A) \subset A$ for all i) if and only if $\tau_D(AO^*) \subset AO^*$.*

Proof. Suppose A is differential under D. Let $\alpha \in AO^*$. Then

$$\alpha = \sum_{i=0}^{\infty} a_i t^i$$

with $a_i \in A$. So for all $j = 0, 1, \ldots$

$$\delta_j(\alpha) = \delta_j\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} \delta_j(a_i)t^i \in AO^*.$$

Therefore

$$\tau_D(\alpha) = \sum_{i=0}^{\infty} \delta_i(\alpha)t^i \in AO^*.$$

Hence $\tau_D(AO^*) \subset AO^*$. If $\tau_D(AO^*) \subset AO^*$, then $\tau_D(A) \subset AO^*$. So let $a \in A$. Then

$$\tau_D(a) = \sum_{i=0}^{\infty} \delta_i(a)t^i \in AO^*.$$

Therefore $\delta_i(a) \in A$ for each i.

We can now prove the following theorem.

THEOREM 1. *Let O be a Noetherian ring and let A be an ideal in O with associated primes $p_1, \ldots, p_s$. Let $D = \{\delta_i\} \in H(O, O)$ such that A is differential* *under D. Then $p_1, \ldots, p_s$ are also differential under D and A can be written as an irredundant intersection $q_1 \cap \ldots \cap q_s$ of primary ideals which are differential under D.*

Proof. This proof is much like A. Seidenberg's Theorem 1 in [**4**]. In place of e^{tD}, we use τ_D. Since A is differential under D, AO^* is invariant under τ_D, i.e., $\tau_D(AO^*) \subset AO^*$. Now it follows from [**1**, p. 33], that $\tau_{D^{-1}}$ (D^{-1} is the inverse of D in the group $H(O, O)$) also maps AO^* into AO^*. Hence $\tau_D(AO^*) = AO^*$. Thus τ_D permutes the associated primes $p_1O^*, \ldots, p_sO^*$ of AO^*. Say $\tau_D(p_iO^*) = p_jO^*$. Then for $a \in p_i$, $\tau_D(a) \in p_jO^*$. Thus $a \in p_j$. So $p_i \subset p_j$. Now $\tau_{D^{-1}}(p_jO^*) = p_iO^*$. So $p_j \subset p_i$. Hence $i = j$ and $\tau_D(p_iO^*) \subset p_iO^*$. Therefore by Lemma 1, p_i is differential under D. The rest of the proof follows exactly as in Seidenberg's result.

Theorem 1 gives us a sufficient condition for a Noetherian ring O to be an integral domain. Namely:

COROLLARY. *Let O be a Noetherian ring in which (0) is the only ideal which is differential under $H(O, O)$, then O is an integral domain.*

The next theorem allows us to pass to a local ring when attempting to decide whether a prime ideal $p \subset O$ is differential under $H(O, O)$. We shall denote by O_p the local ring obtained from O by localizing at p. If we let $n = \{x \in O | rx = 0, r \in O$ but $r \notin p\}$, then the compliment of p/n consists of non zero divisors in O/n and O_p is just $(O/n)_{p/n}$. We need the following proposition:

PROPOSITION 1. *Let p be a prime ideal in a ring O and let $D = \{\delta_i\} \in H(O, O)$. Then D induces a higher derivation $D' = \{\delta_i'\} \in H(O_p, O_p)$.*

Proof. We first note that $n = \{x \in O | rx = 0, r \notin p\}$ is differential under D. We may argue this by induction on i. If $x \in n$, then there exists an element $r \in O$ but not in p such that $rx = 0$. Then $0 = \delta_1(rx) = r\delta_1(x) + x\delta_1(r)$. So $0 = r^2\delta_1(x) + rx\delta_1(r) = r^2\delta_1(x)$. Thus $\delta_1(x) \in n$. So n is differential under δ_1. Assume now that $\delta_1, \ldots, \delta_m$ map n into n. Let $x \in n$, $r \notin p$ such that $rx = 0$. Then

$$0 = \delta_{m+1}(rx) = \sum_{j+k=m+1} \delta_j(r)\delta_k(x).$$

So

$$r\delta_{m+1}(x) = -\ (\delta_1(r)\delta_m(x) + \ldots + x\delta_{m+1}(r)) \in n.$$

Hence there exists an $r' \in O$ but not in p such that

$$r'(r\delta_{m+1}(x)) = 0.$$

Since $r'r \in O$, but not p, $\delta_{m+1}(x) \in n$. Thus n is a differential ideal.

We now form O/n. Since n is differential, D induces a higher derivation $\bar{D} = \{\bar{\delta}_i\} \in H(O/n, O/n)$ in the natural way, i.e.,

$$\bar{\delta}_i(a + n) = \delta_i(a) + n \ (a \in O).$$

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Let $K(O/n)$ denote the total quotient ring of O/n. Then

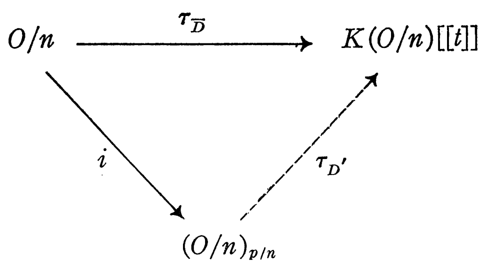
$$O/n \subset O_p = (O/n)_{p/n} \subset K(O/n).$$

Thus $\bar{D}$ can be considered as an element of $H(O/n, K(O/n))$; i.e., we may view $\bar{D}$ as a higher derivation of O/n into $K(O/n)$. We may now apply [**1**, Lemma 2] to uniquely extend $\bar{D}$ to a higher derivation $D' = \{\delta_i'\} \in H(O_p, K(O/n))$. It remains to show that $D' \in H(O_p, O_p)$.

Now $\bar{D} = \{\bar{\delta}_i\} \in H(O/n, K(O/n))$ gives rise to an isomorphism $\tau_{\bar{D}}: O/n \rightarrow K(O/n)[[t]]$ (t an indeterminate) as follows:

$$\tau_{\bar{D}}(b) = \sum_{i=0}^{\infty} \bar{\delta}_i(b)t^i \qquad (b \in O/n).$$

We have the following diagram which we wish to complete with a map $\tau_{D'}$.



So we define

$$\tau_{D'}\left(\frac{b}{c}\right) = \tau_{\bar{D}}(b)\{\tau_{\bar{D}}(c)\}^{-1}$$

where $c, b \in O/n$, $c \notin p/n$. Then

$$\tau_{D'}(\ ) = \sum_{i=0}^{\infty} \delta_i'(\ )t^i,$$

i.e., $\{\delta_i'\}$ are just the component parts of the isomorphism $\tau_{D'}$.

Now if

$$\alpha = \sum_{i=0}^{\infty} a_i t^i \in O/n[[t]]$$

with $a_0 \notin p/n$, then α is a unit in $K(O/n)[[t]]$ and has inverse $\alpha^{-1} \in O_p[[t]]$. Hence it follows that $\tau_{D'}$ actually maps $O_p \rightarrow O_p[[t]]$. Thus $D' = \{\delta_i'\} \in H(O_p, O_p)$.

THEOREM 2. *Let p be a prime ideal in a ring O. If pO_p is differential under $H(O_p, O_p)$, then p is differential under $H(O, O)$.*

Proof. Here pO_p of course means the maximal ideal of the local ring O_p. Suppose p is not differential under $H(O, O)$. Then there exists a higher derivation $D = \{\delta_i\} \in H(O, O)$ such that $\delta_j(p) \not\subset p$ for some $j \geqq 1$. Using Proposition 1, we may extend D to $D' \in H(O_p, O_p)$ by $D \rightarrow \bar{D} \rightarrow D'$. One easily checks that $\bar{D} = \{\bar{\delta}_i\} \in H(O/n, O/n)$ has the property that $\bar{\delta}_j(p/n) \not\subset p/n$. Thus, $\delta_j'(pO_p) \not\subset pO_p$. But this is a contradiction, since pO_p was assumed differential under $H(O_p, O_p)$.

Proposition 1 and Theorem 2 show that higher derivations like derivations can be extended in a canonical way from O to O_p. Going the other way seems to be harder. In the case of rings finitely generated over a field k of characteristic zero, the converse of both Proposition 1 and Theorem 2 follows from [**2**, Theorem 5] and [**4**, Theorem 2]. The general case remains unknown.

We specialize to rings which are finitely generated extensions of k. Any sequence $\delta_0, \ldots, \delta_m$ of abelian group homomorphisms from O to O such that

(1) $\delta_0 =$ identity, and

(2) $\delta_i(ab) = \sum_{j+k=i} \delta_j(a)\delta_k(b) \qquad (i = 1, \ldots, m),$

will be called a higher derivation of rank m on O.

If $O = k[x_1, \ldots, x_n]$, we shall denote by $H_k(O, O)$ the set of all higher derivations $D = \{\delta_i\}$ on O which consist of k-linear mappings $\delta_i : O \to O$. Similarly, a higher derivation of rank m on O will be called a higher k-derivation if it consists of k-linear maps.

We need the following lemma:

LEMMA 2. *Let O be a finitely generated extension of the field k and p a prime ideal of O. Let $n = \{x \in O, rx = 0, r \notin p\}$ and $D = \{\delta_i\} \in H_k(O_p, O_p)$. Then for every positive integer m, there exist elements $k_i \in O/n$, $k_i \notin p/n$, $i = 1, \ldots, m$, such that $\{\delta_0, k_1\delta_1, \ldots, k_m\delta_m\}$ is a k-higher derivation of rank m on O/n.*

Proof. Since O is a finitely generated extension of the field k, O/n is also a finitely generated extension of k. Hence O/n has the form $k[\bar{x}_1, \ldots, \bar{x}_s]$ for some field k. Now let m be a positive integer. Then for each $j = 1, \ldots, m$

$$\delta_j(\bar{x}_i) = \frac{u_{ij}}{v_{ij}},$$

where $u_{ij}, v_{ij} \in O/n$ and $v_{ij} \notin p/n$. So set

$$l_j = \prod_{i=1}^{s} v_{ij}.$$

Then $l_j \in O/n$ but $l_j \notin p/n$. We also note that for each $j = 1, \ldots, m$, $(l_1^m \ldots l^m{}_{j-1} l_j)\delta_j$ is an abelian group homomorphism of $O/n \to O/n$. The fact that $(l_1^m \ldots l^m{}_{j-1} l_j)\delta_j$ maps O/n into O/n can be proven by successive applications of Leibnitz's rule (2).

If we now set $k_i = \{l_1^m \ldots l^m{}_{m-1} l_m\}^i$ for $i = 1, \ldots, m$, then $\{\delta_0, k_1\delta_1, k_2\delta_2, \ldots, k_m\delta_m\}$ forms a k-higher derivation of rank m on O/n.

We may now prove the following partial converse to Theorem 2:

THEOREM 3. *Let $O = k[x_1, \ldots, x_n]$ be a finitely generated extension of the field k and let p be a prime ideal of O. Suppose that for any positive integer m, p is differential under all k-higher derivations of rank m. Then pO_p is differential under all k-derivations of finite or infinite rank.*

Proof. We use the notation that appears in [**4**, Theorem 2].

In O, let $(0) = q_1 \cap \ldots \cap q_s$ be an irredundant primary decomposition of (0). $O/n = k[\bar{x}_1, \ldots, \bar{x}_h]$. Let $q_i \subset p$ for $i = 1, \ldots, t$ and $q_i \not\subset p$ for $i = t+1, \ldots, s$. Let $R = k[X_1, \ldots, X_h]$, and let Π be the natural mapping of R onto O. Let $q_i' = \Pi^{-1}(q_i)$ for $i = 1, \ldots, s$. Set $A = q_1' \cap \ldots \cap q_s'$ and $N = q_1' \cap \ldots \cap q_t'$.

Now suppose pO_p is not differential under $H_k(O_p, O_p)$. Then there exists a $D = \{\delta_i\} \in H_k(O_p, O_p)$ such that $\delta_j(pO_p) \not\subset pO_p$ for some $j \geqq 1$. Hence it suffices to prove pO_p is differential under all k-derivations of finite rank.

Suppose pO_p is not differential under all k-derivations of finite rank. Then there exists a k-higher derivation $\{\delta_0, \ldots, \delta_m\}$ of rank m such that $\delta_m(pO_p) \not\subset pO_p$. Without loss of generality, we may assume $\delta_i(pO_p) \subset pO_p$ for $i = 0, \ldots, m-1$. By lemma 2, we can find elements $k_1, \ldots, k_m \in O/n$ such that $k_i \notin p/n$, $i = 1, \ldots, m$, and $\{\delta_0, k_1\delta_1, \ldots, k_m\delta_m\}$ is a k-higher derivation of rank m on O/n. If $i < m$, then $k_i\delta_i(p/n) \subset pO_p \cap O/n = p/n$. We also know there exists an $\bar{x} \in p/n$ and $\bar{y} \in O/n$ $(\bar{y} \notin p/n)$ such that $\delta_m(\bar{x}/\bar{y}) \notin pO_p$. But

$$\delta_m(\bar{x}/\bar{y}) = \sum_{j+k=m} \delta_j(\bar{x})\delta_k(1/\bar{y}).$$

Hence $\delta_m(\bar{x}) \notin pO_p$. So $k_m\delta_m(\bar{x}) \notin p/n$. Thus using Lemma 2, we have constructed a k-higher derivation $\{\bar{\mu}_0\bar{\mu}_1, \ldots, \bar{\mu}_m\}$ $(\bar{\mu}_i = k_i\delta_i)$ of rank m on O/n such that

$$\bar{\mu}_j(p/n) \subset p/n \ (j = 0, 1, \ldots, m-1)$$
$$\bar{\mu}_m(p/n) \not\subset p/n.$$

We next note that $R/N \cong O/n$. Let Π_0 denote the natural mapping of R onto O/n given by $\Pi_0(X_i) = \bar{x}_i$. We now define a k-higher derivation $\{\mu_0, \ldots, \mu_m\}$ of rank m on R as follows: Let $a_{ij} \in R$ such that $\Pi_0(a_{ij}) = \bar{\mu}_i(\bar{x}_j)$ for $i = 1, \ldots, m$, $j = 1, \ldots, h$. Define for $i = 0, \ldots, m$ $\mu_i{:}R \to R$ by $\mu_0 =$ identity and $\mu_i(X_j) = a_{ij}$. This gives us a well defined k-higher derivation $\{\mu_i\}$ on R [**2**, Proposition (2)]. (One uses Leibnitz's rule to extend μ_i to all of R.) Note that if $f(X_1, \ldots, X_h)$ is any polynomial in R, then $\Pi_0\{\mu_j(f)\} = \bar{\mu}_j(\Pi_0(f)) = \bar{\mu}_j(f(\bar{x}_1, \ldots, \bar{x}_h))$ for $j = 0, \ldots, m$. From this fact, we immediately get that N is differential under $\{\mu_0, \ldots, \mu_m\}$.

Now let p' be the pull back of p to R, i.e., $p' = \Pi^{-1}(p)$. Let $a \in q'_{t+1} \cap \ldots \cap q_s' - p'$, i.e., let a be an element in $q'_{t+1} \subset \ldots \subset q_s'$ which is not in p'. Now consider $\{\mu_0, a\mu_1, \ldots, a^m\mu_m\}$. The $\{a^i\mu_i\}^m{}_{i=0}$ clearly form a k-higher derivation of rank m on R. If $x \in A$, then for all $i = 0, \ldots, m$

$$a^i\mu_i(x) \in N,$$

since N is differential under μ_i. Hence $a^i\mu_i(x) \in q_1' \cap \ldots \cap q_t'$. Since $a \in q'_{t+1} \cap \ldots \cap q_s'$ we get $a^i\mu_i(x) \in q_1' \cap \ldots \cap q_s' = A$. Hence A is differential under $\{\mu_0, \ldots, a^m\mu_m\}$. We also note that $\mu_m(p') \not\subset p'$ since $\bar{\mu}_m(p/n) \not\subset p/n$. Thus $a^m\mu_m(p') \not\subset p'$.

We have now constructed a k-higher derivation $\{a^i\mu_i\}$ of rank m on R such that A is differential under $\{a^i\mu_i\}$ but $a^m\mu_m(p') \not\subset p'$. Thus we get an induced k-higher derivation $\{\bar{\delta}_i\}^m{}_{i=0}$ on $R/A \cong O$ such that $\bar{\delta}_m(p) \not\subset p$. This is a contradiction and the proof is complete.

We shall finish this section by presenting some theorems which will give us examples of ideals which are not differential and ideals which are differential under higher derivations. We note that in [**4**, Theorem 3], A. Seidenberg proved the following result:

Let $O = k[x_1, \ldots, x_n]$ *be a finitely generated extension of the field* k *and* p *a non-minimal prime ideal of* O *such that* O_p *is regular. Then* p *is not differential under* Der(O).

This theorem does not yield an immediate result for higher derivations since every derivation $\delta : O \rightarrow O$ cannot necessarily be embedded as the term of degree one in a higher derivation on O. To see this, consider the following example which appears in [**3**].

Example 1. Let k be a field of characteristic $q \neq 0$ and let X, Y be indeterminates over k. Consider the irreducible polynomial $Y^q - X^q - X^{q+1} \in k[X, Y]$. Let $O = k[X, Y]/(Y^q - X^q - X^{q+1}) = k[x, y]$. Then there exists a $\delta \in$ Der(O) such that $\delta(x) = 0$ and $\delta(y) = 1$. Let $\bar{O}$ denote the integral closure of O in its quotient field. Then $\delta(\bar{O}) \not\subset \bar{O}$. If δ could be embedded as the term of degree one in a higher derivation $\{\delta_0, \delta_1, \delta_2, \delta_3, \ldots\} \in H(O, O)$, then $\delta(\bar{O}) \subset \bar{O}$. For, Seidenberg has shown [**3**, p. 173] that the following result is true for higher derivations:

Let O *be an integral domain with quasi-integral closure* O'. *If* $D = \{\delta_i\}$ *is a higher derivation on* Σ, *the quotient field of* O, *such that* $\delta_i(O) \subset O$ *for all* $i = 1, 2, \ldots,$ *then* $\delta_i(O') \subset O'$ *for all* $i = 1, 2, \ldots.$

Now for Noetherian rings, the quasi-integral closure is equal to the integral closure. Hence δ is an example of a derivation which cannot be imbedded in a higher derivation.

Thus we must work a little harder to obtain a result analogous to [**4**, Theorem 3].

Let k be an arbitrary field and $X_1, \ldots, X_n$ be indeterminates over k. Let Σ denote the quotient field of $k[X_1, \ldots, X_n]$ and $\{u_{ij} | j = 1, \ldots, n, i = 1, 2, \ldots, \infty\}$ and T be indeterminates over Σ. For each $i = 1, 2, \ldots,$ we define a k-linear mapping $q_i : k[X_1, \ldots, X_n] \rightarrow k[X_1, \ldots, X_n][u_{ij}]$ as follows: Given any monomial $X_1^{m_1} \ldots X_n^{m_n}$ in $k[X_1, \ldots, X_n]$, we define $q_i(X_1^{m_1} \ldots X_n^{m_n})$ to be the coefficient of T^i in the following power series in $(k[X_1, \ldots, X_n][u_{ij}])[[T]]$:

$$\left\{X_1 + \sum_{j=1}^{\infty} u_{j1}T^j\right\}^{m_1} \ldots \left\{X_n + \sum_{j=1}^{\infty} u_{jn}T^j\right\}^{m_n}.$$

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Thus q_i is well defined on the monomials of $k[X_1, \ldots, X_n]$. We extend the definition of q_i to all of $k[X_1, \ldots, X_n]$ by linearity. If we define $q_0 = 1$, then a routine computation shows that $\{q_0, q_1, q_2, \ldots\}$ forms a k-higher derivation from $k[X_1, \ldots, X_n]$ to $k[X_1, \ldots, X_n][u_{ij}]$.

Now suppose $\Sigma' = k(x_1, \ldots, x_n)$ is a finitely generated field extension of k. Let $\{\bar{u}_{ij} \in \Sigma' | j = 1, \ldots, n,\ i = 1, 2, \ldots \infty\}$ be a collection of elements of Σ'. Then we have a natural k-algebra homomorphism

$$\bar{\Pi}: k[X_1, \ldots, X_n][u_{ij}] \to k[x_1, \ldots, x_n][\bar{u}_{ij}] \subset \Sigma'$$

given by $\bar{\Pi}(X_i) = x_i$ and $\bar{\Pi}(u_{ij}) = \bar{u}_{ij}$. If $f(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$, we shall say that the $\{\bar{u}_{ij}\}$ solve $q_i(f) = 0$ if $\bar{\Pi}q_i(f) = 0$. Thus $\{\bar{u}_{ij}\}$ solve $q_i(f) = 0$ if, when we substitute for $X_1, \ldots, X_n, x_1, \ldots, x_n$ and for u_{ij} the $\bar{u}_{ij}$ in $q_i(f)$, we get zero. We can now prove the following lemma:

LEMMA 3. *Let $\Sigma' = k(x_1, \ldots, x_n)$ be a finitely generated field extension of k with relations $f_1, \ldots, f_r \in k[X_1, \ldots, X_n]$. If $D = \{\delta_i\} \in H_k(\Sigma', \Sigma')$, then $\{\bar{u}_{ij} = \delta_i(x_j) | j = 1, \ldots, n,\ i = 1, 2, \ldots \infty\}$ is a system of elements of Σ' which solve the equations*

(3) $$q_i(f_k) = 0 \quad (k = 1, \ldots, r, i = 1, 2, \ldots, \infty).$$

Conversely, if $\{\bar{u}_{ij} | j = 1, \ldots, n, i = 1, \ldots, \infty\}$ is a collection of elements of Σ' which solve (3), *then there exists a higher derivation $D = \{\delta_i\} \in H_k(\Sigma', \Sigma')$ such that $\delta_i(x_j) = \bar{u}_{ij}$.*

Proof. $f_1, \ldots, f_r$ being the relations of Σ' of course means that $f_1, \ldots, f_r$ generate an ideal $A \subset k[X_1, \ldots, X_n]$ such that

$$O \to A \to k[X_1, \ldots, X_n] \overset{\Pi_0}{\to} k[x_1, \ldots, x_n] \to O$$

is exact. Thus $f_k(x_1, \ldots, x_n) = 0$ for $k = 1, \ldots, r$. Hence if $D \in H_k(\Sigma', \Sigma')$, then $\bar{u}_{ij} = \delta_i(x_j)$ satisfy (3) by repeated applications of Leibnitz's rule.

Thus let us suppose we have a collection of elements $\{\bar{u}_{ij} \in \Sigma' | j = 1, \ldots, n,\ i = 1, \ldots, \infty\}$ which solve the equations (3). We define k-linear maps $\delta_i: k[x_1, \ldots, x_n] \to \Sigma'$ as follows: Given any $g(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ define

$$\delta_i(g(x_1, \ldots, x_n)) = \bar{\Pi}\{q_i(g(X_1, \ldots, X_n))\} \ (i \geqq 1).$$

Thus δ_i is defined by the following diagram:

$$\begin{array}{ccc} k[X_1, \ldots, X_n] & \xrightarrow{q_i} & k[X_1, \ldots, X_n][u_{ij}] \\ \downarrow \Pi_0 & \delta_i = \bar{\Pi} q_i \Pi_0^{-1} & \downarrow \bar{\Pi} \\ k[x_1, \ldots, x_n] & & k[x_1, \ldots, x_n][\bar{u}_{ij}] \end{array}$$

It is not obvious that δ_i is well defined. So suppose $l(x_1, \ldots, x_n) =$

$g(x_1, \ldots, x_n)$ in $k[x_1, \ldots, x_n]$. Then $l(X_1, \ldots, X_n) - g(X_1, \ldots, X_n) \in A$. So there exists $h_1, \ldots, h_r \in k[X_1, \ldots, X_n]$ such that

$$l(X_1, \ldots, X_n) - g(X_1, \ldots, X_n) = \sum_{k=1}^{r} h_k(X_i) f_k(X_i).$$

Thus, since the q_i's form a higher derivation, we have

$$\begin{aligned}
\delta_i(l(x_1, \ldots, x_n)) - \delta_i(g(x_1, \ldots, x_n)) \\
&= \bar{\Pi}\{q_i(l(X_1, \ldots, X_n))\} - \bar{\Pi}\{q_i(g(X_1, \ldots, X_n))\} \\
&= \bar{\Pi}\{q_i(l(X_1, \ldots, X_n) - g(X_1, \ldots, X_n))\} \\
&= \bar{\Pi}\left\{q_i\left(\sum_{k=1}^{r} h_k f_k\right)\right\} \\
&= \bar{\Pi}\left\{\sum_{k=1}^{r} q_i(h_k f_k)\right\} \\
&= \bar{\Pi}\left\{\sum_{k=1}^{r} \sum_{\lambda_1+\lambda_2=i} q_{\lambda_1}(h_k) q_{\lambda_2}(f_k)\right\} \\
&= \sum_{k=1}^{r} \sum_{\lambda_1+\lambda_2=i} \bar{\Pi} q_{\lambda_1}(h_k) \bar{\Pi} q_{\lambda_2}(f_k) = 0
\end{aligned}$$

since by hypothesis, the $\bar{u}_{ij}$ solve $q_i(f_k) = 0$ for all $k = 1, \ldots, r$ and all $i = 1, 2, \ldots, \infty$. Note

$$\bar{\Pi}_{q_0}(f_k) = \bar{\Pi} f_k = f_k(x_1, \ldots, x_n) = 0.$$

Hence the δ_i are all well defined. It is now clear that $\{\delta_i\}$ form a k-higher derivation of $k[x_1, \ldots, x_n]$ into Σ'. By [**1**, Lemma 2], we may uniquely extend the $\{\delta_i\}$ to a higher derivation $D = \{\delta_i\} \in H_k(\Sigma', \Sigma')$. Finally, we note that $\delta_i(x_j) = \bar{u}_{ij}$ by construction.

We note that the equations which appear in (3) have the following form: For fixed $i = 1, 2, \ldots, \infty$

$$q_i(f_k) = 0 \quad (k = 1, \ldots, r)$$

can be written as

$$\sum_{j=1}^{n} A_{ikj} u_{ij} + B_k = 0 \qquad (k = 1, \ldots, r),$$

with $A_{ikj}, B_k \in k[X_1, \ldots, X_n][u_{lj} | l = 1, \ldots, i-1 \; j = 1, \ldots, n]$. Hence for each i the equations are linear in the $u_{ij} (j = 1, \ldots, n)$. We can now prove the main result.

THEOREM 4. *Let $O = k[x_1, \ldots, x_n]$ be a finitely generated extension of the field k and p a non-minimal prime ideal in O such that O_p is regular. Then pO_p is not differential under $H_k(O_p, O_p)$.*

Proof. Set $n = \{x \in O | rx = 0,\ r \notin p\}$ and form O/n. Since O_p is a regular local ring, it is an integral domain. Thus O/n is an integral domain. Set $O/n = k[\bar{x}_1, \ldots, \bar{x}_n]$ and let Σ' denote the quotient field of O/n. Then we have $O/n \subset O_p \subset \Sigma' = k(\bar{x}_1, \ldots, \bar{x}_n)$. Let $\hat{O}_p$ denote the completion of O_p. Then $\hat{O}_p$ is a complete regular local ring of equal characteristic. Hence $\hat{O}_p$ has the form $K[[t_1, \ldots, t_r]]$ where K is the residue class field of O_p and $t_1, \ldots, t_r$ are a system of parameters of pO_p. The elements $t_1, \ldots, t_r$ are analytically independent over K and K contains an isomorphic copy of k. We may assume that $K \supset k$ without loss of generality.

Since the elements $t_1, \ldots, t_r$ are analytically independent over K, we may easily construct higher derivations on $\hat{O}_p$. Thus there exists a $D = \{\delta_i\} \in H(\hat{O}_p, \hat{O}_p)$ such that: (a) $\delta_i(K) = 0$ for $i \geqq 1$; (b) $\delta_1(t_1) = 1$ and $\delta_1(t_i) = 0$ for $i = 2, \ldots, r$; (c) $\delta_j(t_i) = 0$ for all $j > 1$ and for all $i = 1, \ldots, r$. To construct such a higher derivation, one uses [**2**, Proposition 2] and [**1**, Lemma 2 and Proposition 2]. Without any loss of generality, we may assume that the t_i lie in O/n and that $t_1 = \bar{x}_1$. We shall now construct a k-higher derivation $D' = \{\delta_i'\}$ on O_p such that $\delta_i'(\bar{x}_1) = 1$.

Let $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$ be the generators of the kernel of the mapping $k[X] \rightarrow k[\bar{x}]$. Then by Lemma 3, to construct any k-higher derivation of O/n into Σ', we must find a collection of elements $\{\bar{u}_{ij} \in \Sigma' | j = 1, \ldots, n, i = 1, \ldots, \infty\}$ which solve the equations

$$q_i(f_q) = 0 \ (q = 1, \ldots, m, i = 1, \ldots, \infty).$$

If $\bar{u}_{ij} \in O_p$ for all i and j, then $D' \in H_k(O/n, O_p)$, i.e., D' will be a k-higher derivation of O/n into O_p. We may then extend D' by the usual localization technique to a k-higher derivation on O_p. Thus to construct a k-higher derivation D' on O_p such that $\delta_1'(\bar{x}_1) = 1$, we must find elements $\{\bar{u}_{ij} \in O_p | j = 1, \ldots, n, i = 1, \ldots, \infty\}$ which solve $q_i(f_q) = 0$ and have $\bar{u}_{11} = 1$. We shall show that for any fixed i there exist elements $\{\bar{u}_{lj} | l = 1, \ldots, i, j = 1, \ldots, n\}$ in O_p which solve

$$q_l(f_q) = 0 \ (l = 1, \ldots, i, q = 1, \ldots, m)$$

and have $\bar{u}_{11} = 1$.

We proceed by induction on i. If $i = 1$, we seek a solution to $q_1(f_q) = 0$. As noted after Lemma 3, $q_1(f_q) = 0$ is a system of m linear equations in u_{lj}. Now this linear system has a solution with $\bar{u}_{11} = 1$ in $\hat{O}_p$. In [**4**, Lemma, p. 39], A. Seidenberg showed that if a system of linear equations with coefficients in O_p has a solution in $\hat{O}_p$, then the system has a solution in O_p. Hence it follows that there exist elements $\bar{u}_{lj} \in O_p$ which solve $q_1(f_q) = 0$. We may still take $\bar{u}_{11} = 1$. Let us assume we have constructed $\{\bar{u}_{lj} \in O_p | l = 1, \ldots, N < i; j = 1, \ldots, n\}$ which solve $q_l(f_q) = 0$ for $l = 1, \ldots, N$ and have $\bar{u}_{11} = 1$. Then we have k-linear maps $1, \delta_1, \ldots, \delta_N : O/n \rightarrow O_p$ which form a higher derivation of rank N on O/n. We may then extend these maps to a higher derivation $\{\hat{\delta}_0, \hat{\delta}_1, \ldots, \hat{\delta}_N\}$ of rank N on $\hat{O}_p$. Since $\hat{O}_p = K[[t_1, \ldots, t_r]]$, one easily sees that $\hat{\delta}_0, \hat{\delta}_1, \ldots, \hat{\delta}_N$ can be imbedded as the first $N + 1$ terms in a k-higher derivation on $\hat{O}_p$. In particular, there exists a k-linear map $\hat{\delta}_{N+1}$ on $\hat{O}_p$ such that $1, \hat{\delta}_1, \ldots, \hat{\delta}_{N+1}$ forms a higher derivation of rank $N + 1$.

Now if we substitute $\bar{u}_{lj}$ for u_{lj}, $l = 1, \ldots, N$ and $j = 1, \ldots, n$ into the system of equations $q_{N+1}(f_q) = 0$, we obtain a system of linear equations in $u_{N+1,j}$ with coefficients in O_p. Since $\hat{\delta}_{N+1}$ exists on $\hat{O}_p$, this system has a solution in $\hat{O}_p$. Hence it has a solution $\bar{u}_{N+1,j} \in O_p$. Thus for each i there exist elements $\{\bar{u}_{lj} \in O_p | l = 1, \ldots, i,\ j = 1, \ldots, n\}$ which solve $q_i(f_q) = 0$ and have $\bar{u}_{11} = 1$.

It now easily follows that there exist elements $\{\bar{u}_{ij} \in O_p | i = 1, \ldots, \infty, j = 1, \ldots, n\}$ which solve $q_i(f_q) = 0$ for all i and have $\bar{u}_{11} = 1$. Thus there exists a k-higher derivation $\{\delta_i\} \in H_k(O_p, O_p)$ such that $\delta_1(\bar{x}_1) = 1$. Hence $\delta_1(pO_p) \not\subset pO_p$ and pO_p is not differential.

For examples of ideals which are differential, we have the following result:

THEOREM 5. *Let O be a ring containing a field of characteristic zero. If an ideal A in O is differential under* Der(O), *then A is differential under $H(O, O)$.*

Proof. Let $D = \{\delta_i\} \in H(O, O)$. Then by [**2**, Theorem 5] each δ_i has the form

$$\sum \left\{\frac{d_{j_1} \ldots d_{j_r}}{r!} \,|\, j_1 + \ldots + j_r = i\right\}$$

with $\{d_i\}$ a sequence of derivations on O. Hence if A is differential under Der(O), A is differential under $H(O, O)$.

COROLLARY. *Let V be an irreducible affine variety over a field k of characteristic zero. Let $k[x]$ denote the coordinate ring of V. Let p be the prime ideal in $k[x]$ of a component of the singular locus of V. Set $O = k[x]$. Then pO_p is differential under $H(O_p, O_p)$.*

Proof. The result follows from Theorem 5 and [**4**, Theorem 5].

II. Zariski's Lemma. In this section, we shall give a generalization of Zariski's lemma. We noted in the introduction that the lemma does not permit a straightforward generalization to the characteristic $q \neq 0$ case. That is, if O is a complete local ring containing a field k of characteristic $q \neq 0$ and $\delta \in$ Der(O) such that $\delta(x)$ is a unit for some $x \in m$, the maximal ideal of O, then there may be no subring $O_1 \subset O$ such that $O_1[[x]] = O$, $\delta(O_1) = 0$, and x is analytically independent over O_1. To see this, consider the following example:

Example 2. We use the same example as in Example 1. Let $k[x, y] = k[X, Y]/(Y^q - X^q - X^{q+1})$. Let O be the localization of $k[x, y]$ at the origin (0) and let $\hat{O}$ be the completion of O. As pointed out in Example 1, there exists a derivation $\delta \in$ Der(O) such that $\delta(x) = 0$ and $\delta(y) = 1$. Extend δ by the usual techniques to a derivation on $\hat{O}$. Then δ takes the element y in the maximal ideal of $\hat{O}$ into 1. We shall now show that there exists no subring O_1 of $\hat{O}$ such that y is analytically independent over O_1, $\delta(O_1) = 0$ and $O_1[[y]] = \hat{O}$.

Suppose such a subring O_1 existed in $\hat{O}$. Then $x = \sum \alpha_i y^i$ with $\alpha_i \in O_1$. Therefore $0 = \delta(x) = \sum i\alpha_i y^{i-1}$. Since O_1 is a subring of $\hat{O}$, $i \in O_1$. Since y is analytically independent over O_1, we must have $i\alpha_i = 0$ for $i = 1, 2, \ldots$. Thus $\alpha_i = 0$ if $i \not\equiv 0(q)$. Hence

$$x = \sum_{n=0}^{\infty} \alpha_{nq} y^{nq}$$

with $\alpha_{nq} \in O_1$. Now $y^q = x^q + x^{q+1}$. Hence

$$y^q = \left\{\sum_{n=0}^{\infty} \alpha_{nq}{}^q y^{nq^2}\right\} + \left\{\sum_{n=0}^{\infty} \alpha_{nq}{}^q y^{nq^2}\right\}\left\{\sum_{n=0}^{\infty} \alpha_{nq} y^{nq}\right\}.$$

Using the fact that y is analytically independent over O_1 again, we get $\alpha_0{}^q\alpha_q = 1$. Hence α_0 is a unit in $\hat{O}$. But

$$\alpha_0 = x - \sum_{n=1}^{\infty} \alpha_{nq} y^{nq} \in (x, y)\hat{O}.$$

Since $(x, y)\hat{O}$ is the maximal ideal of $\hat{O}$, we reach the desired contradiction.

Thus the natural extension of Zariski's lemma to the characteristic $q \neq 0$ case is false. However, if we replace $\delta \in \mathrm{Der}(O)$ by $D = \{\delta_i\} \in H(O, O)$ we can obtain some partial results.

THEOREM 6. *Let O be a complete local ring with maximal ideal m. Let $x \in m$ and $D = \{\delta_i\} \in H(O, O)$ such that $\delta_1(x)$ is a unit in O and $\delta_i(x) = 0$ for all $i > 1$. Then there exists a subring O_1 of O such that:* (a) *O_1 is a complete local ring;* (b) *x is analytically independent over O_1; and* (c) $O = O_1[[x]]$.

Proof. If $\delta_1(x) = \epsilon^{-1}$, then $\{\epsilon^i\delta_i\}$ is a higher derivation on O such that $\epsilon\delta_1(x) = 1$ and $\epsilon^i\delta_i(x) = 0$ for $i > 1$. Hence we may assume that $\delta_1(x) = 1$. Set $\tau_x = \sum(-1)^i x^i \delta_i$, i.e., τ_x is a ring endomorphism on O given by $\tau_x(\alpha) = \sum(-1)^i x^i \delta_i(\alpha)$. Let $O_1 = \tau_x(O)$. Since τ_x is a ring homomorphism, O_1 is a complete local ring contained in O.

Now $\tau_x(x) = 0$. Therefore $x \in \ker \tau_x$. If $y \in \ker \tau_x \cap O_1$, then there exists a $z \in O$ such that $y = \tau_x(z) = z - x\delta_1(z) + \ldots = z - xl$ for some $l \in O$. Thus $z = y + xl \in \ker \tau_x$. Therefore $0 = \tau_x(z) = y$. Hence $\ker \tau_x \cap O_1 = (0)$.

We now show that x is analytically independent over O_1. Since x is not zero, x is not in O_1. Suppose

$$\sum_{i=0}^{\infty} \alpha_i x^i = 0$$

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for some coefficients α_i in O_1. Then $\alpha_0 \in O_1 \cap \ker \tau_x$. Thus $\alpha_0 = 0$. Assume we have shown that $\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0$. Then we have

$$*:0 = \alpha_{n+1}x^{n+1} + \alpha_{n+2}x^{n+2} + \ldots .$$

By induction on k, one can easily show that

$$\delta_k(x^k) \equiv 1(x) \ (k \geqq 1)$$

and

$$\delta_i(x^k) = 0(x) \ (i < k).$$

Thus applying δ_{n+1} to $*$, we get $0 = \alpha_{n+1} + z_1'$ for some element $z_1' \in Ox \subset \ker \tau_x$. Therefore $\alpha_{n+1} = 0$. Thus x is analytically independent over O_1.

Finally if $y \in O$, then $\tau_x(y) = y - x\delta_1(y) + x^2\delta_2(y) - \ldots$. So $y = \tau_x(y) - x\delta_1(y) - \ldots$. But $\tau_x(\delta_1(y)) = \delta_1(y) - x\delta_1^2(y) + \ldots$. So

$$y = \tau_x(y) + \tau_x(\delta_1(y))x + (\delta_1^2(y) - \delta_2(y))x^2 + \ldots .$$

If we continue expanding in this manner, we obtain $y = \sum \alpha_i x^i$ with $\alpha_i \in O_1$. Hence $O_1[[x]] = O$.

COROLLARY 1. Zariski's original lemma.

Proof. If O is a complete local ring containing the rationals and $\delta \in \mathrm{Der}(O)$ such that $\delta(x)$ is a unit for some $x \in m$, then $\epsilon^{-1}\delta \in \mathrm{Der}(O)$ such that $\epsilon^{-1}\delta(x) = 1$. Then

$$\left\{\frac{(\epsilon^{-1}\delta)^i}{i!}\right\} \in H(O, O)$$

and satisfies the hypotheses in our theorem. The rest follows easily.

COROLLARY 2. *Let V be an affine algebraic variety over a field k. Let $k[x]$ denote the coordinate ring of V. If p is a prime ideal of $k[x]$ such that the local ring $O = \{k[x]\}_p$ admits a higher derivation $D = \{\delta_i\} \in H(O, O)$ such that $\delta_1(x) = 1$ and $\delta_i(x) = 0$, $i > 1$, for some $x \in pO_p$, then V is analytically a product along the subvariety $\mathscr{V}(p)$.*

Proof. We may extend D to the completion $\hat{O}_p$ and apply Theorem 6.

COROLLARY 3. *Let O be a complete local ring containing the rationals. Let m denote the maximal ideal of O. If m is not differential under $H(O, O)$, then there exists an element $x \in m$ and a subring O_1 of O such that x is analytically independent over O_1 and $O_1[[x]] = O$.*

Proof. If m is not differential under $H(O, O)$, there exists a higher derivation $D = \{\delta_i\} \in H(O, O)$ such that $\delta_j(m) \not\subset m$ for some $j \geqq 1$. By [**2**, Theorem 5],

$$\delta_j = \sum \left\{\frac{d_{i_1} \ldots d_{i_r}}{r!} \,\middle|\, i_1 + \ldots + i_r = j\right\},$$

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where $\{d_i\}$ is a sequence of derivations of O. Hence the result follows from Zariski's original lemma.

We note that Corollary 3 implies that if m is not differential under $H(O, O)$, then there exists an element $x \in m$ and a higher derivation $D = \{\delta_i\} \in H(O,O)$ such that $\delta_1(x) = 1$ and $\delta_i(x) = 0$, $i > 1$. For, we may define

$$\delta_i(x) = \begin{cases} 1, & i = 1 \\ 0, & i > 1, \end{cases}$$

and

$$\delta_i(O_1) = 0 \ (i \geqq 1).$$

Thus in the characteristic zero case, the hypotheses of Theorem 6 are not so restrictive as they look.

In Zariski's original lemma, if there exists an $x \in m$ with $\delta(x) \notin m$, then there exists a subring O_1 of O such that $O_1[[x]] = O$ and x is analytically independent over O_1. This suggests the following conjecture concerning higher derivations: Suppose O is a complete local ring and $D = \{\delta_i\} \in H(O, O)$. Let $x \in m$ such that $\delta_j(x) \notin m$ for some $j \geqq 1$. Then does there exist a subring O_1 of O such that $O_1[[x]] = O$ and x is analytically independent over O_1? We give an example which shows that no such subring O_1 need exist.

Example 3. Let k denote the prime field of characteristic 2. Let X and Y be indeterminates over k. By [**1**, Proposition 2], there exists a higher derivation $D = \{\delta_i\} \in H(k[X, Y], k[X, Y])$ such that

$$\delta_1(X) = \delta_1(Y) = 0$$

and

$$\delta_i(X) = \delta_i(Y) = 1 \ (i > 1).$$

A simple calculation shows that the principle ideal $(X^2 + Y^2)$ in $k[X, Y]$ is differential under D, i.e., $\delta_i(X^2 + Y^2) \subset (X^2 + Y^2)$ for all i. Thus D induces a higher derivation D' on $O = k[X, Y]/(X^2 + Y^2) = k[x, y]$. Let $p = (x, y)$. Then p is a prime ideal in O. Let $n = \{r \in O | rz = 0,\ z \notin p\}$. Then none of the elements x, y or $x + y$ is in n. Since n is differential under D', we may extend D' to a higher derivation on $O_p = (O/n)_{p/n}$ and then to a higher derivation on the completion $\hat{O}_p$. Thus we have a complete local ring $\hat{O}_p$ and a higher derivation $D' = \{\delta_i'\} \in H(\hat{O}_p, \hat{O}_p)$ such that $\delta_2'(x) = 1$. The example will be complete if we show that no subring O_1 of $\hat{O}_p$ exists such that x is analytically independent over O_1 and $O_1[[x]] = O$.

Suppose such a subring did exist. Then $y = \Sigma\alpha_i x^i$ with α_i in O_1. Therefore $\Sigma\alpha_i^2 x^{2i} = y^2$. But $y^2 = x^2$. Hence $0 = \alpha_0^2 + (1 + \alpha_1)^2 x^2 + \alpha_2^2 x^4 + \ldots$. Thus $\alpha_i = 0$ if $i \neq 1$ and $\alpha_1 = 1$. But this says $x = y$, which is a contradiction.

In conclusion, we note that Theorem 6 can also be proven under the slightly more general hypothesis that there exists an $x \in m$ such that $\delta_1(x) = 1$ and $\tau_x(x) = \sum (-1)^i \delta_i(x) x^i = 0$, when O is an integral domain.

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