J. Austral. Math. Soc. (*Series A*) **58** (1995), 134–141

ON THE NUMBER OF SUBGROUPS IN FINITE SOLVABLE GROUPS

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(Received 5 October 1990; revised 10 March 1992)

Communicated by H. Lausch

Abstract

The main result of this paper is an upper bound for the number of maximal subgroups in finite solvable groups. Our result improves an earlier one of Cook, Wiegold and Williamson [1]. At the end, we use our bound to deduce an estimation for the total number of subgroups in finite solvable groups.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 D 10.

1. Complements of minimal normal subgroups

To obtain our bound for the number of maximal subgroups in finite solvable groups, we refer to a result of Gaschütz [2], and introduce some notation. Let U be a complemented minimal normal subgroup of a finite solvable group G. By a Jordan-Hölder type theorem (see [3]), the multiplicity of the isomorphism type of U as a complemented chief-factor does not depend on the given chief series of G. We shall denote this number by $\mathfrak{l}_G(U)$. Furthermore, we use $\overline{\mathfrak{c}_G}(U)$ to denote the number of conjugacy classes of complements of U in G.

THEOREM 1.1 (Gaschütz [1]). *Let U be a complemented minimal normal subgroup of a finite solvable group G. Let $E = \mathrm{End}_G(U)$ be its endomorphism ring and $l = \mathfrak{l}_G(U)$. Then*

$$\overline{\mathfrak{c}_G}(U) = |E|^{l-1}.$$

For our purposes, we only need the following easy consequence of Theorem 1.1.

The authors would like to thank the Deutsche Forschungsgemeinschaft and the Universities of Milano and Tel Aviv for their generous support.

https://doi.org/10.1017/S1446788700038155 Published online by Cambridge University Press

COROLLARY 1.2. *Let U be a complemented minimal normal p-subgroup of a finite solvable group G. Set $l = \mathfrak{l}_G(U)$, let $\mathfrak{c}_G(U)$ denote the number of complements of U in G and let $|G|_p$ be the p-part of $|G|$. Then*

(a) $\mathfrak{c}_G(U) \le |U|^l \le |G|_p$; *and*
(b) *if G is nilpotent, then* $\mathfrak{c}_G(U) \le (1/p) \cdot |G|_p$.

PROOF. (a) The second inequality is trivial. For the first one, observe that $E = \mathrm{End}_G(U)$ is a finite field, and we can consider U as an E-vector space of dimension d, say. Now $|U| = |E|^d \ge |E|$, and since $\mathfrak{c}_G(U) \le |U| \cdot \overline{\mathfrak{c}_G}(U)$, Assertion (a) follows from Theorem 1.1

(b) Since G is assumed to be nilpotent, we have $U \le \mathbf{Z}(G)$ and so $|E| = |U| = p$. By Theorem 1.1 we obtain

$$\mathfrak{c}_G(U) = \overline{\mathfrak{c}_G}(U) = |U|^{l-1} \le (1/p) \cdot |G|_p.$$

2. Counting maximal subgroups

We now apply the results of Section 1 to obtain an upper bound for the number $\mathfrak{m}(G)$ of maximal subgroups in a finite solvable group G. The proof of Theorem 2.1 turns out to be rather technical.

THEOREM 2.1. *Let G be a finite solvable group with Frattini subgroup Φ. We denote by p the largest prime divisor and by q the smallest prime divisor of $|G|$. Then*

$$\mathfrak{m}(G) \le (p|G/\Phi| - q)/(q(p-1)).$$

PROOF. We proceed by induction on $|G|$, reducing in turn to the situations described in Steps 1–5.

Step 1. $\Phi = 1$.

Proof. Since Φ is contained in every maximal subgroup of G, we have $\mathfrak{m}(G) = \mathfrak{m}(G/\Phi)$. As $\pi(G) = \pi(G/\Phi)$, p and q are still the largest and the smallest prime divisors (respectively) of $|G/\Phi|$. Now $\mathbf{\Phi}(G/\Phi) = 1$, and if $\Phi \ne 1$, induction yields

$$\mathfrak{m}(G) = \mathfrak{m}(G/\Phi) \le (p|G/\Phi| - q)/(q(p-1)),$$

the asserted inequality.

In the following, we fix a minimal normal subgroup U of G. Then every maximal subgroup H of G satisfies either $U \le H$, or $G = UH$ and $U \cap H = 1$. We shall therefore freely use that

$$\mathfrak{m}(G) = \mathfrak{m}(G/U) + \mathfrak{c}_G(U).$$

Step 2. $|G|$ is no prime power.

Proof. Otherwise $p = q$, and $|U| = p$. We apply both induction and Corollary 1.2(b) to deduce

$$\mathfrak{m}(G) = \mathfrak{m}(G/U) + \mathfrak{c}_G(U) \leq (|G/U| - 1)/(p-1) + |G|/p = (|G| - 1)/(p-1).$$

Since by Step 1, $\Phi = 1$, this is the desired bound.

Step 3. $|G| \neq pq$.

Proof. Suppose that $|G| = pq$. Since $p > q$, we may assume that $|U| = p$. Then $\mathfrak{m}(G) = \mathfrak{m}(G/U) + \mathfrak{c}_G(U) \leq 1 + p = (p|G| - q)/(q(p-1))$, and the conclusion of the theorem holds.

Let $|U|$ be an r-power for some prime r, and recall that $q \leq r \leq p$.

Step 4. $|G/U|$ is not a prime power.

Proof. Suppose first that $|G/U|$ is a q-group. Then $r = p > q$, by Step 2. We start considering the subcase where $|U| \geq p^2$. It then follows from Corollary 1.2(a) and induction that

$$\begin{aligned} q(p-1) \cdot \mathfrak{m}(G) &= q(p-1) \cdot \mathfrak{m}(G/U) + q(p-1) \cdot \mathfrak{c}_G(U) \\ &\leq q(p-1)(|G|_q - 1)/(q-1) + q(p-1)|G|_p. \end{aligned}$$

Since $p > q$ and $p^2 \,\Big|\, |G|$, this gives

$$\begin{aligned} q(p-1) \cdot \mathfrak{m}(G) &\leq q(p-1)|G|/(p^2(q-1)) - q(p-1)/(q-1) + (p-1)|G| \\ &\leq (p|G| - q) - (1 - 1/(q-1))|G| \\ &\leq p|G| - q, \end{aligned}$$

as required.

By Step 3, it is sufficient to handle secondly the subcase where $|U| = p$ and $|G/U| \geq q^2$. Again we obtain by Corollary 1.2(a) and induction that

$$\begin{aligned} q(p-1) \cdot \mathfrak{m}(G) &\leq q(p-1) \cdot \mathfrak{m}(G/U) + q(p-1) \cdot \mathfrak{c}_G(U) \\ &\leq q(p-1)(|G|_q - 1)/(q-1) + q(p-1)|G|_p \\ &\leq q(p-1)|G|/(p(q-1)) - q(p-1)/(q-1) + (p-1)|G|/q \\ &\leq (q^2 + p(q-1))(p-1)|G|/(pq(q-1)) - q. \end{aligned}$$

Observe now that

$$q^2 \leq q^2 + q - 2 = (q+2)(q-1) \leq (p+1)(q-1) \qquad \text{and}$$
$$(2p+1)(p-1) \leq (qp+1)(p-1) \leq qp^2.$$

Combining these inequalities with the estimate above, we get

$$q(p-1)\cdot \mathfrak{m}(G) \le p|G| - q.$$

To finish Step 4, we still have to consider the case where $|G/U|$ is a p-power. Then $r = q < p$ and

$$\begin{aligned} q(p-1)\cdot \mathfrak{m}(G) &\le q(p-1)\cdot \mathfrak{m}(G/U) + q(p-1)\cdot \mathfrak{c}_G(U) \\ &\le q(p-1)(|G|_p - 1)/(p-1) + q(p-1)|G|_q \\ &\le |G| - q + q(p-1)|G|/p \\ &\le p|G| - q. \end{aligned}$$

This completes Step 4.

From now on, we denote by p_0 and q_0 the largest and smallest prime divisors (respectively) of $|G/U|$.

Step 5. We have $r = q$.

Proof. Assume first that $q < r < p$. Then certainly $p_0 = p$, $q_0 = q$ and $|G|_r \le |G|/pq$. We thus obtain via Corollary 1.2(a) and induction

$$\begin{aligned} q(p-1)\cdot \mathfrak{m}(G) &\le q(p-1)\cdot \mathfrak{m}(G/U) + q(p-1)\cdot \mathfrak{c}_G(U) \\ &\le q(p-1)(p|G/U| - q)/(q(p-1)) + q(p-1)|G|_r \\ &\le p|G|/q - q + q(p-1)|G|/(pq) \\ &\le p|G| - q, \end{aligned}$$

as required.

Assume secondly that $q < r = p$. If $p_0 = p$, then

$$\begin{aligned} q(p-1)\cdot \mathfrak{m}(G) &\le q(p-1)\cdot \mathfrak{m}(G/U) + q(p-1)\cdot \mathfrak{c}_G(U) \\ &\le p|G|/p - q + q(p-1)|G|/q \\ &= p|G| - q. \end{aligned}$$

We are therefore left with the case where $p_0 < p$. Since by Step 4, $|G/U|$ is not a prime power, we actually have $2 \le q = q_0 < p_0 < p = r$. Now $|G|_p \le |G|/(p_0 q)$, and arguing as above, we find

$$\begin{aligned} q(p-1)\cdot \mathfrak{m}(G) &\le q(p-1)\cdot \mathfrak{m}(G/U) + q(p-1)\cdot \mathfrak{c}_G(U) \\ &\le q(p-1)\cdot (p_0|G|/p - q)/(q(p_0-1)) + q(p-1)|G|/(p_0 q) \\ &= ((p-1)p_0/((p_0-1)p) + (p-1)/p_0)|G| - (p-1)q/(p_0-1) \\ &\le ((p-1)/2 + (p-1)/3)|G| - q \\ &\le p|G| - q. \end{aligned}$$

https://doi.org/10.1017/S1446788700038155 Published online by Cambridge University Press

We have thus finished step 5 .

Step 6 . Conclusion.

Proof. By Step 5, we still have to deal with the case $q = r$.

Suppose first that $q_0 > q$, whence by Step 4, $r = q < q_0 < p_0 = p$. Since $|G|_q \leq |G|/(q_0 p)$, again Corollary 1.2(a) and induction yield

$$\begin{aligned} q(p-1)\cdot \mathfrak{m}(G) &\leq q(p-1)\cdot \mathfrak{m}(G/U) + q(p-1)\cdot \mathfrak{c}_G(U) \\ &\leq q(p-1)(p|G|/q - q_0)/(q_0(p-1)) + q(p-1)|G|/(q_0 p) \\ &\leq p|G|/q_0 - q + |G| \\ &\leq (p/3+1)|G| - q \\ &\leq p|G| - q. \end{aligned}$$

For the rest of the proof, we may assume that $r = q = q_0 < p_0 = p$. Again Corollary 1.2(a) and induction imply

$$\begin{aligned} q(p-1)\cdot \mathfrak{m}(G) &\leq q(p-1)\cdot \mathfrak{m}(G/U) + q(p-1)\cdot \mathfrak{c}_G(U) \\ &\leq p|G|/q - q + q(p-1)|G|/p \\ &= ((p^2 + q^2(p-1))/(pq))|G| - q \\ &\leq p|G| - q. \end{aligned}$$

To prove the last inequality, we need only show that $p^2 + q^2(p-1) \leq p^2 q$. This is equivalent to $p^2 - q^2 \leq pq(p-q)$, or $p + q \leq pq$, or $q \leq p(q-1)$, which is certainly true. The proof is complete.

Cook, Wiegold and Williamson [1] obtained the bound

$$\mathfrak{m}(G) \leq (|G| - 1)/(q-1),$$

where as above q denotes the smallest prime divisor of $|G|$. It is achieved if and only if G is an elementary abelian q-group. Our bound is better if G is not elementary abelian, as we shall see below at the end of this section. In fact it is achieved for certain $\{p, q\}$-groups.

EXAMPLE 2.2. (a) Let p and q be prime numbers such that $q \mid p-1$. Let $Q \cong Z_q$, and $P_i \cong Z_p$ for $i = 1, \dots, n$. Suppose that Q acts fixed-point-freely in the same way on each P_i, and consider the semidirect product $G = (P_1 \times \cdots \times P_n)Q$ with respect to this action. Then the Fitting subgroup $P_1 \times \cdots \times P_n$ is elementary abelian and homogeneous.

By Theorem 1.1, we have $\overline{\mathfrak{c}_G}(P_1) = p^{n-1}$, and since Q acts fixed-point-freely on P_1, it follows that $\mathfrak{c}_G(P_1) = p^n$. Thus

$$\mathfrak{m}(G) = \mathfrak{m}(G/P_1) + \mathfrak{c}_G(P_1) = \mathfrak{m}(G/P_1) + p^n,$$

and $G/P_1 \cong (P_2 \times \ldots \times P_n)Q$. Arguing by induction on n, we obtain

$$\mathfrak{m}(G) = 1 + p + p^2 + \ldots + p^n = (p^{n+1} - 1)/(p - 1) = (p|G| - q)/(q(p - 1)).$$

(b) Specializing further, we consider the case $q = 2$ and $p = 3$. Then by (a), $\mathfrak{m}(G) = (3^{n+1} - 1)/2 = (3|G| - 2)/4$. Indeed, the following general estimation holds.

COROLLARY 2.3. *Let q be the smallest prime divisor of $|G|$. If G is not an elementary abelian q-group, then*

$$\mathfrak{m}(G) \leq (q + 1)|G|/q^2 \leq 3|G|/4.$$

PROOF. Suppose that G is a q-group. Since G is not elementary abelian, we have $\Phi(G) \neq 1$, and Theorem 2.1 implies

$$\mathfrak{m}(G) \leq (|G|/q - 1)/(q - 1) \leq |G|/(q(q - 1)) \leq (q + 1)|G|/q^2 \leq 3|G|/4.$$

If on the other hand G is not a q-group, we denote by p the largest prime divisor of $|G|$. Then $p \geq q + 1$, and consequently $p/(p - 1) \leq (q + 1)/q$. It thus follows from Theorem 2.1 that

$$\mathfrak{m}(G) \leq (p|G| - q)/(q(p - 1)) \leq p|G|/(q(p - 1)) \leq (q + 1)|G|/q^2 \leq 3|G|/4.$$

If G is not elementary abelian, then certainly $|G| \geq q^2$ and then

$$(q + 1)|G|/q^2 \leq (|G| - 1)/(q - 1).$$

Thus Corollary 2.3 improves the bound given in [1].

3. Counting subgroups

In this section, we deduce a (rather crude) bound for the total number of subgroups in a finite solvable group G, using our estimations from Section 2 We shall actually count subgroup chains in G which cannot be refined, and start with an analytical lemma.

LEMMA 3.1. *Let $g \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$, defined by $f(r) = g^r/2^{r(r-1)/2}$. Then f is increasing on the interval* $(-\infty, \log_2 g]$.

https://doi.org/10.1017/S1446788700038155 Published online by Cambridge University Press

PROOF. Calculating the derivative f' of f, we obtain

$$\begin{aligned} f'(r) &= (\ln(g) \cdot g^r \cdot 2^{r(r-1)/2} - g^r \cdot \ln(2) \cdot (2r-1)/2 \cdot 2^{r(r-1)/2})/2^{r(r-1)} \\ &= (g^r/2^{r(r-1)/2}) \cdot (\ln(g) - \ln(2) \cdot (2r-1)/2). \end{aligned}$$

If $r \le \log_2 g$, it follows that

$$\begin{aligned} \ln(g) - \ln(2) \cdot (2r-1)/2 &\ge \ln(g) - \ln(2) \cdot (2 \cdot \log_2 g - 1)/2 \\ &= \ln(2) \cdot \log_2 g - \ln(2) \cdot \log_2 g + \ln(\sqrt{2}) \\ &= \ln(\sqrt{2}) > 0. \end{aligned}$$

Therefore, $f'(r) > 0$ for $r \in (-\infty, \log_2 g]$, and the claim holds.

THEOREM 3.2. *Let G be a finite solvable group. Then the number $\mathfrak{f}(G)$ of non-refinable subgroup chains of G satisfies*

$$\mathfrak{f}(G) \le (1/2) \cdot |G|^{(d+1)/2},$$

where $d = \log_2 |G|$.

In particular: The number of subgroups of G is bounded by

$$((d+1)/2) \cdot |G|^{(d+1)/2}.$$

PROOF. Since each subgroup of G is a member of a subgroup chain of G that cannot be refined, and since each such chain contains at most $d+1$ subgroups of G, the last assertion follows immediately from the bound for $\mathfrak{f}(G)$.

If $|G| \le 3$, then the theorem certainly holds. So suppose that $|G| \ge 4$ and let $1 = G_r < G_{r-1} < \cdots < G_1 < G_0 = G$ be a non-refinable subgroup chain of maximal length r. Then $r \le \log_2 |G| = d$ and $|G_i| \le |G|/2^i$. Once G_i has been chosen, there are exactly $\mathfrak{m}(G_i)$ possibilities to choose G_{i+1} in the next step. Since by Corollary 2.3,

$$\mathfrak{m}(G_i) \le |G_i| \le |G|/2^i,$$

and since $G_r = 1$, it follows that

$$\mathfrak{f}(G) \le \mathfrak{m}(G_0) \cdot \mathfrak{m}(G_1) \cdots \mathfrak{m}(G_{r-2}) \le |G|^{r-1}/2^{(r-1)(r-2)/2}.$$

By Lemma 3.1, the function $f(s) = |G|^s/2^{s(s-1)/2}$ is increasing on $(-\infty, d]$. Hence

$$\begin{aligned} \mathfrak{f}(G) &\le |G|^{d-1}/2^{(d-1)(d-2)/2} \\ &= (|G|/2^{(d/2-1)})^{d-1} \\ &= (2|G|^{1/2})^{d-1} \\ &= |G|/2 \cdot |G|^{(d-1)/2} \\ &= (1/2) \cdot |G|^{(d+1)/2}, \end{aligned}$$

as required.

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