NOTE ON THE UNIQUENESS OF THE GREEN'S FUNCTIONS ASSOCIATED WITH CERTAIN DIFFERENTIAL EQUATIONS

D. B. SEARS

CONDITIONS to be imposed on $q(x)$ which ensure the uniqueness of the Green's function associated with the linear second-order differential equation

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\}\, y = 0,$$

the range $0 \leqslant x < \infty$, and suitable boundary conditions, have been obtained recently by Hartman, Wintner and Titchmarsh. It will be shown in this paper that the methods used by these writers may be employed to yield more general theorems. Corresponding results are obtained for the analogous partial second-order differential equation, for which an uniqueness theorem has been obtained by Titchmarsh.

1. I consider first of all the equation

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\}y = 0 \tag{1.1}$$

over $(0, \infty)$, where $q(x)$ is supposed continuous for $0 \leqslant x < \infty$, and λ is a complex parameter independent of x. The Green's function associated with (1.1) and a homogeneous boundary condition at $x=0$ will be unique only if the differential equation is of limit-point type[1]; i.e., if, for any λ, and so for all[2] λ, (1.1) does not possess two linearly independent solutions of class $L^2(0, \infty)$. Instead of considering the Green's function directly, I shall, therefore, consider the number of linearly independent solutions of (1.1) which are of class $L^2(0, \infty)$. This is the procedure adopted by Hartman and Wintner [1], who also take $\lambda = 0$. To keep this paper self-contained I consider general values of λ. This does not unduly complicate the argument, and avoids appeal to the theory of Integral Equations on which Weyl's theorem, by virtue of which one particular value of λ may be chosen, is based. Solutions of (1.1) will, therefore, be functions of both x and λ, but dependence of the symbols on λ will not be shown explicitly. Primes denote differentiation partially with respect to x. Throughout A and K will denote constants, not necessarily the same at each appearance.

Received July 4, 1949.

[1]See e.g. [4]§§2.1, 2.9, or [6].

[2][6] Kap. II, Satz 5. For another proof, see [3] §10.

https://doi.org/10.4153/CJM-1950-029-9 Published online by Cambridge University Press

2. If $q(x)$ satisfies any one of the following conditions, then (1.1) cannot possess two independent solutions of L^2:

(i) $$q(x) > -Ax^2;$$

(ii) $$\int^x q(t)\,dt > -Ax^3;$$

(iii) $$q(x_2) - q(x_1) > A(x_1 - x_2)$$

for all x_1, x_2 sufficiently large, $x_1 < x_2$;

(iv) $q(x)$ is monotonic and

$$\int^\infty |q(x)|^{-\frac{1}{2}}\,dx = \infty;$$

(v) $q(x)$ is differentiable,

$$\int^\infty |q(x)|^{-\frac{1}{2}}\,dx = \infty, \qquad \overline{\lim_{x\to\infty}} \left|\frac{q'(x)}{\{q(x)\}^{3/2}}\right| < \infty.$$

Of these conditions, (i) was given by Titchmarsh [5] and by Hartman and Wintner [2], while the remainder are due to Hartman and Wintner, [1], [2]. Clearly (i) is a special case of (ii). It is easily seen that (iii) is a special case of (i), and so also of (ii).

3. All the above conditions, with the exception of (ii), are contained in the following result:

THEOREM 1. *Let* $q(x) \geqslant -Q(x)$ *where* $Q(x) > \delta > 0$, *and*

(3.1) $$\int^\infty \{Q(x)\}^{-\frac{1}{2}}\,dx = \infty.$$

In addition, either

(3.2) *$Q'(x)$ exists and $Q(x)$ is an integral, while*

$$\overline{\lim_{x\to\infty}} \left|\frac{Q'(x)}{\{Q(x)\}^{3/2}}\right| < \infty,$$

or

(3.3) *$Q(x)$ is monotonic and continuous.*

Then the differential equation (1.1) *cannot, for any real or complex value of* λ, *possess two linearly independent solutions of class* $L^2(0, \infty)$.

There is no loss of generality in assuming, when (3.2) holds, that $Q(x) > 1$, and when (3.3) holds, that $Q(x) \to \infty$.

If the theorem is false, then every solution of (1.1) is $L^2(0, \infty)$. If $\phi(x)$, $\theta(x)$ denote the solutions satisfying the boundary conditions

$$\phi(0) = 1,\ \phi'(0) = 0, \quad \theta(0) = 0,\ \theta'(0) = 1,$$

for all values of λ, then, for all $x \geqslant 0$,

$$\phi(x)\,\theta'(x) - \phi'(x)\,\theta(x) = 1,$$

and

$$\int^\infty \{Q(x)\}^{-\frac{1}{2}}\,dx = \int^\infty \{Q(x)\}^{-\frac{1}{2}}\,\{\phi(x)\,\theta'(x) - \phi'(x)\,\theta(x)\}\,dx$$
$$\leqslant \left\{\int^\infty |\phi(x)|^2 dx \int^\infty \frac{|\theta'(x)|^2}{Q(x)}\,dx\right\}^{\frac{1}{2}} + \left\{\int^\infty |\theta(x)|^2 dx \int^\infty \frac{|\phi'(x)|^2}{Q(x)}\,dx\right\}^{\frac{1}{2}}$$
$$\leqslant K\left\{\int^\infty \frac{|\theta'(x)|^2}{Q(x)}\,dx\right\}^{\frac{1}{2}} + K'\left\{\int^\infty \frac{|\phi'(x)|^2}{Q(x)}\,dx\right\}^{\frac{1}{2}}$$

for some positive K, K'. By (3.1) it will therefore be sufficient to show that for any solution $y(x)$ of (1.1) which is of class $L^2(0, \infty)$, the integral

$$\int^\infty \frac{|y'(x)|^2}{Q(x)}\,dx \tag{3.4}$$

is convergent. Alternately it will be sufficient to show that

$$\int^\infty \frac{|y'(x)|^2}{Q(2x)}\,dx \tag{3.5}$$

is convergent.

Accordingly let $\lambda = u + iv$ be fixed, and let $y(x)$ be any such solution of L^2. Let $\bar{y}(x)$ denote its conjugate. By (1.1),

$$\bar{y}(x)\,y''(x) + \{\lambda - q(x)\}\,|y(x)|^2 = 0,$$

and, taking the real part,

$$\frac{1}{2}\frac{d^2}{dx^2}|y(x)|^2 - |y'(x)|^2 + \{u - q(x)\}\,|y(x)|^2 = 0. \tag{3.6}$$

Assume first of all that (3.1) and (3.2) hold. For any $T>0$, it follows from (3.6) that

$$\int_0^T\left(1 - \frac{x}{T}\right)\{Q(x)\}^{-1}\left\{|y'(x)|^2 - \frac{1}{2}\frac{d^2}{dx^2}|y(x)|^2\right\}dx$$
$$= \int_0^T\left(1 - \frac{x}{T}\right)\{Q(x)\}^{-1}\{u - q(x)\}\,|y(x)|^2\,dx$$
$$\leqslant \int_0^T\left(1 - \frac{x}{T}\right)\{Q(x)\}^{-1}\{u + Q(x)\}\,|y(x)|^2\,dx$$
$$\leqslant (1 + |u|)\int_0^T |y(x)|^2\,dx < K$$

as $T \to \infty$, since $Q(x) > 1$. Also, writing $\mathbf{R}(z)$ for the real part of z,

$$\frac{1}{2}\int_0^T \left(1-\frac{x}{T}\right)\{Q(x)\}^{-1}\frac{d^2}{dx^2}|y(x)|^2\,dx$$

$$= -\frac{\mathbf{R}\{y(0)\,\bar{y}'(0)\}}{Q(0)} + \frac{1}{2}\int_0^T \left\{\frac{d}{dx}|y(x)|^2\right\}\left\{\left(1-\frac{x}{T}\right)\frac{Q'(x)}{Q^2(x)} + \frac{1}{TQ(x)}\right\}dx$$

$$= -\frac{\mathbf{R}\{y(0)\bar{y}'(0)\}}{Q(0)} + \mathbf{R}\int_0^T y(x)\bar{y}'(x)\left(1-\frac{x}{T}\right)\frac{Q'(x)}{Q^2(x)}\,dx$$

$$+\frac{|y(T)|^2}{2TQ(T)} - \frac{|y(0)|^2}{2TQ(0)} + \frac{1}{2T}\int_0^T |y(x)|^2\frac{Q'(x)}{Q^2(x)}\,dx.$$

Substituting this result in the above inequality, it follows that

$$\int_0^T \left(1-\frac{x}{T}\right)\{Q(x)\}^{-1}|y'(x)|^2\,dx$$

$$< K + \frac{|y(T)|^2}{2TQ(T)} + \int_0^T |y(x)y'(x)|\left(1-\frac{x}{T}\right)\frac{|Q'(x)|}{Q^2(x)}\,dx$$

$$+\frac{1}{2T}\int_0^T |y(x)|^2\frac{|Q'(x)|}{Q^2(x)}\,dx.$$

By the inequality $2|ab| \leqslant a^2 + b^2$, with $a^2 = |y'|^2Q^{-1}$ and $b^2 = |y|^2Q'^2Q^{-3}$,

$$\int_0^T |y(x)\,y'(x)|\left(1-\frac{x}{T}\right)\frac{|Q'(x)|}{Q^2(x)}\,dx$$

$$\leqslant \frac{1}{2}\int_0^T \left(1-\frac{x}{T}\right)\left\{\frac{|y'(x)|^2}{Q(x)} + \frac{|y(x)|^2\,[Q'(x)]^2}{Q^3(x)}\right\}dx;$$

and, since $Q(x) > 1$,

$$\int_0^T |y(x)|^2\frac{|Q'(x)|}{Q^2(x)}\,dx \leqslant \int_0^T |y(x)|^2\left(\frac{[Q'(x)]^2}{Q^3(x)}\right)^{\frac{1}{2}}dx$$

so that

$$\frac{1}{2}\int_0^T \left(1-\frac{x}{T}\right)\{Q(x)\}^{-1}|y'(x)|^2\,dx$$

$$< K + \frac{|y(T)|^2}{2TQ(T)} + \frac{1}{2}\int_0^T \left(1-\frac{x}{T}\right)\frac{|y(x)|^2\,[Q'(x)]^2}{Q^3(x)}\,dx$$

$$+\frac{1}{2T}\int_0^T |y(x)|^2\left(\frac{[Q'(x)]^2}{Q^3(x)}\right)^{\frac{1}{2}}dx$$

$$< K + \frac{|y(T)|^2}{2TQ(T)}$$

https://doi.org/10.4153/CJM-1950-029-9 Published online by Cambridge University Press

by (3.2). Since $|y(x)|^2$ is $L(0, \infty)$, a sequence of values of T, tending to infinity, may be found for which

$$|y(T)|^2 < 2KTQ(T),$$

and thus, as $T \to \infty$ through this sequence,

$$\int_0^T \left(1 - \frac{x}{T}\right) \{Q(x)\}^{-1} |y'(x)|^2 \, dx = O(1).$$

It follows that the integral (3.4) is convergent, and hence the first part of the theorem has been proved.

Now let (3.1) and (3.3) hold, and again let $y(x)$ be any solution of (1.1) which is of class $L^2(0, \infty)$. Integrating by parts,

$$\int_0^T \left(1 - \frac{x}{T}\right) |y'(x)|^2 \, dx$$

$$= \left[y(x)\, \bar{y}'(x) \left(1 - \frac{x}{T}\right)\right]_0^T - \int_0^T y(x) \left\{\left(1 - \frac{x}{T}\right) \bar{y}''(x) - \frac{1}{T} \bar{y}'(x)\right\} dx$$

$$= - y(0)\, \bar{y}'(0) - \int_0^T \left(1 - \frac{x}{T}\right) |y(x)|^2 \{q(x) - \bar{\lambda}\} \, dx + \frac{1}{T} \int_0^T y(x)\, \bar{y}'(x) \, dx.$$

Taking real parts,

$$\int_0^T \left(1 - \frac{x}{T}\right) |y'(x)|^2 \, dx = - \mathbf{R}\{y(0)\, \bar{y}'(0)\}$$

$$- \int_0^T \left(1 - \frac{x}{T}\right) \{q(x) - u\} \, |y(x)|^2 \, dx + \frac{1}{2T} \{|y(T)|^2 - |y(0)|^2\}.$$

Now replace T by t, multiply through by $2t$, and integrate with respect to t over $(0, T)$. Then

$$\int_0^T (T - x)^2 |y'(x)|^2 \, dx = - T^2 \mathbf{R}\{y(0)\, \bar{y}'(0)\} - T|y(0)|^2$$

$$- \int_0^T (T - x)^2 \{q(x) - u\} \, |y(x)|^2 \, dx + \int_0^T |y(x)|^2 \, dx.$$

It follows that

$$\int_0^T \left(1 - \frac{x}{T}\right)^2 |y'(x)|^2 \, dx < K + \int_0^T |y(x)|^2 Q(x) \, dx.$$

Also

$$\int_0^T \left(1 - \frac{x}{T}\right)^2 |y'(x|^2 \, dx \geqslant \int_0^{\frac{1}{2}T} \left(1 - \frac{x}{T}\right)^2 |y'(x)|^2 \, dx$$

$$\geqslant \frac{1}{4} \int_0^{\frac{1}{2}T} |y'(x)|^2 \, dx.$$

Hence, replacing T by $2T$,

$$\int_0^T |y'(x)|^2\,dx < K + 4\int_0^{2T} |y(x)|^2\,Q(x)\,dx. \tag{3.7}$$

It will be convenient to write

$$\xi(T) = K + 4\int_0^{2T} |y(x)|^2\,Q(x)\,dx,$$

$$\eta(T) = \int_0^T |y'(x)|^2\,dx,$$

so that (3.7) is the inequality $\eta(T) < \xi(T)$. Now

$$\int_0^T |y'(x)|^2 \{Q(2x)\}^{-1}\,dx = \int_0^T \{Q(2x)\}^{-1}\,d\eta(x)$$

$$= \frac{\eta(T)}{Q(2T)} + \int_0^T \eta(x)\,d\left\{-\frac{1}{Q(2x)}\right\}.$$

Since $Q(x)$ is non-decreasing, it follows from (3.7) that

$$\int_0^T |y'(x)|^2 \{Q(2x)\}^{-1}\,dx < \frac{\xi(T)}{Q(2T)} + \int_0^T \xi(x)\,d\left\{-\frac{1}{Q(2x)}\right\}$$

$$< \frac{K}{Q(0)} + \int_0^T \frac{d\xi(x)}{Q(2x)}$$

$$< K + 8\int_0^T |y(2x)|^2\,dx < K$$

as $T \to \infty$, so that the integral (3.5) is convergent. This completes the proof of the theorem.

4. A similar theorem may be proved for the partial differential equation

$$\nabla^2 w + \{\lambda - q(x, y)\}\,w = 0 \tag{4.1}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, and the region to be considered is the whole (x, y) plane. Polar coordinates will be written (r, θ), and denoted in the functional symbols by square brackets, thus $q(x, y) = q[r, \theta]$. With this notation it has been shown by Titchmarsh [5] that the Green's function is unique when $q(x, y) > -Ar^2 - B$, where A, B are constants. This is the analogue of condition (i) of §2, and is the only result known for the equation (4.1).

For the partial differential equation no analogues are known for the limit-point and limit-circle cases of (1.1), so now the theorem must be formulated in terms of the Green's function. For the properties of this function and for certain theorems in two-dimensional analysis required for the proof, I refer to

[5, §4]. It will be assumed that $q(x, y)$ is continuous and has continuous partial derivatives of the first order, and that $\lambda = u + iv$, $v \neq 0$. Then the Green's function $G(x, y, \xi, \eta, \lambda)$ exists, satisfies (4.1) except at $x = \xi$, $y = \eta$, and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x, y, \xi, \eta, \lambda)|^2 \, dx \, dy$$

is convergent.

THEOREM 2. *Let* $q(x, y) \geqslant -Q(r)$, *where* $Q(r) > \delta > 0$, *and*

(4.2) $$\int^{\infty} \{Q(r)\}^{-\frac{1}{2}} dr = \infty.$$

In addition, either

(4.3) $Q'(r)$ *is continuous and*

$$\overline{\lim_{r \to \infty}} \left| \frac{Q'(r)}{\{Q(r)\}^{3/2}} \right| < \infty,$$

or

(4.4) $Q(r)$ *is monotonic and continuous, for* $0 < r < \infty$.

Then, for any λ *which is not real, no solution* $g(x, y, \lambda)$ *of* (4.1) *which has bounded second-order partial derivatives* $\frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial y^2}$ *can be* L^2 *over the whole plane. In particular, the Green's function is unique.*

If there exist two Green's functions, let $g(x, y) = g(x, y, \lambda)$ denote their difference, where now g is L^2 over the whole plane. Alternatively let g denote any solution of L^2 with the specified properties. For either case,

(4.5) $$\iint_{r \leqslant R} |g(x, y)|^2 \, dx \, dy = \frac{1}{v} \mathrm{I} \int_0^{2\pi} g[R, \theta] \, \bar{g}_R \, [R, \theta] \, R \, d\theta$$

for any $R > 0$, $\bar{g}$ being the conjugate of g, and suffixes denoting partial differentiation [5, p. 196].

It must be shown that $g(x, y)$ is identically zero. Let $f(x)$ denote (temporarily) a positive function, integrable over any finite interval $(0, T)$, and

$$F(T) = \int_0^T f(x) \, dx.$$

Multiply (4.5) by $f(R)$, and integrate over $(0, T)$. Then

$$\iint_{r \leqslant T} \{F(T) - F(r)\} \, |g(x, y)|^2 \, dx \, dy = \frac{1}{v} \mathrm{I} \iint_{r \leqslant T} g[r, \theta] \bar{g}_r \, [r, \theta] r f(r) dr \, d\theta,$$

R being now replaced by r. It follows that, as $T \to \infty$,

$$\iint_{r \leqslant T} \left\{1 - \frac{F(r)}{F(T)}\right\} |g(x, y)|^2 \, dx \, dy \tag{4.6}$$
$$= O\left\{\frac{1}{F^2(T)} \iint_{r \leqslant T} r|g\,[r, \theta]|^2 \, dr \, d\theta \iint_{r \leqslant T} rf^2(r)|g_r[r, \theta]|^2 dr \, d\theta\right\}^{\frac{1}{2}}$$
$$= O\left\{\frac{1}{F^2(T)} \iint_{r \leqslant T} rf^2(r)|g_r\,[r, \theta]|^2 \, dr \, d\theta\right\}^{\frac{1}{2}}.$$

Now if $g(x, y)$ is not identically zero, constants K, R_0 may be found such that, for all $R > R_0$,

$$\iint_{r \leqslant R} |g(x, y)|^2 \, dx \, dy \geqslant K > 0.$$

Multiply by $f(R)$, and integrate with respect to R between R_0 and T, where $T > R_0$. Then

$$K\left\{F(T) - F(R_0)\right\} \leqslant \int_{R_0}^{T} f(R) \, dR \iint_{r \leqslant R} |g\,[r, \theta]|^2 \, rdr \, d\theta$$
$$\leqslant \int_{0}^{T} f(R) \, dR \iint_{r \leqslant R} |g\,[r, \theta]|^2 \, rdr \, d\theta$$
$$\leqslant \iint_{r \leqslant T} \left\{F(T) - F(r)\right\} |g\,[r, \theta]|^2 \, rdr \, d\theta.$$

Hence

$$\iint_{r \leqslant T} \left\{1 - \frac{F(r)}{F(T)}\right\} |g(x, y)|^2 \, dx \, dy \geqslant K\left\{1 - \frac{F(R_0)}{F(T)}\right\}. \tag{4.7}$$

If a function $f(x)$ may be found such that $F(T) \rightarrow \infty$ as $T \rightarrow \infty$, and

$$\iint_{r \leqslant T} rf^2(r) \, |g_r[r, \theta]|^2 \, dr \, d\theta = O(1),$$

then (4.6), (4.7) lead to a contradiction, so that g must vanish identically. Actually $f(x)$ will be chosen as either $\{Q(x)\}^{-\frac{1}{2}}$ or $\{Q(2x)\}^{-\frac{1}{2}}$, so that, in order to prove the theorem, it will be sufficient to show that either

$$\iint_{r \leqslant T} r\left\{Q(r)\right\}^{-1} |g_r[r, \theta]|^2 \, dr \, d\theta = O(1), \tag{4.8}$$

or

$$\iint_{r \leqslant T} r\left\{Q(2r)\right\}^{-1} |g_r[r, \theta]|^2 \, dr \, d\theta = O(1). \tag{4.9}$$

One further preliminary result will be required. Let $\phi(x)$ denote a real function with continuous first derivative in any finite interval $(0, T)$. Then, proceeding as in [5, p.197] and omitting for brevity the arguments (x, y),

$$\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)\phi(r)\,g\,\nabla^2\bar{g}\,dx\,dy$$

$$= -\iint\limits_{r\leqslant T}\left[\frac{\partial\bar{g}}{\partial x}\frac{\partial}{\partial x}\left\{\left(1-\frac{r}{T}\right)\phi(r)g\right\}+\frac{\partial\bar{g}}{\partial y}\frac{\partial}{\partial y}\left\{\left(1-\frac{r}{T}\right)\phi(r)g\right\}\right]dx\,dy$$

$$= -\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)\phi(r)\,(|g_x|^2+|g_y|^2)\,dx\,dy+\frac{1}{T}\iint\limits_{r\leqslant T}r\phi(r)g[r,\theta]\bar{g}_r[r,\theta]\,dr\,d\theta$$

$$-\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)r\,\phi'(r)\,g[r,\theta]\,\bar{g}_r[r,\theta]\,dr\,d\theta.$$

Substituting from (4.1), and taking real parts,

$$\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)\phi(r)\,(|g_x|^2+|g_y|^2)\,dx\,dy$$

(4.10)
$$= -\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)\phi(r)\{q(x,y)-u\}\,|g(x,y)|^2\,dx\,dy$$
$$+\frac{1}{2T}\iint\limits_{r\leqslant T}r\,\phi(r)\{g[r,\theta]\bar{g}_r[r,\theta]+\bar{g}[r,\theta]g_r[r,\theta]\}\,dr\,d\theta$$
$$-\frac{1}{2}\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)r\,\phi'(r)\{g[r,\theta]\,\bar{g}_r[r,\theta]+\bar{g}[r,\theta]\,g_r[r,\theta]\}\,dr\,d\theta.$$

Assume now that $Q(r)$ satisfies (4.2) and (4.3). As before, there is no loss of generality in taking $Q(r) > 1$. Then

$$\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)r\{Q(r)\}^{-1}\,|g_r[r,\theta]|^2\,dr\,d\theta$$
$$\leqslant \iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)r\,\{Q(r)\}^{-1}\{|g_r[r,\theta]|^2+r^{-2}|g_\theta[r,\theta]|^2\}\,dr\,d\theta$$
$$\leqslant \iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)\{Q(r)\}^{-1}\,\{|g_x|^2+|g_y|^2\}\,dx\,dy.$$

By (4.10) with $\phi(x) = \{Q(x)\}^{-1}$, this cannot exceed

$$\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)\{Q(r)\}^{-1}\,\{Q(r)+u\}\,|g(x,y)|^2\,dx\,dy$$
$$+\frac{1}{2T}\iint\limits_{r\leqslant T}r\,\{Q(r)\}^{-1}\{g\bar{g}_r+\bar{g}g_r\}\,dr\,d\theta$$
$$+\frac{1}{2}\iint\limits_{r\leqslant T}\left(1-\frac{r}{T}\right)r\,\frac{Q'(r)}{Q^2(r)}\,\{g\bar{g}_r+\bar{g}g_r\}\,dr\,d\theta.$$

Since g is L^2 and $Q > 1$, this, in turn, cannot exceed

$$K + \frac{1}{2T}\iint_{r \leqslant T} r\,\{Q(r)\}^{-1}\left\{\frac{\partial}{\partial r}|g[r,\theta]|^2\right\} dr\,d\theta$$
$$+\frac{1}{2}\iint_{r \leqslant T} r\left(1-\frac{r}{T}\right)\frac{Q'(r)}{Q^2(r)}\{g\bar{g}_r + \bar{g}g_r\}\,dr\,d\theta.$$

The first of these integrals is now integrated by parts, and to the second is applied the inequality $2|ab| \leqslant a^2 + b^2$, with $a^2 = |g_r|^2Q^{-1}$ and $b^2 = |g|^2Q'^2Q^{-3}$. Then

$$\iint_{r \leqslant T} r\left(1-\frac{r}{T}\right)\{Q(r)\}^{-1}|g_r[r,\theta]|^2\,dr\,d\theta$$
$$< K + \frac{1}{2T}\int_0^{2\pi}\left[\frac{r}{Q(r)}|g[r,\theta]|^2\right]_0^T d\theta - \frac{1}{2T}\iint_{r \leqslant T}|g[r,\theta]|^2\left\{\frac{1}{Q(r)} - \frac{rQ'(r)}{Q^2(r)}\right\} dr\,d\theta$$
$$+\frac{1}{2}\iint_{r \leqslant T} r\left(1-\frac{r}{T}\right)\left\{\frac{|g_r[r,\theta]|^2}{Q(r)} + \frac{|g[r,\theta]|^2[Q'(r)]^2}{Q^3(r)}\right\} dr\,d\theta.$$

It follows from (4.3) that

$$\iint_{r \leqslant T} r\left(1-\frac{r}{T}\right)\{Q(r)\}^{-1}|g_r[r,\theta]|^2\,dr\,d\theta$$
$$< K + \frac{1}{Q(T)}\int_0^{2\pi}|g[T,\theta]|^2\,d\theta.$$

Omitting the $Q(T)$ appearing on the right, replace T by R, multiply by $2R$ and integrate again with respect to R over $(0, T)$. Then

$$\iint_{r \leqslant T} r\,(T-r)^2\{Q(r)\}^{-1}|g_r[r,\theta]|^2\,dr\,d\theta$$
$$< KT^2 + 2\iint_{r \leqslant T} r\,|g[r,\theta]|^2\,dr\,d\theta,$$

and hence, as $T \to \infty$,

$$\iint_{r \leqslant T} r\left(1-\frac{r}{T}\right)^2\{Q(r)\}^{-1}|g_r[r,\theta]|^2\,dr\,d\theta = O(1).$$

Thus (4.8) holds, and the theorem is true in this case.

Now assume the conditions (4.2) and (4.4), and let $Q(r) \to \infty$ as $r \to \infty$. Taking $\phi(r) = 1$ in (4.10), the last term vanishes, so that, on integrating the second term on the right-hand side by parts,

$$\iint\limits_{r \leqslant T} \left(1 - \frac{r}{T}\right) (|g_x|^2 + |g_y|^2)\, dx\, dy \leqslant \iint\limits_{r \leqslant T} \left(1 - \frac{r}{T}\right) \{Q(r) + u\}\, |g|^2\, dx\, dy$$

$$+ \frac{1}{2} \int_0^{2\pi} |g[T, \theta]|^2\, d\theta - \frac{1}{2T} \int_0^{2\pi} \int_0^T |g[r, \theta]|^2\, dr\, d\theta$$

$$\leqslant K + \iint\limits_{r \leqslant T} \left(1 - \frac{r}{T}\right) Q(r) |g|^2\, dx\, dy + \frac{1}{2} \int_0^{2\pi} |g[T, \theta]|^2\, d\theta.$$

Now replace T by R, multiply by $2R$, and integrate again. Then it follows immediately that

$$\iint\limits_{r \leqslant T} \left(1 - \frac{r}{T}\right)^2 (|g_x|^2 + |g_y|^2)\, dx\, dy < K + \iint\limits_{r \leqslant T} Q(r) |g|^2\, dx\, dy.$$

Writing

$$\xi(T) = K + \iint\limits_{r \leqslant 2T} Q(r)\, |g|^2\, dx\, dy,$$

$$\eta(T) = \tfrac{1}{4} \iint\limits_{r \leqslant T} r\, |g_r[r, \theta]|^2\, dr\, d\theta,$$

it follows that

$$\eta(T) \leqslant \tfrac{1}{4} \iint\limits_{r \leqslant T} (|g_x|^2 + |g_y|^2)\, dx\, dy$$

$$\leqslant \iint\limits_{r \leqslant 2T} \left(1 - \frac{r}{2T}\right)^2 (|g_x|^2 + |g_y|^2)\, dx\, dy < \xi(T).$$

Now, since $Q(r)$ is non-decreasing,

$$\tfrac{1}{4} \iint\limits_{r \leqslant T} \{Q(2r)\}^{-1}\, r\, |g_r[r, \theta]|^2\, dr\, d\theta$$

$$= \tfrac{1}{4} \int_0^T \{Q(2r)\}^{-1} r\, dr \int_0^{2\pi} |g_r[r, \theta]|^2\, d\theta$$

$$= \int_0^T \frac{d\eta(r)}{Q(2r)}$$

$$= \frac{\eta(T)}{Q(2T)} + \int_0^T \eta(r)\ d\left(-\frac{1}{Q(2r)}\right)$$

$$< \frac{\xi(T)}{Q(2T)} + \int_0^T \xi(r)\ d\left(-\frac{1}{Q(2r)}\right)$$

$$< K + 4 \int_0^T r\, dr \int_0^{2\pi} |g\,[2r, \theta]|^2\, d\theta$$

$$< K + \iint\limits_{r \leqslant 2T} r |g[r, \theta]|^2\, dr\, d\theta.$$

REFERENCES

[1] P. Hartman and A. Wintner, *Criteria of non-degeneracy for the wave equation*, Amer. J. of Math., vol. 70 (1948), 295-308.

[2] ——— *A criterion for the non-degeneracy of the wave equation*, Amer. J. of Math., vol. 71 (1949), 206-213.

[3] E. C. Titchmarsh, *On expansions in eigenfunctions* (IV), Quart. J. of Math. (Oxford), vol. 12 (1941), 33-50.

[4] ——— *Eigenfunction expansions associated with second-order differential equations* (Oxford, 1946).

[5] ——— *On the uniqueness of the Green's function associated with a second-order differential equation*, Can. J. of Math., vol. 1 (1949), 191-198.

[6] H. Weyl, *Uber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehorigen Entwicklungen willkürlicher Funktionen*, Math. Ann. 68 (1910), 220-269.

University of Cape Town,
South Africa