Proceedings of the Edinburgh Mathematical Society (2013) **56**, 805–827
DOI:10.1017/S0013091513000370

SOLUTIONS AND MULTIPLE SOLUTIONS FOR SUPERLINEAR PERTURBATIONS OF THE PERIODIC SCALAR p-LAPLACIAN

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(Received 4 September 2011)

Abstract We consider a nonlinear periodic problem driven by the scalar p-Laplacian and with a reaction term which exhibits a $(p-1)$-superlinear growth near $\pm\infty$ but need not satisfy the Ambrosetti–Rabinowitz condition. Combining critical point theory with Morse theory we prove an existence theorem. Then, using variational methods together with truncation techniques, we prove a multiplicity theorem establishing the existence of at least five non-trivial solutions, with precise sign information for all of them (two positive solutions, two negative solutions and a nodal (sign changing) solution).

Keywords: scalar p-Laplacian; critical groups; mountain pass theorem; C condition; p-superlinearity; AR condition

2010 *Mathematics subject classification:* Primary 34B15; 34B18
Secondary 34C25; 58E05

1. Introduction

In this paper, we study the following nonlinear periodic problem driven by the scalar p-Laplacian:

$$\left.\begin{aligned} -(|u'(t)|^{p-2}u'(t))' = f(t,u(t)) \quad \text{almost everywhere (a.e.) on } T=[0,b],\\ u(0)=u(b), \quad u'(0)=u'(b), \quad 1<p<\infty. \end{aligned}\right\} \tag{1.1}$$

Here, $f\colon T\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory reaction, i.e. for all $x\in\mathbb{R}$, $t\to f(t,x)$ is measurable and, for almost all (a.a.) $t\in T$, $x\to f(t,x)$ is continuous.

The aim of this work is to prove existence and multiplicity results for (1.1) when the reaction $f(t,\cdot)$ exhibits $(p-1)$-superlinear growth but does not necessarily satisfy the well-known Ambrosetti–Rabinowitz (AR) condition, which is very common when studying 'superlinear' problems. We recall that the AR condition requires that there exist $\mu > p$ and $M > 0$ such that

$$0 < \mu F(t,x) \leqslant f(t,x)x \quad \text{for a.a. } t \in T, \text{ all } |x| \geqslant M, \tag{1.2}$$

where

$$F(t,x) = \int_0^x f(t,s)\,\mathrm{d}s$$

(see [**5**]). Integrating (1.2), we obtain the weaker condition

$$\hat{c}_0|x|^\mu \leqslant F(t,x) \quad \text{for a.a. } t \in T, \text{ all } |x| \geqslant M \text{ and some } \hat{c}_0 > 0. \tag{1.3}$$

This implies the much weaker condition

$$\lim_{x\to\pm\infty} \frac{F(t,x)}{|x|^p} = +\infty \quad \text{uniformly for a.a. } t \in T. \tag{1.4}$$

Evidently, (1.4) is implied by the condition

$$\lim_{x\to\pm\infty} \frac{f(t,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.a. } t \in T. \tag{1.5}$$

Condition (1.5) implies that for a.a. $t \in T$, $f(t,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$.

The AR condition ensures that the Palais–Smale sequences of the energy functional of (1.1) are bounded. Therefore, the energy functional satisfies the Palais–Smale condition and we can apply the minimax methods of critical point theory. However, the AR condition is rather restrictive and excludes many functions which exhibit slower growth near $\pm\infty$, as is evident from (1.3). For this reason, there have been efforts to replace (1.2) by a weaker condition. We refer the reader to the recent works of Miyagaki and Souto [**19**] and Li and Yang [**18**] for a discussion of the literature in this direction. In this paper, motivated by the aforementioned works, we employ a condition involving the quantity $\vartheta(t,x) = f(t,x)x - pF(t,x)$ (see Hypotheses (H) in § 3), which is more general than (1.2) and incorporates more reaction terms $f(t,x)$ in our framework.

Existence and multiplicity results for the periodic p-Laplacian can be found in the works of Aizicovici *et al*. [**1**, **2**], del Pino *et al*. [**11**], Gasiński and Papageorgiou [**15**], Jiang and Wang [**17**], Motreanu *et al*. [**20**], Papageorgiou and Papageorgiou [**22**] and Rynne [**24**]. Of these works, only [**15**] treats problems with a $(p-1)$-superlinear reaction. They prove the existence of three non-trivial solutions using a stronger 'superlinearity' condition near $\pm\infty$.

In this paper, combining variational methods based on the critical point theory with Morse theory, we prove an existence theorem and a multiplicity theorem. In the multiplicity theorem, we produce five non-trivial solutions and, in addition, we provide precise sign information for all of them. For both theorems, we assume a similar behaviour of $f(t,\cdot)$ near zero, namely we require that it grows $(p-1)$-linearity near zero.

In the next section, for the convenience of the reader, we recall some of the main mathematical tools which we use in this paper.

https://doi.org/10.1017/S0013091513000370 Published online by Cambridge University Press

2. Mathematical background

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that $x \in X$ is a critical point of φ if $\varphi'(x) = 0$. If $x \in X$ is a critical point of φ, then $c = \varphi(x)$ is a critical value of φ. We say that φ satisfies the C condition if the following is true.

Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|x_n\|)\varphi'(x_n) \to 0$ in X^* as $n \to \infty$ admits a strongly convergent subsequence.

Evidently, the C condition is more general than the well-known Palais–Smale condition. However, as was shown by Bartolo *et al*. [**6**] (see also [**21**]), it suffices to have the minimax theorems of critical point theory. In particular, we have the following slightly more general version of the mountain pass theorem (see [**5**]).

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies the C condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > \varrho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf[\varphi(x)\colon \|x - x_0\| = \varrho] = \eta_\varrho,$$
$$c = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1}\varphi(\gamma(t)), \quad \text{where } \Gamma = \{\gamma \in C([0,1], X)\colon \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geqslant \eta_\varrho$ and c is a critical value of φ.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following notation:

$$\begin{aligned}\varphi^c &= \{x \in X\colon \varphi(x) \leqslant c\},\\ K_\varphi &= \{x \in X\colon \varphi'(x) = 0\},\\ K_\varphi^c &= \{x \in K_\varphi\colon \varphi(x) = c\}.\end{aligned}$$

If $Y_2 \subseteq Y_1 \subseteq X$, then, for every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2), with integer coefficients. The critical groups of φ at an isolated critical point $x_0 \in X$, with $c = \varphi(x_0)$, are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}) \quad \text{for all } k \geqslant 0.$$

Here, U is a neighbourhood of x_0 such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighbourhood U of x_0.

Suppose that $\varphi \in C^1(X)$ satisfies the C condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geqslant 0.$$

The second deformation theorem (see, for example, [**21**, **23**]) implies that the above definition of critical groups at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. If φ satisfies the C condition, has a finite critical set K_φ and, for some $k \geqslant 0$, we have $C_k(\varphi, 0) \neq 0$ and $C_k(\varphi, \infty) = 0$, then φ has a non-trivial critical point (see [**23**]).

In the study of (1.1), we use the following two spaces:

$$W^{1,p}_{\mathrm{per}}(0,b) = \{u \in W^{1,p}(0,b) \colon u(0) = u(b)\}$$

and

$$\hat{C}^1(T) = C^1(T) \cap W^{1,p}_{\mathrm{per}}(0,b).$$

Recall that $W^{1,p}(0,b)$ is embedded continuously (in fact compactly) in $C(T)$, and so the evaluations at $t = 0$ and $t = b$ in the definition of $W^{1,p}_{\mathrm{per}}(0,b)$ make sense. The space $\hat{C}^1(T)$ is an ordered Banach space with positive cone

$$\hat{C}_+ = \{u \in \hat{C}^1(T) \colon u(t) \geqslant 0 \text{ for all } t \in T\}.$$

This cone has a non-empty interior given by

$$\operatorname{int} \hat{C}_+ = \{u \in \hat{C}_+ \colon u(t) > 0 \text{ for all } t \in T\}.$$

Next, we recall some facts about the spectrum of the negative periodic scalar p-Laplacian. So, we consider the nonlinear eigenvalue problem

$$-(|u'(t)|^{p-2}u'(t))' = \hat{\lambda}|u(t)|^{p-2}u(t) \quad \text{on } T = [0,b],\ u(0) = u(b),\ u'(0) = u'(b). \tag{2.1}$$

A number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of the negative periodic scalar p-Laplacian if (2.1) has a non-trivial solution, which is an eigenfunction corresponding to $\hat{\lambda}$. It is easy to see that a necessary condition for $\hat{\lambda} \in \mathbb{R}$ to be an eigenvalue is that $\hat{\lambda} \geqslant 0$. In fact, $\hat{\lambda}_0 = 0$ is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (i.e. the space of constant functions). Note that $\hat{\lambda}_0 = 0$ is the only eigenvalue with eigenfunctions of constant sign. All eigenvalues $\hat{\lambda} > 0$ have nodal (i.e. sign changing) eigenfunctions.

Let $\pi_p = 2\pi(p-1)^{1/p}/p\sin(\pi/p)$. Then, the sequence

$$\left\{\hat{\lambda}_n = \left(\frac{2n\pi_p}{b}\right)^p\right\}_{n\geqslant 0}$$

is the set of all eigenvalues for (2.1). If $p = 2$ (linear eigenvalue problem), then $\pi_2 = \pi$ and we have the well-known spectrum of the negative periodic scalar Laplacian, which is

$$\left\{\hat{\lambda}_n = \left(\frac{2n\pi}{b}\right)^2\right\}_{n\geqslant 0}.$$

If $u \in C^1(T)$ is an eigenfunction of (2.1), then $u(t) \neq 0$ a.e. on T and, in fact, the zero set of $u(\cdot)$ is finite. The L^p-normalized principal eigenfunction is denoted by $\hat{u}_0$ and $\hat{u}_0(t) = 1/b^{1/p}$ for all $t \in T$. The sequence of eigenvalues $\{\hat{\lambda}_n\}_{n\geqslant 0}$ can be obtained using the Ljusternik–Schnirelmann theory (see, for example, [**12**]). In this way, we have minimax characterizations of the eigenvalues. An alternative minimax expression for $\hat{\lambda}_1 > 0$ (the first non-trivial eigenvalue) is the following (see [**20**]).

Proposition 2.2. *If $\partial B_1^{L^p} = \{u \in L^p(T)\colon \|u\|_p = 1\}$, $M = W^{1,p}_{\mathrm{per}}(0,b) \cap \partial B_1^{L^p}$ and*

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M)\colon \hat{\gamma}(-1) = -\hat{u}_0,\ \hat{\gamma}(1) = \hat{u}_0\},$$

then

$$\hat{\lambda}_1 = \inf_{\hat{\gamma}\in\Gamma} \max_{-1\leqslant s\leqslant 1} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\hat{\gamma}(s) \right\|_p^p .$$

A detailed study of the spectrum of the negative periodic scalar p-Laplacian can be found in [**7**].

Let $A\colon W^{1,p}_{\mathrm{per}}(0,b) \to W^{1,p}_{\mathrm{per}}(0,b)^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_0^b |u'(t)|^{p-2}u'(t)y'(t)\,\mathrm{d}t \quad \text{for all } u, y \in W^{1,p}_{\mathrm{per}}(0,b). \tag{2.2}$$

The next proposition summarizes the properties of A (see, for example, [**2**]).

Proposition 2.3. *The nonlinear map $A\colon W^{1,p}_{\mathrm{per}}(0,b) \to W^{1,p}_{\mathrm{per}}(0,b)^*$ defined by (2.2) is continuous, bounded (i.e. maps bounded sets to bounded ones), strictly monotone (hence maximal monotone too) and of type $(S)_+$ (i.e. if $u_n \xrightarrow{w} u$ in $W^{1,p}_{\mathrm{per}}(0,b)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W^{1,p}_{\mathrm{per}}(0,b)$).*

In what follows, by $\|\cdot\|$ we denote the standard norm of $W^{1,p}_{\mathrm{per}}(0,b)$. Moreover, for $u \in W^{1,p}_{\mathrm{per}}(0,b)$, we set $u^{\pm} = \max\{\pm u, 0\}$. Recall that $u = u^+ - u^-$ and $|u| = u^+ + u^-$. Finally, by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$.

3. The existence theorem

For the existence theorem, the hypotheses on the reaction term $f(t,x)$ are the following.

(H) $f\colon T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that, for a.a. $t \in T$, $f(t,0) = 0$ and the following hold.

(i) $|f(t,x)| \leqslant \alpha(t)(1 + |x|^{r-1})$ for a.a $t \in T$, all $x \in \mathbb{R}$, with $\alpha \in L^1(T)_+$, $p < r < \infty$.

(ii) If

$$F(t,x) = \int_0^x f(t,s)\,\mathrm{d}s,$$

then

$$\lim_{x\to\pm\infty} \frac{F(t,x)}{|x|^p} = +\infty \text{ uniformly for a.a. } t \in T,$$

and if $\vartheta(t,x) = f(t,x)x - pF(t,x)$, then there exists $\beta^* > 0$ such that

$$\vartheta(t,x) \leqslant \vartheta(t,y) + \beta^* \quad \text{for a.a. } t \in T,\ \text{all } 0 \leqslant x \leqslant y \text{ or } y \leqslant x \leqslant 0. \tag{3.1}$$

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(iii) One of the following alternatives holds:

(a) there exist $m \geqslant 0$ and $\eta, \hat{\eta} \in L^\infty(T)$ such that

$$\hat{\lambda}_m \leqslant \eta(t) \leqslant \hat{\eta}(t) \leqslant \hat{\lambda}_{m+1} \quad \text{a.e. on } T,\ \hat{\lambda}_m \neq \eta,\ \hat{\eta} \neq \hat{\lambda}_{m+1}$$

and

$$\eta(t) \leqslant \liminf_{x \to 0} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \limsup_{x \to 0} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \hat{\eta}(t)$$

uniformly for a.a. $t \in T$;

(b) there exists $\eta_0 \in L^\infty(T)$ such that $\eta_0(t) \leqslant 0$ a.e. on T, $\eta_0 \neq 0$ and

$$\limsup_{x \to 0} \frac{pF(t,x)}{|x|^p} \leqslant \eta_0(t) \quad \text{uniformly for a.a. } t \in T.$$

Remark 3.1. Hypothesis (H) (ii) classifies the problem as p-superlinear (the superlinearity condition is imposed on the potential function $F(t,x)$). However, we do not employ the AR condition. Instead we use (3.1), which allows us to consider functions with slower growth near $\pm\infty$, as the following example illustrates. Hypothesis (H) (iii) (both options) implies that asymptotically at zero we have non-uniform non-resonance with respect to any eigenvalue.

Example 3.2. The function

$$f(x) = |x|^{p-2}x(\ln(1+|x|) + \eta), \quad \text{with } \eta \in (\hat{\lambda}_m, \hat{\lambda}_{m+1}),$$

satisfies Hypotheses (H) (for the sake of simplicity we drop the t-dependence) for some $m \geqslant 0$ or $\eta < 0$.

Note that this $f(\cdot)$ does not satisfy the AR condition.

Let $\varphi\colon W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ be the energy functional for (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|u'\|_p^p - \int_0^b F(t,u(t))\,\mathrm{d}t \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b).$$

We know that $\varphi \in C^1(W^{1,p}_{\mathrm{per}}(0,b))$.

Proposition 3.3. *If Hypotheses (H) hold, then φ satisfies the C condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \quad \text{for some } M_1 > 0,\ \text{all } n \geqslant 1, \tag{3.2}$$

and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b)^* \text{ as } n \to \infty. \tag{3.3}$$

From (3.3) we have that

$$\left|\langle A(u_n), h\rangle - \int_0^b f(t,u_n)h\,\mathrm{d}t\right| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in W^{1,p}_{\mathrm{per}}(0,b), \tag{3.4}$$

with $\varepsilon_n \to 0^+$.

In (3.4), we choose $h = u_n \in W^{1,p}_{\mathrm{per}}(0,b)$ and we have that

$$-\|u_n'\|_p^p + \int_0^b f(t,u_n)u_n \,\mathrm{d}t \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1. \tag{3.5}$$

On the other hand, from (3.2) we have that

$$\|u_n'\|_p^p - \int_0^b pF(t,u_n)\,\mathrm{d}t \leqslant pM_1 \quad \text{for all } n \geqslant 1. \tag{3.6}$$

Adding (3.5) and (3.6), we obtain that

$$\int_0^b \vartheta(t,u_n)\,\mathrm{d}t = \int_0^b [f(t,u_n)u_n - pF(t,u_n)]\,\mathrm{d}t \leqslant M_2 \quad \text{for some } M_2 > 0, \text{ all } n \geqslant 1. \tag{3.7}$$

Claim 3.4. *$\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded.*

We argue indirectly. So, suppose that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is unbounded. By passing to a subsequence if necessary, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. We set $y_n = u_n/\|u_n\|$, $n \geqslant 1$. Then, $\|y_n\| = 1$ for all $n \geqslant 1$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \qquad \text{and} \qquad y_n \to y \quad \text{in } C(T). \tag{3.8}$$

First suppose that $y \neq 0$. We set $Z(y) = \{t \in T : y(t) = 0\}$. Then, $|T \setminus Z(y)|_1 > 0$ and $|u_n(t)| \to +\infty$ for a.a. $t \in T \setminus Z(y)$. Hypothesis (H) (ii) implies that

$$\frac{F(t,u_n(t))}{\|u_n\|^p} = \frac{F(t,u_n(t))}{|u_n(t)|^p}|y_n(t)|^p \to +\infty \quad \text{for a.a. } t \in T \setminus Z(y). \tag{3.9}$$

From (3.9) and Fatou's lemma we have that

$$\int_0^b \frac{F(t,u_n(t))}{\|u_n\|^p}\,\mathrm{d}t \to +\infty \quad \text{as } n \to \infty. \tag{3.10}$$

But, from (3.2) we know that

$$-\frac{1}{p}\|y_n'\|_p^p + \int_0^b \frac{F(t,u_n(t))}{\|u_n\|^p}\,\mathrm{d}t \leqslant \frac{M_1}{\|u_n\|^p} \quad \text{for all } n \geqslant 1. \tag{3.11}$$

Passing to the limit as $n \to \infty$ in (3.11), and using (3.8) and (3.10), we reach a contradiction.

Therefore, we may assume that $y = 0$. To treat this case, first note that by passing to a suitable subsequence if necessary, we may assume that

$$\|y_n'\|_p \geqslant \beta > 0 \quad \text{for some } \beta > 0, \text{ all } n \geqslant 1. \tag{3.12}$$

Indeed, otherwise we have $\|y_n'\|_p \to 0$, which, in conjunction with (3.8), implies that $y_n \to 0$ in $W^{1,p}_{\mathrm{per}}(0,b)$ (recall that $y = 0$), which contradicts the fact that $\|y_n\| = 1$ for all $n \geqslant 1$.

For every $n \geqslant 1$, we consider the continuous function $\sigma_n\colon [0,1] \to \mathbb{R}$ defined by

$$\sigma_n(\tau) = \varphi(\tau u_n) \quad \text{for all } \tau \in [0,1].$$

Let $\tau_n \in [0,1]$ such that

$$\sigma_n(\tau_n) = \max[\sigma_n(\tau)\colon \tau \in [0,1]], \quad n \geqslant 1. \tag{3.13}$$

For $\lambda > 0$, let $v_n = (2\lambda p/\beta)^{1/p} y_n \in W^{1,p}_{\text{per}}(0,b)$, $n \geqslant 1$. Then, $v_n \to 0$ in $C(T)$ (see (3.8) and recall that $y = 0$). From the dominated convergence theorem (see Hypothesis (H) (i)), we have that

$$\int_0^b F(t, v_n(t))\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Since $\|u_n\| \to \infty$, we can find an integer $n_0 \geqslant 1$ such that $(2\lambda p/\beta)^{1/p} 1/\|u_n\| \in (0,1)$ for all $n \geqslant n_0$. Then, from (3.13) we have that

$$\begin{aligned}
\sigma_n(\tau_n) \geqslant \sigma\biggl(\biggl(\frac{2\lambda p}{\beta}\biggr)^{1/p} \frac{1}{\|u_n\|}\biggr) \quad \text{for all } n \geqslant n_0 \\
\implies \varphi(\tau_n u_n) &\geqslant \varphi(v_n) \\
&= \frac{1}{p}\|v_n'\|_p^p - \int_0^b F(t, v_n)\,\mathrm{d}t \\
&\geqslant 2\lambda - \int_0^b F(t, v_n)\,\mathrm{d}t \qquad \text{(see (3.12))} \\
&\geqslant \lambda > 0 \quad \text{for all } n \geqslant n_1 \geqslant n_0 \quad \text{(see (3.14))}.
\end{aligned} \tag{3.15}$$

Since $\lambda > 0$ is arbitrary, from (3.15) we infer that

$$\varphi(\tau_n u_n) \to +\infty \quad \text{as } n \to \infty. \tag{3.16}$$

Note that $0 \leqslant \tau_n u_n^+ \leqslant u_n^+$ and $-u_n^- \leqslant -\tau_n u_n^- \leqslant 0$ for all $n \geqslant 1$. So, from (3.1) we have that

$$\int_0^b \vartheta(t, \tau_n u_n^+)\,\mathrm{d}t \leqslant \int_0^b \vartheta(t, u_n^+)\,\mathrm{d}t + \beta^* b, \tag{3.17}$$

$$\int_0^b \vartheta(t, -\tau_n u_n^-)\,\mathrm{d}t \leqslant \int_0^b \vartheta(t, -u_n^-)\,\mathrm{d}t + \beta^* b. \tag{3.18}$$

Since $\vartheta(t,0) = 0$ for a.a. $t \in T$, adding (3.17) and (3.18), we obtain that

$$\int_0^b \vartheta(t, \tau_n u_n)\,\mathrm{d}t \leqslant \int_0^b \vartheta(t, u_n)\,\mathrm{d}t + 2\beta^* b \quad \text{for all } n \geqslant 1. \tag{3.19}$$

Note that $\varphi(0) = 0$ and $|\varphi(u_n)| \leqslant M_1$ for all $n \geqslant 1$ (see (3.2)). These facts together with (3.16) imply that $\tau_n \in (0,1)$ for all $n \geqslant 1$ large, say $n \geqslant n_2$. From (3.13) we have that

$$0 = \tau_n \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi(\tau u_n)\bigg|_{\tau=\tau_n} = \langle \varphi'(\tau_n u_n), \tau_n u_n\rangle = \|\tau_n u_n'\|_p^p - \int_0^b f(t,\tau_n u_n)(\tau_n u_n)\,\mathrm{d}t$$
$$\implies \|\tau_n u_n'\|_p^p = \int_0^b f(t,\tau_n u_n)(\tau_n u_n)\,\mathrm{d}t \quad \text{for all } n \geqslant n_2. \tag{3.20}$$

From (3.19) we have that

$$\int_0^b [f(t,\tau_n u_n)(\tau_n u_n) - pF(t,\tau_n u_n)]\,\mathrm{d}t \leqslant \int_0^b \vartheta(t,u_n)\,\mathrm{d}t + 2\beta^* b \quad \text{for all } n \geqslant 1$$
$$\implies \|\tau_n u_n'\|_p^p - \int_0^b pF(t,\tau_n u_n)\,\mathrm{d}t \leqslant \int_0^b \vartheta(t,u_n)\,\mathrm{d}t + 2\beta^* b \quad \text{for all } n \geqslant n_2$$
$$\text{(see (3.20))}$$
$$\implies p\varphi(\tau_n u_n) \leqslant \int_0^b \vartheta(t,u_n)\,\mathrm{d}t + 2\beta^* b \quad \text{for all } n \geqslant n_2$$
$$\implies \int_0^b \vartheta(t,u_n)\,\mathrm{d}t \to +\infty \quad \text{as } n\to\infty \quad \text{(see (3.16))}.$$

However, this contradicts (3.7). This proves the claim.

By virtue of the claim, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \qquad \text{and} \qquad u_n \to u \quad \text{in } C(T). \tag{3.21}$$

In (3.4), we choose $h = u_n - u$, pass to the limit as $n\to\infty$ and use (3.21). Then,

$$\lim_{n\to\infty}\langle A(u_n), u_n - u\rangle = 0 \implies u_n \to u \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b)$$
$$\implies \varphi \text{ satisfies the C condition.}$$

□

Proposition 3.5. *If Hypotheses (H) hold, then $C_k(\varphi,\infty) = 0$ for all $k \geqslant 0$.*

Proof. Let $\partial B_1 = \{u \in W^{1,p}_{\mathrm{per}}(0,b)\colon \|u\| = 1\}$ and $u \in \partial B_1$. Hypothesis (H) (ii) implies that

$$\varphi(\tau u) \to -\infty \quad \text{as } \tau \to +\infty. \tag{3.22}$$

By virtue of (3.1), for every $u \in W^{1,p}_{\mathrm{per}}(0,b)$, we have that

$$0 = \vartheta(t,0) \leqslant \vartheta(t,u^+(t)) + \beta^*$$

and

$$0 = \vartheta(t,0) \leqslant \vartheta(t,-u^-(t)) + \beta^* \quad \text{for a.a. } t \in T$$
$$\implies 0 \leqslant \vartheta(t,u(t)) + 2\beta^* \quad \text{for a.a. } t\in T$$
$$\implies -\vartheta(t,u(t)) = pF(t,u(t)) - f(t,u(t))u(t) \leqslant 2\beta^* \quad \text{for a.a. } t \in T. \tag{3.23}$$

Then, for $u \in W^{1,p}_{\mathrm{per}}(0,b)$ and $\tau > 0$, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\tau u) &= \langle \varphi'(\tau u), u\rangle \\ &= \frac{1}{\tau}\langle \varphi'(\tau u), \tau u\rangle \\ &= \frac{1}{\tau}\biggl[\|\tau u'\|_p^p - \int_0^b f(t,\tau u)(\tau u)\,\mathrm{d}t\biggr] \\ &\leqslant \frac{1}{\tau}\biggl[\|\tau u'\|_p^p - \int_0^b pF(t,\tau u)\,\mathrm{d}t + 2\beta^* b\biggr] \quad \text{(see (3.23))} \\ &= \frac{1}{\tau}[p\varphi(\tau u) + 2\beta^* b]. \end{aligned} \tag{3.24}$$

By virtue of (3.22), we see that, for $\tau > 0$ large, we have $\varphi(\tau u) \leqslant \mu < -2\beta^* b/p$, and so from (3.24) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\tau u) < 0. \tag{3.25}$$

Then, for $u \in \partial B_1$, we can find a unique $\gamma(u) > 0$ such that $\varphi(\gamma(u)u) = \mu$. Moreover, invoking the implicit function theorem (see (3.25)), we have $\gamma \in C(\partial B_1)$. We extend γ to $W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\}$ by setting

$$\hat{\gamma}(u) = \frac{1}{\|u\|}\gamma\biggl(\frac{1}{\|u\|}\biggr) \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\}.$$

Clearly, $\hat{\gamma} \in C(W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\})$ and $\varphi(\hat{\gamma}(u)u) = \mu$. Moreover, $\varphi(u) = \mu$ implies that $\hat{\gamma}(u) = 1$. So, if we set

$$\hat{\gamma}_0(u) = \begin{cases} 1 & \text{if } \varphi(u) < \mu, \\ \hat{\gamma}(u) & \text{if } \varphi(u) \geqslant \mu, \end{cases} \tag{3.26}$$

then $\hat{\gamma}_0 \in C(W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\})$.

We consider the homotopy $h\colon [0,1] \times (W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\}) \to W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\}$ defined by

$$h(s,u) = (1-s)u + s\hat{\gamma}_0(u)u.$$

Note that

$$h(0,u) = u, \quad h(1,u) \in \varphi^{\mu} \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\} \quad \text{(see (3.26))}$$

and

$$h(s,\cdot)|_{\varphi^{\mu}} = \mathrm{id}\,|_{\varphi^{\mu}}, \quad s \in [0,1]. \tag{3.27}$$

From (3.27) it follows that φ^{μ} is a strong deformation retract of $W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\}$. Using the radial retraction we see that ∂B_1 is a deformation retract of $W^{1,p}_{\mathrm{per}}(0,b) \setminus \{0\}$ (see [**13**, Theorem 6.5, p. 325]). Therefore, we infer that

$$\begin{aligned} &\varphi^{\mu} \text{ and } \partial B_1 \text{ are homotopy equivalent} \\ &\qquad \implies H_k(W^{1,p}_{\mathrm{per}}(0,b), \varphi^{\mu}) = H_k(W^{1,p}_{\mathrm{per}}(0,b), \partial B_1) \quad \text{for all } k \geqslant 0. \end{aligned} \tag{3.28}$$

Since $W^{1,p}_{\mathrm{per}}(0,b)$ is infinite dimensional, ∂B_1 is contractible in itself. Hence,

$$\begin{aligned}H_k(W^{1,p}_{\mathrm{per}}(0,b),\partial B_1)=0 \quad &\text{for all } k\geqslant 0 \quad (\text{see [}\mathbf{16}\text{, p. 389]})\\ &\implies H_k(W^{1,p}_{\mathrm{per}}(0,b),\varphi^{\mu})=0 \quad \text{for all } k\geqslant 0 \quad (\text{see (3.28)})\\ &\implies C_k(\varphi,\infty)=0 \quad \text{for all } k\geqslant 0 \quad (\text{choose } \mu<0 \text{ with } |\mu| \text{ big}).\end{aligned}$$

□

Proposition 3.6. *If Hypotheses (H) hold, then $C_{m+1}(\varphi,0)\neq 0$ or $C_k(\varphi,0)=\delta_{k,0}\mathbb{Z}$ for all $k\geqslant 0$.*

Proof. First assume that (H) (iii) (a) is in effect. Let $\beta\in(\hat{\lambda}_m,\hat{\lambda}_{m+1})$ and consider the C^1-functional $\psi\colon W^{1,p}_{\mathrm{per}}(0,b)\to\mathbb{R}$ defined by

$$\psi(u)=\frac{1}{p}\|u'\|_p^p-\frac{\beta}{p}\|u\|_p^p \quad \text{for all } u\in W^{1,p}_{\mathrm{per}}(0,b).$$

Since $\beta\notin\sigma(p)$ (the spectrum of the negative periodic scalar p-Laplacian), it follows that ψ satisfies the C condition.

We consider the homotopy $h\colon[0,1]\times W^{1,p}_{\mathrm{per}}(0,b)\to W^{1,p}_{\mathrm{per}}(0,b)$ defined by

$$h(s,u)=(1-s)\varphi(u)+s\psi(u) \quad \text{for all } (s,u)\in[0,1]\times W^{1,p}_{\mathrm{per}}(0,b).$$

Note that $h(0,\cdot)=\varphi$ and $h(1,\cdot)=\psi$ and that both functionals satisfy the C condition (see Proposition 3.3).

Suppose that we can find $\{s_n\}_{n\geqslant 1}\subseteq[0,1]$ and $\{u_n\}_{n\geqslant 1}\subseteq W^{1,p}_{\mathrm{per}}(0,b)\setminus\{0\}$ such that

$$s_n\to s\in[0,1],\ u_n\to 0 \text{ in } W^{1,p}_{\mathrm{per}}(0,b) \text{ and } h'_u(s_n,u_n)=0 \quad \text{for all } n\geqslant 1. \tag{3.29}$$

From (3.29), we have that

$$A(u_n)=(1-s_n)N_f(u_n)+s_n\beta|u_n|^{p-2}u_n \quad \text{for all } n\geqslant 1, \tag{3.30}$$

with $N_f(u)(\cdot)=f(\cdot,u(\cdot))$ for all $u\in W^{1,p}_{\mathrm{per}}(0,b)$.

We set $y_n=u_n/\|u_n\|$, $n\geqslant 1$. Then, $\|y_n\|=1$ for all $n\geqslant 1$, and so we may assume that

$$y_n\xrightarrow{w} y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \qquad \text{and} \qquad y_n\to y \quad \text{in } C(T). \tag{3.31}$$

From (3.30), we have that

$$A(y_n)=(1-s_n)\frac{N_f(u_n)}{\|u_n\|^{p-1}}+s_n\beta|y_n|^{p-2}y_n \quad \text{for all } n\geqslant 1. \tag{3.32}$$

By virtue of Hypotheses (H) (i), (ii), we can find $\hat{\alpha}\in L^1(T)_+$ such that

$$\begin{aligned}&|f(t,x)|\leqslant\hat{\alpha}(t)(|x|^{p-1}+|x|^{r-1}) \quad \text{for a.a. } t\in T,\ \text{all } x\in\mathbb{R}\\ &\qquad\implies \left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}\subseteq L^1(T) \text{ is uniformly integrable (recall that } p<r).\end{aligned}$$

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Thus, by virtue of the Dunford–Pettis theorem, we may assume that

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} g \quad \text{in } L^1(T). \tag{3.33}$$

Using Hypothesis (H) (iii) and reasoning as in [**3**, Proof of Proposition 31], we have that

$$g(t) = \xi(t)|y|^{p-2}y \quad \text{for a.a. } t \in T, \text{ with } \eta \leqslant \xi \leqslant \hat{\eta}. \tag{3.34}$$

On (3.32) we act with $y_n - y \in W^{1,p}_{\mathrm{per}}(0,b)$, pass to the limit as $n \to \infty$ and use (3.31) and (3.33). We obtain that

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0 \\ &\qquad \implies y_n \to y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \quad \text{and so } \|y\| = 1 \quad \text{(see Proposition 2.3).} \end{aligned} \tag{3.35}$$

So, if in (3.32) we pass to the limit as $n \to \infty$ and use (3.33)–(3.35), we obtain that

$$\begin{aligned} &A(y) = ((1-s)\xi + s\beta)|y|^{p-2}y \\ &\qquad \implies -(y'(t)^{p-2}y'(t))' = \xi_s|y(t)|^{p-2}y(t) \quad \text{a.e. on } T,\ y(0) = y(b),\ y'(0) = y'(b), \end{aligned} \tag{3.36}$$

where $\xi_s = (1-s)\xi + s\beta$ (see [**2**]).

Note that

$$\hat{\lambda}_m \leqslant \xi_s(t) \leqslant \hat{\lambda}_{m+1} \quad \text{a.e. on } T,\ \hat{\lambda}_m \neq \xi_s,\ \hat{\lambda}_{m+1} \neq \xi_s.$$

Invoking [**1**, Proposition 2], we infer that $y = 0$ (see (3.36)), which contradicts (3.35).

This argument shows that we can find $\varrho \in (0,1)$ small such that $u = 0$ is the only critical point of the family $\{h(s,\cdot)\}_{s\in[0,1]}$ in $\bar{B}_\varrho = \{u \in W^{1,p}_{\mathrm{per}}(0,b)\colon \|u\| \leqslant \varrho\}$. Invoking the homotopy invariance property of critical groups (see [**9**, p. 334]), we have that

$$C_k(h(0,\cdot),0) = C_k(h(1,\cdot),0) \quad \text{for all } k \geqslant 0 \implies C_k(\varphi,0) = C_k(\psi,0) \quad \text{for all } k \geqslant 0. \tag{3.37}$$

Let $\varrho' > 0$ and introduce the two sets

$$C_0 = \{u \in W^{1,p}_{\mathrm{per}}(0,b)\colon \|u'\|_p^p < \beta\|u\|_p^p,\ \|u\| = \varrho'\}$$

and

$$D = \{u \in W^{1,p}_{\mathrm{per}}(0,b)\colon \|u'\|_p^p \geqslant \beta\|u\|_p^p\}.$$

Evidently, both are symmetric sets and $C_0 \cap D \neq \emptyset$, $0 \in D$. The set $\partial B_\varrho = \{u \in W^{1,p}_{\mathrm{per}}(0,b)\colon \|u\| = \varrho'\}$ is a Banach C^1-manifold of codimension 1, and, hence, it is locally contractible. The set C_0 is an open subset of ∂B_ϱ. So, it follows that C_0 is locally contractible too. Also, it is clear that the open set $W^{1,p}_{\mathrm{per}}(0,b) \setminus D$ is locally contractible. If by 'ind' we denote the Fadell–Rabinowitz cohomological index [**14**], we have $\operatorname{ind} C_0 = m+1$ and $\operatorname{ind} C = m+1$ (see [**23**, p. 68]). Then, invoking [**10**, Theorem 3.6], we can find $K \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ such that the pair $(C \cup K, C_0)$ and D homologically link in dimension

$m+1$. So, from [**8**, p. 89], we have that $C_{m+1}(\psi,0)\neq 0$ and by virtue of (3.37) we conclude that $C_{m+1}(\varphi,0)\neq 0$.

Now, suppose that Hypothesis (H) (iii) (b) is in effect. Then, by virtue of Hypotheses (H) (i) and (iii) (b), given $\varepsilon>0$, we can find $\alpha_\varepsilon\in L^1(T)_+$ such that

$$F(t,x)\leqslant \frac{1}{p}(\eta_0(t)+\varepsilon)|x|^p+\alpha_\varepsilon(t)|x|^r \quad \text{for a.a. } t\in T,\ \text{all } x\in\mathbb{R}.$$

Then, for all $u\in W^{1,p}_{\mathrm{per}}(0,b)$, we have that

$$\begin{aligned}
\varphi(u)&=\frac{1}{p}\|u'\|_p^p-\int_0^b F(t,u(t))\,\mathrm{d}t\\
&\geqslant \frac{1}{p}\|u'\|_p^p-\int_0^b \eta_0(t)|u|^p\,\mathrm{d}t-\frac{\varepsilon}{p}\|u\|^p-\tilde{c}\|u\|^r \quad \text{for some } \tilde{c}>0\\
&\geqslant \frac{\tilde{\xi}_0-\varepsilon}{p}\|u\|^p-\tilde{c}\|u\|^r \quad \text{for some } \tilde{\xi}_0>0 \quad (\text{see [\textbf{2}, Proposition 7]})\\
&\qquad\qquad\qquad \implies u=0 \text{ is a local minimizer of } \varphi \ (\text{recall that } p<r)\\
&\qquad\qquad\qquad \implies C_k(\varphi,0)=\delta_{k,0}\mathbb{Z} \quad \text{for all } k\geqslant 0 \quad (\text{see [\textbf{8}]}).
\end{aligned}$$

□

Propositions 3.3, 3.5 and 3.6 lead to the following existence theorem (see § 2).

Theorem 3.7. *If Hypotheses (H) hold, then (1.1) has a non-trivial solution $u_0\in C^1(T)$.*

4. The multiplicity theorem

For the multiplicity theorem, the hypotheses on the reaction $f(t,x)$ are the following.

(H′) $f\colon T\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that for a.a. $t\in T$, $f(t,0)=0$ and the following hold.

(i) $|f(t,x)|\leqslant\alpha(t)(1+|x|^{r-1})$ for a.a $t\in T$, all $x\in\mathbb{R}$, with $\alpha\in L^1(T)_+$, $p<r<\infty$.

(ii) $\lim_{x\to\pm\infty}F(t,x)/|x|^p=+\infty$ uniformly for a.a. $t\in T$ and there exists $\beta^*>0$ such that

$$\vartheta(t,x)\leqslant\vartheta(t,y)+\beta^* \quad \text{for a.a. } t\in T,\ \text{all } 0\leqslant x\leqslant y \text{ or } y\leqslant x\leqslant 0.$$

(iii) There exist $\lambda^*>\hat{\lambda}_1$ and $\hat{\eta}\in L^1(T)_+$ such that

$$\lambda^*\leqslant\liminf_{x\to 0}\frac{f(t,x)}{|x|^{p-2}x}\leqslant\limsup_{x\to 0}\frac{f(t,x)}{|x|^{p-2}x}\leqslant\hat{\eta}(t)$$

uniformly for a.a. $t\in T$.

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(iv) There exist real numbers $c_- < 0 < c_+$ such that

$$f(t, c_-) \leqslant \beta_- < 0 < \beta_+ \leqslant f(t, c_+) \quad \text{for a.a. } t \in T.$$

(v) For every $\varrho > 0$, there exists $\xi_\varrho^* > 0$ such that, for a.a. $t \in T$, $x \to f(t,x) + \xi_\varrho^* |x|^{p-2}x$ is non-decreasing on $[-\varrho, \varrho]$.

Remark 4.1. The asymptotic condition at $\pm\infty$ (see (H′) (ii)) remains the same. The asymptotic condition at 0 (see (H′) (iii)) is somewhat weaker than (H) (iii), since we do not require that the quotient $f(t,x)/|x|^{p-2}x$ asymptotically stays in the spectral interval $[\hat{\lambda}_k, \hat{\lambda}_{k+1}]$. We only require that, for a.a. $t \in T$, $f(t, \cdot)$ is $(p-1)$-linear near 0 and the quotient $f(t,x)/|x|^{p-2}x$ near zero stays above $\hat{\lambda}_1 > 0$. Of course, we also added Hypotheses (H′) (iv) and (H′) (v). Hypothesis (H′) (iv) states that the reaction has non-trivial zeros.

Example 4.2. The following function satisfies Hypotheses (H′) (as before, for the sake of simplicity, we drop the t-dependence):

$$f(x) = \begin{cases} \eta(|x|^{p-2}x - 2|x|^{q-2}x) & \text{if } |x| \leqslant 1, \\ |x|^{p-2}x \ln|x| - \eta|x|^{\tau-2}x & \text{if } |x| > 1, \end{cases}$$

with $\eta > \hat{\lambda}_1$, $1 < \tau < p < q < \infty$.

We start by producing two constant sign solutions. To this end, we introduce the following truncations–perturbations of the reaction $f(t,x)$:

$$\hat{f}_+(t,x) = \begin{cases} 0 & \text{if } x < 0, \\ f(t,x) + x^{p-1} & \text{if } 0 \leqslant x \leqslant c_+, \\ f(t,c_+) + c_+^{p-1} & \text{if } c_+ < x \end{cases}$$

and

$$\hat{f}_-(t,x) = \begin{cases} f(t,c_-) + |c_-|^{p-2}c_- & \text{if } x < c_-, \\ f(t,x) + |x|^{p-2}x & \text{if } c_- \leqslant x \leqslant 0, \\ 0 & \text{if } 0 < x. \end{cases} \tag{4.1}$$

Both are Carathéodory functions. We set

$$\hat{F}_\pm(t,x) = \int_0^b \hat{f}_\pm(t,s)\,\mathrm{d}s$$

and consider the C^1-functionals $\hat{\varphi}_\pm \colon W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \frac{1}{p}[\|u'\|_p^p + \|u\|_p^p] - \int_0^b \hat{F}_\pm(t, u(t))\,\mathrm{d}t \quad \text{for all } u \in W^{1,p}_{\text{per}}(0,b).$$

Proposition 4.3. *If Hypotheses (H′) hold, then (1.1) has at least two non-trivial constant sign solutions $u_0 \in \operatorname{int}\hat{C}_+$, $v_0 \in -\operatorname{int}\hat{C}_+$ and $c_- < v_0(t) < 0 < u_0(t) < c_+$ for all $t \in T$.*

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Proof. We show the proof for the positive solution u_0, the proof for the negative solution v_0 being similar.

Evidently, $\hat{\varphi}_+$ is coercive (see (4.1)) and it is sequentially weakly lower semi-continuous. So, by the Weierstrass Theorem, we can find $u_0 \in W^{1,p}_{\mathrm{per}}(0,b)$ such that

$$\hat{\varphi}_+(u_0) = \inf[\hat{\varphi}_+(u) \colon u \in W^{1,p}_{\mathrm{per}}(0,b)] = \hat{m}_+. \tag{4.2}$$

By virtue of Hypothesis (H′) (iii), we have that

$$\hat{m}_+ = \hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \quad \text{and so} \quad u_0 \neq 0.$$

Also, from (4.2) we have that

$$A(u_0) + |u_0|^{p-2}u_0 = N_{\hat{f}_+}(u_0), \quad \text{with } N_{\hat{f}_+}(u)(\cdot) = \hat{f}_+(\cdot, u(\cdot)), \tag{4.3}$$

for all $u \in W^{1,p}_{\mathrm{per}}(0,b)$.

Acting on (4.3) with $-u_0^- \in W^{1,p}_{\mathrm{per}}(0,b)$, we obtain $u_0 \geqslant 0$. Next, we act on (4.3) with $(u_0 - c_+)^+ \in W^{1,p}_{\mathrm{per}}(0,b)$ and obtain that

$$\begin{aligned}
&\langle A(u_0), (u_0 - c_+)^+\rangle + \int_0^b u_0^{p-1}(u_0 - c_+)^+ \,\mathrm{d}t \\
&\quad = \int_0^b f(t, c_+)(u_0 - c_+)^+ \,\mathrm{d}t + \int_0^b c_+^{p-1}(t, c_+)(u_0 - c_+)^+ \,\mathrm{d}t \quad (\text{see } (4.1)) \\
&\quad \implies \langle A(u_0) - A(c_+), (u_0 - c_+)^+\rangle + \int_0^b (u_0^{p-1} - c_+^{p-1})(u_0 - c_+)^+ \,\mathrm{d}t \leqslant 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{see (H}') \text{ (iv)}) \\
&\quad \implies |\{u_0 > c_+\}|_1 = 0, \text{ i.e. } u_0 \leqslant c_+.
\end{aligned}$$

Hence, $0 \leqslant u_0 \leqslant c_+$ and so (4.3) becomes

$$A(u_0) = N_f(u_0) \text{ (see (4.1))} \implies u_0 \in \hat{C}_+ \setminus \{0\} \text{ solves (1.1).}$$

Let $\varrho = c_+$ and let $\xi^*_\varrho > 0$ be as postulated by Hypothesis (H′) (v). Then,

$$-(|u_0'(t)|^{p-2}u_0'(t))' + \xi^*_\varrho u_0(t)^{p-1} \geqslant 0 \text{ a.e. on } T \implies u_0 \in \operatorname{int} \hat{C}_+ \quad (\text{see } [\mathbf{25}]).$$

For $\tau > 0$, set $u_\tau = u_0 + \tau \in \operatorname{int} \hat{C}_+$. We have that

$$\begin{aligned}
&-(|u_\tau'(t)|^{p-2}u_\tau'(t))' + \xi^*_\varrho u_\tau(t)^{p-1} \\
&\qquad \leqslant -(|u_0'(t)|^{p-2}u_0'(t))' + \xi^*_\varrho u_0(t)^{p-1} + \lambda(\tau) \quad \text{with } \lambda(\tau) \to 0^+ \text{ as } \tau \to 0^+ \\
&\qquad = f(t, u_0(t)) + \xi^*_\varrho u_0(t)^{p-1} + \lambda(\tau) \\
&\qquad \leqslant f(t, c_+) + \xi^*_\varrho c_+^{p-1} + \lambda(\tau) \quad (\text{see (H}') \text{ (v)}) \\
&\qquad \leqslant \beta_+ + \xi^*_\varrho c_+^{p-1} + \lambda(\tau) \qquad\quad (\text{see (H}') \text{ (iv)}).
\end{aligned}$$

Since $\beta_+ < 0$ and $\lambda(\tau) \to 0^+$ as $\tau \to 0^+$, for $\tau > 0$ small we have that

$$\begin{aligned} A(u_\tau) + \xi_\varrho^* u_\tau(t)^{p-1} &\leqslant A(c_+) + \xi_\varrho^* c_+^{p-1} \text{ in } W^{1,p}_{\text{per}}(0,b) \\ &\implies u_\tau \leqslant c_+ \qquad \text{for all } \tau > 0 \text{ small} \\ &\implies u_0(t) < c_+ \quad \text{for all } t \in T. \end{aligned}$$

Similarly, working with $\hat{\varphi}_-$ we produce a negative solution $v_0 \in -\operatorname{int} \hat{C}_+$ such that $c_- < v_0(t) < 0$ for all $t \in T$. □

Remark 4.4. Let

$$\begin{aligned} [0, c_+] &= \{u \in W^{1,p}_{\text{per}}(0,b)\colon 0 \leqslant u(t) \leqslant c_+ \text{ for all } t \in T\}, \\ [c_-, 0] &= \{u \in W^{1,p}_{\text{per}}(0,b)\colon c_- \leqslant u(t) \leqslant 0 \text{ a.e. on } T\}. \end{aligned}$$

From the proof of Proposition 4.3, we have that

$$u_0 \in \operatorname{int}_{\hat{C}^1(T)}[0, u_0] \quad \text{and} \quad v_0 \in \operatorname{int}_{\hat{C}^1(T)}[v_0, 0].$$

Invoking [**2**, Proposition 9], we infer that u_0 and v_0 are both local minimizers of φ (see (4.1)).

Reasoning as in [**4**, Proposition 8], we can have extremal solutions of (1.1) in the order intervals $[0, c_+]$ and $[c_-, 0]$.

Proposition 4.5. *If Hypotheses (H′) hold, then (1.1) has a smallest non-trivial solution $\tilde{u}_0 \in \operatorname{int} \hat{C}_+$ and a biggest solution $u_0 \in \operatorname{int} \hat{C}_+$, with $u_0(t) < c_+$ for all $t \in T$ in the order interval $[0, c_+]$; similarly in the order interval $[c_-, 0]$.*

By virtue of this proposition, we may assume that the two solutions u_0 and v_0 obtained in Proposition 4.3 are extremal, namely that $u_0 \in \operatorname{int} \hat{C}_+$ is the biggest solution of (1.1) in the order interval $[0, c_+]$ and $v_0 \in -\operatorname{int} \hat{C}_+$ is the smallest solution of (1.1) in the order interval $[c_-, 0]$. Using these two solutions together with variational methods and truncation techniques, we produce two more non-trivial solutions of constant sign.

Proposition 4.6. *If Hypotheses (H′) hold, then (1.1) has two more non-trivial solutions of constant sign, $\hat{u} \in \operatorname{int} \hat{C}_+$ and $\hat{v} \in -\operatorname{int} \hat{C}_+$, such that $u_0 \leqslant \hat{u}$, $u_0 \neq \hat{u}$ and $\hat{v} \leqslant v_0$, $\hat{v} \neq v_0$.*

Proof. As already mentioned, we assume that the solutions u_0 and v_0 from Proposition 4.3 are extremal in the order intervals $[0, c_+]$ and $[c_-, 0]$, respectively.

We show the proof for the positive solution $\hat{u}$, the proof for the negative solution $\hat{v}$ being similar.

We consider the following truncation–perturbation of $f(t, x)$:

$$\hat{g}_+(t,x) = \begin{cases} f(t, u_0(t)) + u_0(t)^{p-1} & \text{if } x \leqslant u_0(t), \\ f(t,x) + x^{p-1} & \text{if } u_0(t) < x. \end{cases} \tag{4.4}$$

This is a Carathéodory function. We set

$$\hat{G}_+(t,x) = \int_0^x \hat{g}_+(t,s)\,\mathrm{d}s$$

and introduce the C^1-functional $\hat{\psi}_+ \colon W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) = \frac{1}{p}[\|u'\|_p^p + \|u\|_p^p] - \int_0^b \hat{G}_+(t,u(t))\,\mathrm{d}t \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b).$$

Reasoning as in the proof of Proposition 3.3 and using Hypothesis (H′) (ii), we show that

$$\hat{\psi}_+ \text{ satisfies the C condition.} \tag{4.5}$$

Moreover, Hypothesis (H′) (ii) implies that

$$\hat{\psi}_+(\xi) \to -\infty \quad \text{as } \xi \to +\infty,\ \xi \in \mathbb{R}. \tag{4.6}$$

We consider the following truncation of $\hat{g}_+(t,x)$:

$$g_+(t,x) = \begin{cases} \hat{g}_+(t,x) & \text{if } x < c_+, \\ \hat{g}_+(t,c_+) & \text{if } c_+ \leqslant x. \end{cases} \tag{4.7}$$

We set

$$G_+(t,x) = \int_0^x g_+(t,s)\,\mathrm{d}s$$

and consider the C^1-functional $\psi_+ \colon W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}[\|u'\|_p^p + \|u\|_p^p] - \int_0^b G_+(t,u)\,\mathrm{d}t \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b).$$

It is clear from (4.7) that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0 \in W^{1,p}_{\mathrm{per}}(0,b)$ such that

$$\begin{aligned} \psi_+(\hat{u}_0) &= \inf[\psi_+(\hat{u}) \colon u \in W^{1,p}_{\mathrm{per}}(0,b)] \\ &\implies \psi_+'(\hat{u}_0) = 0 \\ &\implies A(\hat{u}_0) = N_{g_+}(\hat{u}_0) \quad \text{with } N_{g_+}(u)(\cdot) = g_+(\cdot,u(\cdot)) \text{ for all } u \in W^{1,p}_{\mathrm{per}}(0,b). \end{aligned} \tag{4.8}$$

From (4.8), as before (see the proof of Proposition 4.3), we show that

$$\hat{u}_0 \in [u_0, c_+] = \{u \in W^{1,p}_{\mathrm{per}}(0,b) \colon u_0(t) \leqslant u(t) \leqslant c_+ \text{ for all } t \in T\}.$$

The maximality of u_0 implies that $\hat{u}_0 = u_0$. From Proposition 4.3 we know that $u_0(t) < c_+$ for all $t \in T$. Since $\psi_+|_{[0,c_+]} = \hat{\psi}_+|_{[0,c_+]}$ (see (4.7)), it follows that u_0 is a local $\hat{C}^1(T)$-minimizer of $\hat{\psi}_+$. Hence, by virtue of [**2**, Proposition 9], we have that u_0 is a local $W^{1,p}_{\mathrm{per}}(0,b)$-minimizer of $\hat{\psi}_+$. We may assume that u_0 is an isolated critical point of $\hat{\psi}_+$

https://doi.org/10.1017/S0013091513000370 Published online by Cambridge University Press

(otherwise we have a whole sequence of distinct critical points of $\hat{\psi}_+$ converging to u_0 and since

$$K_{\hat{\psi}_+} \subseteq [u_0) = \{u \in W^{1,p}_{\mathrm{per}}(0,b) \colon u_0(t) \leqslant u(t) \text{ for all } t \in T\}$$

we are done; see (4.4)). Then, reasoning as in [**3**, Proof of Proposition 29], we can find $\varrho \in (0,1)$ small such that

$$\hat{\psi}_+(u_0) < \inf[\hat{\psi}_+(u) \colon \|u-u_0\| = \varrho] = \hat{\eta}^+_\varrho. \tag{4.9}$$

Then, (4.5), (4.6) and (4.9) allow us to use Theorem 2.1 (the mountain pass theorem). So, we obtain $\hat{u} \in W^{1,p}_{\mathrm{per}}(0,b)$ such that

$$\hat{\psi}_+(u_0) < \hat{\eta}^+_\varrho \leqslant \hat{\psi}_+(\hat{u}) \tag{4.10}$$

and

$$\hat{\psi}'_+(\hat{u}) = 0. \tag{4.11}$$

From (4.10) we have that $\hat{u} \neq u_0$, while from (4.11) we have that

$$A(\hat{u}) + |\hat{u}|^{p-2}\hat{u} = N_{\hat{g}_+}(\hat{u}), \quad \text{with } N_{\hat{g}_+}(u)(\cdot) = \hat{g}_+(\cdot, u(\cdot)), \text{ for all } u \in W^{1,p}_{\mathrm{per}}(0,b). \tag{4.12}$$

Acting on (4.12) with $(u_0 - \hat{u})^+ \in W^{1,p}_{\mathrm{per}}(0,b)$ and using (4.4), we show that $u_0 \leqslant \hat{u}$. So, (4.12) becomes

$$A(\hat{u}) = N_f(\hat{u}) \text{ (see (4.4))} \implies \hat{u} \in \operatorname{int} \hat{C}_+,\ u_0 \leqslant \hat{u},\ u_0 \neq \hat{u} \text{ is a solution of (1.1).}$$

Similarly, using $v_0 \in -\operatorname{int} \hat{C}_+$ as the smallest solution of (1.1) in the order interval $[c_-, 0]$, we produce a second negative solution $\hat{v} \in -\operatorname{int} \hat{C}_+$, $\hat{v} \leqslant v_0$, $\hat{v} \neq v_0$. □

Next, we produce a nodal (sign changing) solution for (1.1).

Proposition 4.7. *If Hypotheses (H′) hold, then (1.1) admits a nodal solution* $y_0 \in \hat{C}^1(T)$.

Proof. Let $\tilde{u}_0 \in \operatorname{int} \hat{C}_+$ be the smallest positive solution of (1.1) and let $\tilde{v}_0 \in -\operatorname{int} \hat{C}_+$ be the biggest negative solution of (1.1). Also, let $\varrho = \max(\|u_0\|_\infty, \|v_0\|_\infty)$ (with u_0, v_0 the extremal solutions from Proposition 4.5) and let $\xi^*_\varrho > 0$ be as postulated by Hypothesis (H′) (v). We introduce the following truncation–perturbation of $f(t,x)$:

$$h(t,x) = \begin{cases} f(t,\tilde{v}_0(t)) + \xi^*_\varrho|\tilde{v}_0(t)|^{p-2}\tilde{v}_0(t) & \text{if } x < \tilde{v}_0(t), \\ f(t,x) + \xi^*_\varrho|x|^{p-2}x & \text{if } \tilde{v}_0(t) \leqslant x \leqslant \tilde{u}_0(t), \\ f(t,\tilde{u}_0(t)) + \xi^*_\varrho\tilde{u}_0(t)^{p-1} & \text{if } \tilde{u}_0(t) < x. \end{cases} \tag{4.13}$$

This is a Carathéodory function. We set

$$H(t,x) = \int_0^x h(t,s)\,\mathrm{d}s$$

and introduce the C^1-functional $\sigma\colon W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\sigma(u) = \frac{1}{p}[\|u'\|_p^p + \xi_\varrho^*\|u\|_p^p] - \int_0^b H(t,u(t))\,\mathrm{d}t \quad \text{for all } u \in W^{1,p}_{\text{per}}(0,b).$$

Also, let $h_\pm(t,x) = h(t,\pm x^\pm)$,

$$H_\pm(t,x) = \int_0^x h_\pm(t,s)\,\mathrm{d}s$$

and

$$\sigma_\pm(u) = \frac{1}{p}[\|u'\|_p^p + \xi_\varrho^*\|u\|_p^p] - \int_0^b H_\pm(t,u(t))\,\mathrm{d}t$$

for all $u \in W^{1,p}_{\text{per}}(0,b)$. Both are C^1-functionals.

As before, we easily check that

$$K_\sigma \subseteq [\tilde{v}_0, \tilde{u}_0]. \tag{4.14}$$

Moreover, the extremality of the solutions $\tilde{u}_0$, $\tilde{v}_0$ implies that

$$K_{\sigma_+} = \{0, \tilde{u}_0\} \quad \text{and} \quad K_{\sigma_-} = \{\tilde{v}_0, 0\}. \tag{4.15}$$

Clearly, σ_+ is coercive (see (4.13) and recall that $h_+(t,x) = h(t,x^+)$). Also, σ_+ is sequentially weakly semi-continuous. So, σ_+ admits a minimizer that, by virtue of Hypothesis (H′) (iii), is non-trivial. Hence, (4.15) implies that this minimizer equals $\tilde{u}_0 \in \operatorname{int}\hat{C}_+$. If

$$W_+ = \{u \in W^{1,p}_{\text{per}}(0,b)\colon u(t) \geqslant 0 \text{ for all } t \in T\},$$

then $\sigma|_{W_+} = \sigma_+|_{W_+}$. Since $\tilde{u}_0 \in \operatorname{int}\hat{C}_+$, it follows that $\tilde{u}_0$ is a local $\hat{C}^1(T)$-minimizer of σ; hence, it is also a local $W^{1,p}_{\text{per}}(0,b)$-minimizer of φ (see [**2**]). Similarly, using σ_-, we show that $\tilde{v}_0 \in -\operatorname{int}\hat{C}_+$ is a local minimizer of σ. We may assume that $\sigma(\tilde{v}_0) \leqslant \sigma(\tilde{u}_0)$ and, as before, we can find $\varrho \in (0,1)$ small such that

$$\sigma(\tilde{v}_0) \leqslant \sigma(\tilde{u}_0) < \inf[\sigma(u)\colon \|u - \tilde{u}_0\| = \varrho] = \tilde{\eta}_\varrho. \tag{4.16}$$

Since σ is coercive (see (4.13)), it satisfies the C condition. This fact together with (4.16) allows us to use Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in W^{1,p}_{\text{per}}(0,b)$ such that

$$\sigma(\tilde{v}_0) \leqslant \sigma(\tilde{u}_0) < \tilde{\eta}_\varrho \leqslant \sigma(y_0) = \inf_{\gamma \in \Gamma} \max_{-1 \leqslant t \leqslant 1} \sigma(\gamma(t)) \quad (\text{see } (4.16)), \tag{4.17}$$

where $\Gamma = \{\gamma \in C([-1,1], W^{1,p}_{\text{per}}(0,b))\colon \gamma(-1) = \tilde{v}_0,\ \gamma(1) = \tilde{u}_0\}$ and

$$\sigma'(y_0) = 0. \tag{4.18}$$

From (4.17) we have $y_0 \notin \{\tilde{v}_0, \tilde{u}_0\}$, while from (4.18) and (4.14) we have $y_0 \in [\tilde{v}_0, \tilde{u}_0]$. So, if we show that $y_0 \neq 0$, then the extremality of $\tilde{u}_0$, $\tilde{v}_0$ implies that y_0 is nodal. According to (4.17), in order to establish the non-triviality of y_0 it suffices to produce a path $\gamma_* \in \Gamma$ such that $\sigma|_{\gamma_*} < 0 = \sigma(0)$.

To this end, let $M = W^{1,p}_{\mathrm{per}}(0,b) \cap \partial B_1^{L^p}$ furnished with the $W^{1,p}_{\mathrm{per}}(0,b)$-topology and let $M_{\mathrm{c}} = M \cap \hat{C}^1(T)$ furnished with the $\hat{C}^1(T)$-topology. Then, M_{c} is dense in M for the $W^{1,p}_{\mathrm{per}}(0,b)$-topology. We consider the two sets of paths

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1],M) \colon \hat{\gamma}(-1) = -\hat{u}_0,\ \hat{\gamma}(1) = \hat{u}_0\},$$
$$\hat{\Gamma}_{\mathrm{c}} = \{\hat{\gamma} \in C([-1,1],M_{\mathrm{c}}) \colon \hat{\gamma}(-1) = -\hat{u}_0,\ \hat{\gamma}(1) = \hat{u}_0\}.$$

Evidently, $\hat{\Gamma}_{\mathrm{c}}$ is dense in $\hat{\Gamma}$ for the $C([-1,1],M)$-topology. Hypothesis (H′) (iii) implies that we can find $\mu^* \in (\hat{\lambda}_1, \lambda^*)$ and $\delta_0 \in (0, \min\{\min_T |\tilde{v}_0|, \min_T \tilde{u}_0\})$ such that

$$\frac{\mu^*}{p}|x|^p \leqslant F(t,x) \quad \text{for a.a. } t \in T,\ \text{all } |x| \leqslant \delta_0. \tag{4.19}$$

The density of $\hat{\Gamma}_{\mathrm{c}}$ in $\hat{\Gamma}$ for the $C([-1,1],M)$-topology and Proposition 2.2 imply that we can find $\hat{\gamma} \in \hat{\Gamma}_{\mathrm{c}}$ such that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\hat{\gamma}(s)\right\|_p^p \leqslant \hat{\lambda}_1 + \varepsilon \quad \text{for all } s \in [-1,1],\ \text{with } \varepsilon \in (0, \mu^* - \hat{\lambda}_1). \tag{4.20}$$

Note that $\hat{\gamma}([-1,1]) \subseteq \hat{C}^1(T)$ is compact and recall that $\tilde{u}_0 \in \operatorname{int}\hat{C}_+$, $\tilde{v}_0 \in -\operatorname{int}\hat{C}_+$. So, we can find $\vartheta_0 \in (0,1)$ small such that

$$|\vartheta_0 u(t)| \leqslant \delta_0 \quad \text{for all } t \in T \qquad \text{and} \qquad \vartheta_0 u \in [\tilde{v}_0, \tilde{u}_0] \quad \text{for all } u \in \hat{\gamma}([-1,1]). \tag{4.21}$$

For any $u \in \hat{\gamma}([-1,1])$, we have that

$$\begin{aligned}
\sigma(\vartheta_0 u) &= \frac{\vartheta_0^p}{p}\|u'\|_p^p - \int_0^b F(t, \vartheta_0 u(t))\,\mathrm{d}t \\
&\qquad\qquad\qquad\qquad \text{(see (4.13), (4.21) and recall the choice of } \delta_0 > 0) \\
&\leqslant \frac{\vartheta_0^p}{p}[\hat{\lambda}_1 + \varepsilon - \mu^*] \quad \text{(see (4.20), (4.19) and recall that } \|u\|_p = 1) \\
&< 0 \qquad\qquad\qquad\quad \text{(see (4.20))}.
\end{aligned}$$

Let $\hat{\gamma}_0 = \vartheta_0\hat{\gamma}$. Then,

$$\sigma|_{\hat{\gamma}_0} < 0 \tag{4.22}$$

and the continuous path $\hat{\gamma}_0$ connects $-\vartheta_0\hat{u}_0$ and $\vartheta_0\hat{u}_0$.

Let $\alpha = \sigma_+(u_0) = \inf \sigma_+ < 0 = \sigma_+(0)$. Note that $K^{\alpha}_{\sigma_+} = \{u \in K_{\sigma_+} \colon \varphi(u) = \alpha\} = \{\tilde{u}_0\}$ (see (4.15)). Apply the second deformation theorem (see, for example, [**21**, p. 349] and [**23**, p. 3]) to produce a deformation $h \colon [0,1] \times (\sigma_+^0 \setminus \{0\}) \to \sigma_+^0$ such that $h(0,\cdot) = \mathrm{id}$ and

$$h(1, \sigma_+^0 \setminus \{0\}) = \{\tilde{u}_0\}, \tag{4.23}$$
$$\sigma_+(h(s,u)) \leqslant \sigma_+(h(\tau,u)) \quad \text{for all } s,\tau \in [0,1],\ \tau \leqslant s,\ u \in \sigma_+^0 \setminus \{0\}. \tag{4.24}$$

We set $\hat{\gamma}_+(s) = h(s, \vartheta\hat{u}_0)$, $s \in [0,1]$. Then,

$$\hat{\gamma}_+(0) = h(0, \vartheta\hat{u}_0) = \vartheta\hat{u}_0 \text{ and } \hat{\gamma}_+(1) = h(1, \vartheta\hat{u}_0) = \tilde{u}_0 \quad \text{(see (4.23))}$$
$$\implies \hat{\gamma}_+ \text{ is a continuous path connecting } \vartheta\hat{u}_0 \text{ and } \tilde{u}_0.$$

From (4.22) and (4.24), it follows that

$$\sigma_+|_{\hat{\gamma}_+} < 0. \tag{4.25}$$

For $u \in \hat{\gamma}_+([0,1])$, we have that

$$\begin{aligned}\sigma(u) &= \frac{1}{p}[\|u'\|_p^p + \xi_\varrho^*\|u\|_p^p] - \int_0^b (H(t,u^+) + H(t,-u^-))\,\mathrm{d}t \\ &= \sigma_+(u^+) - \int_0^b H(t,-u^-)\,\mathrm{d}t. \end{aligned} \tag{4.26}$$

From (4.13) and Hypothesis (H′) (v), $x = 0$ is a global minimizer of $x \to f(t,x) + (\xi_\varrho^*/p)|x|^p$ on $[-\varrho,\varrho]$ for a.a. $t \in T$. So,

$$\int_0^b H(t,-u^-)\,\mathrm{d}t \geqslant 0.$$

Hence,

$$\sigma(u) \leqslant \sigma_+(u^+) \quad \text{(see (4.26))}$$
$$\implies \sigma|_{\hat{\gamma}_+} < 0 \quad \text{(see (4.25) and recall that } \sigma_+(u) = \sigma_+(u^+)). \tag{4.27}$$

Similarly, we produce a continuous path $\hat{\gamma}_-$ that connects $-\vartheta\hat{u}_0$ and $\tilde{v}_0$ such that

$$\sigma|_{\hat{\gamma}_-} < 0. \tag{4.28}$$

We concatenate $\hat{\gamma}_-$, $\hat{\gamma}_0$, $\hat{\gamma}_+$ and produce $\gamma_* \in \Gamma$ such that

$$\sigma|_{\hat{\gamma}_*} < 0 \quad \text{(see (4.22), (4.27), (4.28))}$$
$$\implies y_0 \neq 0 \quad \text{and so } y_0 \in C^1(T) \text{ is a nodal solution of (1.1).}$$

□

So, summarizing, we have the following multiplicity theorem for (1.1).

Theorem 4.8. *If Hypotheses (H′) hold, then (1.1) has a smallest non-trivial solution* $u_0 \in \operatorname{int}\hat{C}_+$, *a biggest non-trivial solution* $v_0 \in -\operatorname{int}\hat{C}_+$ *such that*

$$c_- < v_0(t) < 0 < u_0(t) < c_+ \quad \textit{for all } t \in T,$$

at least two more solutions of constant sign $\hat{u} \in \operatorname{int}\hat{C}_+$, $\hat{v} \in -\operatorname{int}\hat{C}_+$ *such that*

$$u_0 \leqslant \hat{u}, \quad u_0 \neq \hat{u} \qquad \textit{and} \qquad \hat{v} \leqslant v_0, \quad \hat{v} \neq v_0,$$

and at least one nodal solution $y_0 \in C^1(T)$.

Acknowledgements. The authors thank the referee for corrections and remarks.

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