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STOCHASTIC COMPARISON OF DISCOUNTED REWARDS

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Abstract

It is well know that the expected exponentially discounted total reward for a stochastic process can also be defined as the expected total undiscounted reward earned before an independent exponential stopping time (let us call this the stopped reward). Feinberg and Fei (2009) recently showed that the variance of the discounted reward is smaller than the variance of the stopped reward. We strengthen this result to show that the discounted reward is smaller than the stopped reward in the convex ordering sense.

Keywords: Total discounted reward; stopping time; stochastic ordering

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Consider the following definitions of Feinberg and Fei (2009). Let

$$J_1 = \int_0^\infty \mathrm{e}^{-\alpha t} r_t \,\mathrm{d}t + \sum_{n=1}^{\infty} \mathrm{e}^{-\alpha T_n} R_n,$$

where r_t is an $\mathcal{F}_t$-adapted stochastic process representing the reward rate at time t, $\mathcal{F}_t$ is an increasing filtration on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, T_n is an $\mathcal{F}_t$-adapted stopping time, and R_n is an $\mathcal{F}_{T_n}$-adapted stochastic sequence representing lump sum rewards, and let

$$J_2 = \int_0^T r_t \,\mathrm{d}t + \sum_{n=1}^{N(T)} R_n,$$

where T has an exponential distribution with rate α, independent of $\mathcal{F}_\infty$, and

$$N(t) = \sup\{n : T_n \le t\}$$

is the number of stopping times before T. Then J_1 represents the total discounted reward and J_2 represents the total undiscounted reward earned before an exponential time T. It is well known that $\mathrm{E}\, J_1 = \mathrm{E}\, J_2$, and Feinberg and Fei (2009) showed that $\mathrm{var}(J_1) \le \mathrm{var}(J_2)$. We show the following more general result, where, for two random variables X and Y, $X \le_{\mathrm{cx}} Y$ if and only if $\mathrm{E}\, f(X) \le \mathrm{E}\, f(Y)$ for all convex functions f.

Theorem 1. *It holds that* $J_2 \ge_{\mathrm{cx}} J_1$.

Proof. Let us make the dependency on T explicit, and write $J_2(T)$. We have

$$\mathrm{E}[J_2(T) \mid \mathcal{F}_\infty] = J_1 \mid \mathcal{F}_\infty,$$

so $J_2(T) \mid \mathcal{F}_\infty \ge_{\mathrm{cx}} J_1 \mid \mathcal{F}_\infty$, because, from Jensen's inequality, for any random variable X,

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$X \geq_{\text{cx}} \mathrm{E}\, X$. Therefore, for any convex function f,

$$\mathrm{E}[f(J_2(T)) \mid \mathcal{F}_\infty] \geq_{\text{cx}} \mathrm{E}[f(J_1) \mid \mathcal{F}_\infty],$$

so

$$\mathrm{E}[f(J_2(Y))] = \mathrm{E}[\mathrm{E}[f(J_2(T)) \mid \mathcal{F}_\infty]] \geq_{\text{cx}} \mathrm{E}[f(J_1) \mid \mathcal{F}_\infty] = \mathrm{E}[f(J_1)],$$

and, therefore, $J_2(T) \geq_{\text{cx}} J_1$.

Note that the argument goes through with arbitrary discounting, not necessarily exponential discounting. That is, if the reward rate at time t is $\alpha(t)$, where $\alpha(0) = 1$ and $\lim_{t\to\infty} \alpha(t) = 0$, so

$$J_1 = \int_0^\infty \alpha(t) r_t \, \mathrm{d}t + \sum_{n=1}^\infty \alpha(T_n) R_n,$$

then, again, $J_2(T) \geq_{\text{cx}} J_1$, where $J_2(T)$ is as defined above with T having tail distribution α: $\alpha(t) = \mathrm{P}\{T > t\}$.

Reference

Feinberg, E. A. and Fei, J. (2009). An inequality for variances of the discounted rewards. *J. Appl. Prob.* **46,** 1209–1212.