Glasgow Math. J. **45** (2003) 53–67. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089502008984. Printed in the United Kingdom

INVARIANT SUBSPACES FOR SOME FAMILIES OF UNBOUNDED SUBNORMAL OPERATORS

E. ALBRECHT

Fachrichtung 6.1 – Mathematik, Universität des Saarlandes, Postfach 15 11 50, D-66041 Saarbrücken, Germany
e-mail: ernstalb@math.uni-sb.de

and F.-H. VASILESCU

U.F.R. de Mathématiques, Université des Sciences et Technologies de Lille, U.M.R. du C.N.R.S. 8524, 59655 Villeneuve d'Ascq Cedex, France
e-mail: fhvasil@gat.univ-lillel.fr

(Received 29 May, 2001; accepted 22 March, 2002)

Abstract. Extending results and methods of Thomson and Trent, we prove the existence of non trivial quasi-invariant subspaces for subnormal families of unbounded operators having sufficiently rich domains. In some special cases, proper invariant subspaces are obtained.

2000 *Mathematics Subject Classification.* 47A15, 47B20.

1. Preliminaries. The problem of existence of proper invariant subspaces for arbitrary bounded linear operators has been and still is an obsessive question for operator theorists. While the counterexamples of Enflo and Read have completely settled this problem in the frame of general Banach spaces, the Hilbert space case, as well as that of some particular Banach spaces, offer hopes for a positive answer to some optimistic scholars.

Although the case of bounded operators remains a permanent temptation, we shall try in the following to turn the discussion to some classes of unbounded operators. Also, because the Scott Brown technique [**3**] seems to be very much related to bounded operators, we shall exploit the resources of Thomson's and Trent's techniques [**10**], [**12**] to get some information concerning the existence of invariant subspaces for the unbounded ones. This starting point will force us to restrict ourselves to some families of (unbounded) subnormal operators in Hilbert spaces. Even for the concept of "invariant subspace" it turns out that one has to formulate (at least) two possible definitions.

Let $\mathcal{H}$ be a complex Hilbert space and let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a closed, densely defined linear operator. Also, let $\mathcal{L} \subset \mathcal{H}$ be a closed linear subspace.

Definition 1. We say that $\mathcal{L}$ is *invariant* under T if $D_0(T;\mathcal{L}) := D(T) \cap \mathcal{L}$ is dense in $\mathcal{L}$ and $TD_0(T;\mathcal{L}) \subset \mathcal{L}$.

This work has been prepared while the second named author was a guest of the University of the Saarland as a Humboldt Fellow. He would like to express his gratitude to the Alexander von Humboldt-Foundation for its support and to the University of the Saarland for its hospitality.

We say that $\mathcal{L}$ is *quasi-invariant* under T if the linear subspace

$$D(T;\mathcal{L}) := \{x \in D_0(T;\mathcal{L});\, Tx \in \mathcal{L}\}$$

is dense in $\mathcal{L}$.

If $\mathcal{T}$ is an arbitrary family of linear operators in $\mathcal{H}$, a closed subspace $\mathcal{L} \subset H$ is said to be *(quasi-)invariant* under $\mathcal{T}$ if $\mathcal{L}$ is (quasi-)invariant under each $T \in \mathcal{T}$.

Clearly, every invariant subspace under T is quasi-invariant, and the two notions coincide when T is bounded. Moreover, if $\mathcal{L}$ is invariant (resp. quasi-invariant) under T, then the operator $T : D_0(T;\mathcal{L}) \to \mathcal{L}$ (resp. $T : D(T;\mathcal{L}) \to \mathcal{L}$) is closed.

EXAMPLE 1. Let $N : D(N) \subset \mathcal{H} \to \mathcal{H}$ be a normal operator and let E be the spectral measure of N. Then, for every Borel set $B \subset \mathbb{C}$, the subspace $\mathcal{L} := E(B)\mathcal{H}$ is invariant under N. Indeed, if $x = E(B)x \in D(N) \cap \mathcal{L}$ then $Nx = E(B)Nx \in \mathcal{L}$.

2. Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a closed operator and let $\mathcal{D} \subset D(T)$ be a linear subspace such that $T\mathcal{D} \subset \mathcal{D}$. If $\mathcal{L} := \bar{\mathcal{D}}$, then $\mathcal{L}$ is quasi-invariant. In particular, if there exists a non-null vector $x_0 \in \cap_{k\geq 0} D(T^k)$, then the closure of the subspace generated by the set $\{T^k x_0; k \geq 0\}$ is quasi-invariant under T.

3. We shall show that there are quasi-invariant subspaces which are not invariant. Let $\mathcal{H} := L^2[0, 1]$ and let $\mathcal{H}_0 := \{f \in \mathcal{H}; \int_0^1 f(s)\,ds = 0\}$. We consider the derivation operator $Tf = f'$, defined on the space $D(T) := \{f \in \mathcal{H};\ f' \in \mathcal{H}\}$. T is a closed operator. We shall show that $\mathcal{H}_0$ is quasi-invariant but it is not invariant.

First of all, let us show that the subspace $D(T;\mathcal{H}_0) = \{f \in D(T) \cap \mathcal{H}_0;\ Tf \in \mathcal{H}_0\}$ is dense in $\mathcal{H}_0$. Indeed, since $C_0^\infty(0, 1)$ is dense in $\mathcal{H}$, we can choose a sequence $(f_k)_{k\geq 1}$ in $C_0^\infty(0, 1)$ convergent to a given $f \in \mathcal{H}_0$. Then the sequence $g_k := f_k - \int_0^1 f_k(s)\,ds$, $k \geq 1$, is also convergent to f. Moreover, $g_k' = f_k' \in C_0^\infty(0, 1)$, and $\int_0^1 f_k'(s)\,ds = 0$ for all k. Therefore,

$$D(T;\mathcal{H}_0) \supset \left(C_0^\infty(0, 1) + \mathbb{C}\right) \cap \mathcal{H}_0,$$

and the latter is dense in $\mathcal{H}_0$ as observed before. Consequently, $\mathcal{H}_0$ is quasi-invariant.

On the other hand, the function $f(s) = 1/2 - s$ belongs to $\mathcal{H}_0 \cap D(T)$ but $Tf(s) = -1$ is not in $\mathcal{H}_0$.

Now let $\mathcal{S}$ be a family of densely defined operators in a Hilbert space $\mathcal{H}$. The family $\mathcal{S}$ is said to be *subnormal* **[9]** if there exists a Hilbert space $\mathcal{K} \supset H$ and a family $\mathcal{N}$ consisting of commuting normal operators in $\mathcal{K}$ such that for every $S \in \mathcal{S}$ there is some $N \in \mathcal{N}$ with $S \subset N$. In this case $\mathcal{N}$ is said to be a *normal extension* of $\mathcal{S}$. From this definition it follows, in particular, that the space $\mathcal{H}$ is quasi-invariant under each $N \in \mathcal{N}$.

With $\mathcal{S}$ and $\mathcal{N}$ as above, note that the family $\bar{\mathcal{S}}$ of the closures $\bar{S}$ of all $S \in \mathcal{S}$ is also subnormal, and that $\mathcal{N}$ is a normal extension of $\bar{\mathcal{S}}$.

The aim of this paper is to prove the existence of (quasi-)invariant subspaces for some subnormal families of (not necessarily bounded) operators. We shall mainly use the techniques developed in **[10]**, **[12]** and **[13]** for subnormal bounded operators, respectively for subnormal tuples of bounded operators, adapted to our conditions. We only remark that the Cauchy transform methods for a compactly supported measure (see **[4]**, **[10]**, **[12]**, **[13]**) can be extended to the larger class of finite measures, and that positive measures on $\mathbb{C}^n$ having finite moments of all orders provide interesting examples of subnormal tuples, which, in particular, have (quasi-)invariant subspaces. Nevertheless, examples related to not necessarily finite measures also exist (see the last section).

2. Subalgebras of the algebra $L^\omega(\mu)$ of R. Arens. Throughout this text, if not otherwise specified, let Ω be a locally compact Hausdorff space and let μ be a positive, finite Borel measure on Ω. As in [**1**] we form the space $L^\omega(\mu) := \bigcap_{p\geq 1} L^p(\mu)$. Endowed with the topology given by the family $(\|\cdot\|_p)_{p\geq 1}$ of all L^p-norms on $L^\omega(\mu)$, $(1 \leq p < \infty)$, this is a complete metrizable locally convex topological vector space which is actually a commutative topological algebra. This follows from the generalized Hölder inequality

$$\|fg\|_r \leq \|f\|_p \|g\|_q, \quad f, g \in L^\omega(\mu),$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. As $L^\omega(\mu) = \operatorname{proj}_{\infty\leftarrow p} L^p(\mu)$ is a reduced projective limit, its dual space may be identified (c.f. [**6**, Satz 1.6, p. 143]) with

$$L^{1+}(\mu) := \bigcup_{p>1} L^p(\mu) = \operatorname*{ind}_{p\to 1} L^p(\mu),$$

the duality being given by $\langle g, f\rangle := \int_\Omega f(w)g(w)\,d\mu, f \in L^\omega(\mu), g \in L^{1+}(\mu)$. Conversely, by [**6**, Satz 1.2, p. 142] we have (up to canonical topological isomorphisms) $L^{1+}(\mu)^* = L^\omega(\mu)$. Thus $L^\omega(\mu)$ is reflexive. Obviously, we have the following inclusions (with dense ranges)

$$L^\infty(\mu) \subset L^\omega(\mu) \subset L^p(\mu) \subset L^{1+}(\mu) \subset L^1(\mu).$$

Note, that all these inclusions can be strict (see [**1**] for the case of the Lebesgue measure on the interval [0, 1]).

Let now $\mathcal{A}$ be a subalgebra of $L^\omega(\mu)$ of dimension ≥ 2, containing the constant functions, and let $\mathcal{A}^p(\mu)$ denote the closure of $\mathcal{A}$ in $L^p(\mu)$. The closure $\mathcal{A}^\omega(\mu)$ of $\mathcal{A}$ in $L^\omega(\mu)$ is then a subalgebra of $L^\omega(\mu)$ that actually coincides with $\bigcap_{p>1} \mathcal{A}^p(\mu)$. For $a \in \mathcal{A}^\omega(\mu)$, let N_a and M_a denote the operators of multiplication by a in $L^2(\mu)$ and $\mathcal{A}^2(\mu)$, respectively, with the domains $D(N_a) := \{f \in L^2(\mu); af \in L^2(\mu)\}$ and $D(M_a; \mathcal{A}^2(\mu)) := \{f \in \mathcal{A}^2(\mu); af \in \mathcal{A}^2(\mu)\}$. Thus, the operators M_a are subnormal and $\mathcal{A}^2(\mu)$ is a quasi-invariant subspace for all M_a with $a \in \mathcal{A}^\omega(\mu)$. One of our aims will be to find joint quasi-invariant subspaces for the family of operators $\mathfrak{M}(\mathcal{A}^\omega(\mu)) := \{M_a; a \in \mathcal{A}^\omega(\mu)\}$.

Lemma 2. *Assume that $\mathcal{A}$ is an algebra of Borel measurable functions (with pointwise multiplication) that is contained in $L^1(\mu)$. Then $\mathcal{A}$ is a subalgebra of $L^\omega(\mu)$.*

Proof. Fix an arbitrary $p \in [1, \infty)$ and let $m > p$ be an integer. If $a \in \mathcal{A}$, then $a^m \in \mathcal{A} \subset L^1(\mu)$; i.e. $a \in L^m(\mu) \subset L^p(\mu)$. □

Let us mention some examples of subalgebras of the Arens algebra in the special case that $\Omega = \mathbb{C}^n$. These are of particular interest.

Examples. Let μ be a finite positive Borel measure on $\mathbb{C}^n$ such that the complex algebra $\mathcal{P}_a = \mathcal{P}_{a,n}$ of all analytic polynomials is contained in $L^1(\mu)$. Then $\mathcal{P}_a$ and the algebra $\mathcal{R}(\mu)$ of all analytic rational functions without singularities in the support of μ are subalgebras of $L^\omega(\mu)$.

The next result is a strengthened form of [**4**, Lemma V.4.3].

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

LEMMA 3. *Let $\mathcal{A}$ be a subalgebra of $L^\omega(\mu)$. Let $k \geq 1$ be an integer, let $p = 2k+1$, and let $q = (2k+1)/(2k)$. Then, for each $h \in L^q(\mu)$ such that*

$$\|h\|_q = \sup\left\{\left|\int_\Omega fh\,d\mu\right|;\ f \in \mathcal{A}^p(\mu),\ \|f\|_p \leq 1\right\},$$

we can find $v \in \mathcal{A}^{2+1/k}(\mu)$ such that $|h| = |v|^2\mu$-a.e.

Proof. Without loss of generality we may assume $\|h\|_q = 1$. As the unit ball of $\mathcal{A}^p(\mu)$ is weakly compact, there exists some $u \in \mathcal{A}^p(\mu)$ such that $\|u\|_p = 1$ and $1 = \int uh\,d\mu \leq \|u\|_p\|h\|_q = 1$. Because of the Hölder inequality, we obtain $|u|^p = |h|^q\mu$-a.e., which implies $|u|^{2k} = |h|\mu$-a.e. Put $v := u^k \in L^{2+1/k}(\mu)$. Let $(a_m)_{m\geq 1}$ be a sequence in $\mathcal{A}$ converging to u in $L^p(\mu)$. Fix $C > 0$ with $\sup_m \|a_m\|_p \leq C$. If α, β are nonnegative integers such that $\alpha + \beta = k-1$, then $1/(2+1/k) = 1/(2k+1) + \alpha/(2k+1) + \beta/(2k+1)$. From the general Hölder inequality we obtain

$$\begin{aligned}\left\|(u-a_m)u^\alpha a_m^\beta\right\|_{2+1/k} &\leq \|u-a_m\|_p\|u^\alpha\|_{(2k+1)/\alpha}\left\|a_m^b\right\|_{(2k+1)/\beta}\\ &\leq \|u-a_m\|_p\|u\|_p^\alpha\left\|a_m^\beta\right\|_p \leq \|u-a_m\|_p\|u\|_p^\alpha C^\beta,\end{aligned}$$

and the last term tends to zero as $m \to \infty$. Hence

$$\left\|u^k - a_m^k\right\|_{2+1/k} = \left\|(u-a_m)\sum_{\alpha+\beta=k-1} u^\alpha a_m^\beta\right\|_{2+1/k} \to 0$$

as $m \to \infty$, showing that $u^k \in \mathcal{A}^{2+1/k}(\mu)$. □

3. Some auxiliary results. In this section we develop the necessary function theoretic machinery, following the corresponding results from [**10**], [**12**], and [**13**]; (see also [**4**]).

Let $n \geq 1$ be a fixed integer. We denote by λ_n the Lebesgue measure in the complex Euclidean space $\mathbb{C}^n$.

Let ν be a finite complex measure on $\mathbb{C}^n$. For arbitrary $w = (w_1, \ldots, w_n), z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set

$$\tilde{\nu}(w) := \int_{\mathbb{C}^n}\prod_{j=1}^n |z_j - w_j|^{-1}\,d|\nu|(z).$$

It is easily seen that the assignment $w \to \tilde{\nu}(w)$ is a function defined λ_n-a.e. in $\mathbb{C}^n$ that is, moreover, locally integrable. Indeed, if $r > 0$ and if we set

$$\prod(r) := \{z \in \mathbb{C}^n; |z_j| \leq r, j = 1, \ldots, n\},$$

then we have

$$\int_{\Pi(r)}\tilde{\nu}(w)\,d\lambda_n(w) = \int_{\mathbb{C}^n}\left(\prod_{j=1}^n\int_{|w_j|\leq r}\frac{d\lambda_1(w_j)}{|z_j - w_j|}\right)d|\nu|(z) \leq C_r|\nu|(\mathbb{C}^n) < \infty,$$

where $C_r > 0$ depends only on r (see [**4**, Lemma V.2.1]). As $r > 0$ is arbitrary, we readily infer the assertion.

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

This shows, in particular, that the function

$$\hat{\nu}(w) := \int_{\mathbb{C}^n} \prod_{j=1}^{n} (z_j - w_j)^{-1} \, d\nu(z),$$

which may be called the *Cauchy transform of* ν (see also **[4]**, **[13]** etc.), is defined λ_n-a.e. in $\mathbb{C}^n$ and it is locally integrable. Therefore, we may regard $\hat{\nu}$ as a distribution.

The following two lemmas are almost proved in **[13]**. We give them here for the convenience of the reader.

LEMMA 4. *We have the equality*

$$\frac{\partial^n \hat{\nu}}{\partial \bar{z}_1 \cdots \partial \bar{z}_n} = (-\pi)^n \nu$$

in the sense of the theory of distributions.

Proof. If $\phi \in C_0^\infty(\mathbb{C}^n)$ is arbitrary, we have

$$\phi(w) = \left(-\frac{1}{\pi}\right)^n \int \frac{\partial^n \phi(z)}{\partial \bar{z}_1 \cdots \partial \bar{z}_n} \frac{d\lambda_n(z)}{(z_1 - w_1) \cdots (z_n - w_n)},$$

via a well-known classical representation formula. Therefore,

$$\begin{aligned}
\left(\frac{\partial^n \hat{\nu}}{\partial \bar{z}_1 \cdots \partial \bar{z}_n}\right)(\phi) &= (-1)^n \int_{\mathbb{C}^n} \hat{\nu}(w) \frac{\partial^n \phi}{\partial \bar{w}_1 \cdots \partial \bar{w}_n} \, d\lambda_n(w) \\
&= (-1)^n \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} \prod_{j=1}^{n} (z_j - w_j)^{-1} \frac{\partial^n \phi}{\partial \bar{w}_1 \cdots \partial \bar{w}_n} \, d\lambda_n(w) \right) d\nu(z) \\
&= (-\pi)^n \int_{\mathbb{C}^n} \phi(z) \, d\nu(z).
\end{aligned}$$

□

LEMMA 5. *Let* μ *be a positive finite measure on* $\mathbb{C}^n$ *and take* $g \in L^q(\mu)$*, with* $1 \leq q < 2$*. Then the function* $z \to (z_1 - w_1)^{-1} \cdots (z_n - w_n)^{-1} g(z)$ *is in* $L^q(\mu)$ *except for* w *in a* λ_n*-null set.*

Proof. If $r > 0$ is fixed, for all $z \in \Pi(r)$, we have

$$\int_{\Pi(r)} |w_1 - z_1|^{-q} \cdots |w_n - z_n|^{-q} \, d\lambda_n(w) \leq \int_{\Pi(2r)} |u_1 \cdots u_n|^{-q} \, d\lambda_n(u) = C_{q,r}$$

with $C_{q,r} = (2\pi/(2-q))^n (2r)^{(2-q)^n}$. Hence

$$\begin{aligned}
&\int_{\Pi(r)} \left(\int_{\mathbb{C}^n} |w_1 - z_1|^{-q} \cdots |w_n - z_n|^{-q} |g(z)|^q \, d\mu(z) \right) d\lambda_n(w) \\
&\quad \leq C_{q,r} \int_{\mathbb{C}^n} |g(z)|^q \, d\mu(z) < \infty.
\end{aligned}$$

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

Hence, for each $r > 0$, the function

$$w \to \int_{\mathbb{C}^n} |w_1 - z_1|^{-q} \cdots |w_n - z_n|^{-q} |g(z)|^q \, d\mu(z)$$

is integrable on $\Pi(r)$, and therefore has finite values except for w in a λ_n-null set $\mathcal{N}_r \subset \Pi(r)$. As $\mathcal{N} := \bigcup_{m=1}^{\infty} \mathcal{N}_m$ is a λ_n-null set, our assertion holds for all $w \in \mathbb{C}^n \backslash \mathcal{N}$. □

The next result is a version of a theorem of Brennan; (see [**2**]).

LEMMA 6. *Let μ be a finite positive Borel measure on $\mathbb{C}^n$ such that $\mathcal{P}_a \subset L^1(\mu)$. For a fixed $w \in \mathbb{C}^n$ let us denote by $\mathcal{F}_w$ the linear space of all polynomials $P \in \mathcal{P}_a$ having the form $P(z) = c_0 + \prod_{j=1}^{n}(z_j - w_j)Q(z)$, $z \in \mathbb{C}^n$, $c_0 \in \mathbb{C}$, $Q \in \mathcal{P}_a$.*

If $p > 2$ and $\mathcal{P}_a^p(\mu) \neq L^p(\mu)$, then there exists some point $w \in \mathbb{C}^n$ such that the linear map $\mathcal{F}_w \ni P \to P(w) \in \mathbb{C}$ is continuous with respect to the norm of $L^p(\mu)$.

Proof. Since $\mathcal{P}_a^p(\mu) \neq L^p(\mu)$, there exists a function $g \in L^q(\mu) \backslash \{0\}$ $(1/p + 1/q = 1)$ such that $\int P g \, d\mu = 0$ for all $P \in \mathcal{P}_a$. We clearly have $g \in L^1(\mu)$. Therefore, the measure $\mu_g := g\mu$ is finite on $\mathbb{C}^n$. It follows from Lemma 4 that $\widehat{\mu_g} \neq 0$ on a set E of positive Lebesgue measure. The equality

$$\widehat{\mu_g}(w) = \int_{\mathbb{C}^n} \prod_{j=1}^{n} (z_j - w_j)^{-1} g(z) \, d\mu(z)$$

and Lemma 5 allow us to find a point $w \in E$ such that the function given by $z \to \prod_{j=1}^{n} (z_j - w_j)^{-1} g(z)$ is in $L^q(\mu)$.

Now, let $P \in \mathcal{F}_w$ be fixed. Therefore $P(z) = P(w) + \prod_{j=1}^{n}(z_j - w_j)Q(z)$ for some polynomial Q. By virtue of the choice of g,

$$\int_{\mathbb{C}^n} (P(z) - P(w)) \prod_{j=1}^{n} (z_j - w_j)^{-1} g(z) \, d\mu(z) = 0,$$

which implies that

$$e_w(P) := P(w) = \frac{1}{\widehat{\mu_g}(w)} \int_{\mathbb{C}^n} P(z) h_w(z) \, d\mu(z),$$

where $h_w(z) := \prod_{j=1}^{n} (z_j - w_j)^{-1} g(z)$. Moreover,

$$|e_w(P)| \leq \frac{1}{|\widehat{\mu_g}(w)|} \|P\|_p \|h_w\|_q,$$

which is precisely our assertion. □

In the framework of Section 2, a general Cauchy transform is not available. Therefore, as for the bounded case in [**12**], we shall use a localized version.

Let ν be a finite, complex Borel measure on the locally compact Hausdorff space Ω and let a be a Borel measurable function on Ω that is integrable with respect to $|\nu|$. We show now that, for λ_1-a.e. $z \in \mathbb{C}$, the expression

$$\tilde{\nu}[a](z) := \int_{\Omega} \frac{1}{|z - a(w)|} \, d|\nu|(w)$$

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

is finite. Indeed, if $r > 0$ is arbitrarily given and if we set

$$A_{2r} := \{w \in \Omega; |a(w)| \leq 2r\}, \quad D(0,r) := \{z \in \mathbb{C}; |z| \leq r\}, \tag{1}$$

then using the Fubini theorem and Proposition V.2.2 in [**4**], we have

$$\begin{aligned}
\int_{D(0,r)} \tilde{\nu}[a](z)\, d\lambda_1(z) &= \int_{\Omega\setminus A_{2r}} \int_{D(0,r)} \frac{1}{|z-a(w)|}\, d\lambda_1(z)\, d|\nu|(w) \\
&\quad + \int_{A_{2r}} \int_{D(0,r)} \frac{1}{|z-a(w)|}\, d\lambda_1(z)\, d|\nu|(w) \\
&\leq \int_{\Omega\setminus A_{2r}} \int_{D(0,r)} \frac{1}{r}\, d\lambda_1(z)\, d|\nu|(w) \\
&\quad + \int_{A_{2r}} \int_{D(a(w),2r)} \frac{1}{|z-a(w)|}\, d\lambda_1(z)\, d|\nu|(w) \\
&\leq 5\pi r |\nu|(\Omega).
\end{aligned}$$

As r is arbitrary, we obtain the assertion.

This fact shows that the function

$$z \mapsto \hat{\nu}[a](z) := \int_\Omega \frac{1}{z-a(w)}\, d\nu(w),$$

which may be called the *a-Cauchy transform of* ν, is defined λ_1-a.e. and is locally integrable, and so can be considered as a distribution on $\mathbb{C}$. The following will be our replacement for Lemma 4.

LEMMA 7. *For all* $\varphi \in C_0^\infty(\mathbb{C})$ *we have the equality*

$$\left\langle \frac{\partial \hat{\nu}[a]}{\partial \bar{z}}, \varphi \right\rangle = \pi \int_\Omega \varphi(a(w))\, d\nu(w).$$

Proof. Indeed, by the classical integral representation formula we obtain

$$\begin{aligned}
\left\langle \frac{\partial \hat{\nu}[a]}{\partial \bar{z}}, \varphi \right\rangle &= -\int_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{z}}(z) \int_\Omega \frac{1}{z-a(w)}\, d\nu(w)\, d\lambda_1(z) \\
&= \pi \int_\Omega \frac{-1}{\pi} \int_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{z}}(z) \frac{1}{z-a(w)}\, d\lambda_1(z)\, d\nu(w) \\
&= \pi \int_\Omega \varphi(a(w))\, d\nu(w).
\end{aligned}$$

□

In the next section we shall also need the following fact.

LEMMA 8. *Let* μ *be a positive finite Borel measure on* Ω *and let a be in* $L^\omega(\mu)$ *and* $g \in L^2(\mu)$. *Then the function*

$$w \mapsto \frac{g(w)}{z-a(w)}$$

is in $L^{3/2}(\mu)$ *for* λ_1*-a.e.* $z \in \mathbb{C}$.

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

Proof. With the notation of (1) we have for all $r > 1$,

$$\begin{aligned}
&\int_{D(0,r)} \int_{\Omega} \frac{|g(w)|^{3/2}}{|z-a(w)|^{3/2}}\, d\mu(w)\, d\lambda_1(z) \\
&\quad = \int_{\Omega\setminus A_{2r}} \int_{D(0,r)} \frac{|g(w)|^{3/2}}{|z-a(w)|^{3/2}}\, d\lambda_1(z)\, d\mu(w) \\
&\qquad + \int_{A_{2r}} |g(w)|^{3/2} \int_{D(0,r)} \frac{1}{|z-a(w)|^{3/2}}\, d\lambda_1(z)\, d\mu(w) \\
&\quad \le \pi r^{1/2} \int_{\Omega\setminus A_{2r}} |g(w)|^{3/2}\, d\mu(w) \\
&\qquad + \int_{A_{2r}} |g(w)|^{3/2} \int_{D(a(w),4r)} \frac{1}{|z-a(w)|^{3/2}}\, d\lambda_1(z)\, d\mu(w) \\
&\quad \le \int_{\Omega\setminus A_{2r}} |g(w)|^{3/2}\, d\mu(w) + 2\pi \int_{A_{2r}} |g(w)|^{3/2} \int_0^{4r} \rho^{-1/2}\, d\rho\, d\mu(w) \\
&\quad \le 9\pi r^{1/2} \|g\|_{3/2}^{3/2}.
\end{aligned}$$

Our statement now follows from Fubini's theorem. □

4. Existence of quasi-invariant subspaces. As before, Ω will denote a locally compact Hausdorff space and μ a positive finite Borel measure on Ω. We are now ready to prove our next result.

THEOREM 9. *Let $\mathcal{A}$ be a subalgebra of the Arens algebra $L^\omega(\mu)$ having dimension at least 2. Then the multiplication operators M_a, $a \in \mathcal{A}^\omega(\mu)$, have a proper quasi-invariant subspace in $\mathcal{A}^2(\mu)$.*

Proof. (a) Assume first that for all $a, b \in \mathcal{A}^\omega(\mu)$ and all $g \in \mathcal{A}^2(\mu)^\perp$ the function $z \mapsto \int_\Omega \overline{g(w)} b(w)/(z-a(w))\, d\mu(w)$ vanishes for λ_1-a.e. $z \in \mathbb{C}$. Because of $\dim \mathcal{A} \ge 2$, the algebra $\mathcal{A}$ contains a function a which is not constant μ-a.e. Hence, there exist two points z_1, z_2 in the essential range of a with $z_1 \ne z_2$. Fix two disjoint open sets $U, V \subset \mathbb{C}$ with $z_1 \in U$, $z_2 \in V$. Then $\mu(a^{-1}(U)) \ne 0$ and $\mu(a^{-1}(V)) \ne 0$ by the definition of the essential range. Let now $(\varphi_m)_{m\ge 1}$, $(\psi_m)_{m \ge 1}$ be two sequences in $C_0^\infty(\mathbb{C})$ such that $0 \le \varphi_m(x) \nearrow \chi_U$ and $0 \le \psi_m(x) \nearrow \chi_V$ pointwise on $\mathbb{C}$ for $m \to \infty$. Then $0 \le \varphi_m(a(w)) \nearrow \chi_A$ and $0 \le \psi_m(a(w)) \nearrow \chi_B$ pointwise on Ω, where $A := a^{-1}(U)$, $B := a^{-1}(V)$ are disjoint. It follows from our assumption and Lemma 7 applied to the complex measure $\nu := b\bar{g}\mu$ that

$$0 = \int_\Omega \varphi_m(a(w)) \overline{g(w)} b(w)\, d\mu(w) \to \int_\Omega \chi_A(w) \overline{g(w)} b(w)\, d\mu(w) \qquad \text{as } m \to \infty,$$

for all $g \in \mathcal{A}^2(\mu)^\perp$, $b \in \mathcal{A}^\omega(\mu)$. Hence, $\chi_A b \in \mathcal{A}^2(\mu)$, for all $b \in \mathcal{A}^\omega(\mu)$, and we see that $\chi_A \mathcal{A}^\omega(\mu) \subset D(M_c; \mathcal{A}^2(\mu))$, for all $c \in \mathcal{A}^\omega(\mu)$. Hence, the closure $\mathcal{M}_1$ of $\chi_A \mathcal{A}^\omega(\mu)$ in $\mathcal{A}^2(\mu)$ is a common quasi-invariant subspace for $\{M_c; c \in \mathcal{A}^\omega(\mu)\}$ containing the non-zero element χ_A. In the same way one obtains that the closure $\mathcal{M}_2$ of $\chi_B \mathcal{A}^\omega(\mu)$ in $\mathcal{A}^2(\mu)$ is a common quasi-invariant subspace for $\{M_c; c \in \mathcal{A}^\omega(\mu)\}$ containing the non-zero element χ_B. Since the two spaces are orthogonal, they must be proper quasi-invariant subspaces for $\{M_c; c \in \mathcal{A}^\omega(\mu)\}$.

(b) Therefore we may now assume that there exist some $a, b \in \mathcal{A}^\omega(\mu)$ and some $g \in \mathcal{A}^2(\mu)^\perp$ such that

$$\int_\Omega \frac{\overline{g(w)}b(w)}{z - a(w)}\, d\mu(w) \neq 0, \quad \text{for all } z \in E,$$

where E is not a λ_1-null set. In particular, a cannot be a constant function. Because of Lemma 8, we may also assume that $w \mapsto \frac{\overline{g(w)}}{z - a(w)} \in L^{3/2}(\mu)$, for all $z \in E$. We fix a point $z \in E$. Then, the linear functional

$$k_{a,z} : f \mapsto \int_\Omega \frac{\overline{g(w)}f(w)}{z - a(w)}\, d\mu(w)$$

is continuous on $\mathcal{A}^3(\mu)$. By the Hahn–Banach theorem, we find some $h \in L^{3/2}(\mu)$ such that $k_{a,z}(f) = \int_\Omega h(w)f(w)\, d\mu(w)$ for all $f \in \mathcal{A}^3(\mu)$ and $\|k_{a,z}\| = \|h\|_{3/2}$. Hence, by Lemma 3, there is some $v \in \mathcal{A}^3(\mu)$ such that $|h| = |v|^2\mu$-a.e. Notice that for all $u \in \mathcal{A}^\omega(\mu)$ we have

$$\|uv\|_2 \leq \|v\|_3\|u\|_6,$$

by the generalized Hölder inequality. Therefore, $v \in \bigcap_{u \in \mathcal{A}^\omega(\mu)} D(M_u; \mathcal{A}^2(\mu))$. Let now $\mathcal{M}$ be the closure in $\mathcal{A}^2(\mu)$ of $\mathcal{M}_0 := \{(z - a)uv; u \in \mathcal{A}^\omega(\mu)\}$. Obviously, $\mathcal{M}$ is quasi-invariant for all $M_u, u \in \mathcal{A}^\omega(\mu)$. To show that $\mathcal{M} \neq \mathcal{A}^2(\mu)$, consider the function ϕ defined by $\phi(w) := h(w)/v(w)$ if $v(w) \neq 0$ and $\phi(w) := 0$ if $v(w) = 0$. Note that $|\phi| = |v|$, so that $\phi \in L^3(\mu) \subset L^2(\mu)$. For all $u \in \mathcal{A}^\omega(\mu)$ we obtain

$$\begin{aligned}\int_\Omega \phi(w)(z - a(w))u(w)v(w)\, d\mu(w) &= \int_\Omega (z - a(w))u(w)h(w)\, d\mu(w)\\ &= k_{a,z}((z - a)u)\\ &= \int_\Omega \frac{(z - a(w))u(w)\overline{g(w)}}{z - a(w)}\, d\mu(w) = 0,\end{aligned}$$

as $g \in \mathcal{A}^2(\mu)^\perp$ and $u \in \mathcal{A}^2(\mu)$.

Note also that,

$$\int_\Omega \phi(w)b(w)v(w)\, d\mu(w) = \int_\Omega h(w)b(w)\, d\mu(w) = k_{a,z}(b) = \int_\Omega \frac{\overline{g(w)}b(w)}{z - a(w)}\, d\mu(w) \neq 0.$$

In particular, $k_{a,z} \neq 0$. Hence also $v \neq 0$ and we have shown that $\mathcal{M} \neq \mathcal{A}^2(\mu)$.

If $(z - a(w))v(z) \neq 0$ for μ-a.e. $w \in \Omega$, then $\mathcal{M}$ is not trivial. Otherwise we have $v \in \ker M_{z-a}$ and $1 \notin \ker M_{z-a}$ since a is not constant. Thus, in this case, $\ker M_{z-a}$ will be a nontrivial common invariant subspace, for all M_u with $u \in \mathcal{A}^\omega(\mu)$. □

Remark. In the situation of Theorem 9, let $\mathcal{B}$ be any subalgebra of $L^\omega(\mu)$ such that $\mathcal{B} \subset \mathcal{A}^2(\mu)$. Then $\mathcal{B}^2(\mu) \subset \mathcal{A}^2(\mu)$ and, applying Theorem 9 to $\mathcal{B}$ instead of $\mathcal{A}$, we see that the family of all M_b with $b \in \mathcal{B}$ has a proper quasi-invariant subspace. In particular, one may choose $\mathcal{B}$ maximal with the property $\mathcal{A} \subset \mathcal{B} \subset \mathcal{A}^2(\mu)$.

If $\mathcal{A}$ is a subalgebra of $L^\infty(\mu)$, then $L^\infty(\mu) \cap \mathcal{A}^2(\mu)$ is an algebra. We obtain from Theorem 9 and the preceding remark the following result of T. T. Trent [**12**].

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

Corollary 10. *Let $\mathcal{A}$ be a subalgebra of $L^\infty(\mu)$ of dimension at least 2. Then the family $\mathcal{S}_\mathcal{B}$ of all M_b with $b \in \mathcal{B} := L^\infty(\mu) \cap \mathcal{A}^2(\mu)$ has a proper invariant subspace in $\mathcal{A}^2(\mu)$.*

Proof. Indeed, as noted in the remark, the family $\mathcal{S}_\mathcal{B}$ has a nontrivial common quasi-invariant subspace $\mathcal{M}$. As $\mathcal{S}_\mathcal{B}$ consists of bounded linear operators, the subspace $\mathcal{M}$ is invariant for $\mathcal{S}_\mathcal{B}$. □

The next result is an abstract form of Theorem 9.

Theorem 11. *Let $\mathcal{S}$ be a subnormal family of closed linear operators in the Hilbert space $\mathcal{H}$ with* $\dim \mathcal{H}$ *at least 2. Assume also that there exists a non null linear subspace $\mathcal{D} \subset \bigcap_{S\in\mathcal{S}} D(S)$ such that $S\mathcal{D} \subset \mathcal{D}$, for all $S \in \mathcal{S}$. Then $\mathcal{S}$ has a proper quasi-invariant subspace.*

Proof. Of course, if $\mathcal{S}$ has a joint eigenvector v, then the one dimensional subspace spanned by v is a proper common invariant subspace for $\mathcal{S}$. Hence, from now on, we shall assume that $\mathcal{S}$ has no joint eigenvectors. This forces $\mathcal{D}$ to be infinite dimensional.

Let $\mathcal{N} = \{N_s; S \in \mathcal{S}\}$ be a normal extension of $\mathcal{S}$ in the Hilbert space $\mathcal{K} \supset \mathcal{H}$. For all $S \in \mathcal{S}$ denote the spectral measure of its normal extension N_S by $E_S(\cdot)$. We also write $\mathfrak{B}$ for the C^*-subalgebra of $\mathcal{L}(\mathcal{K})$ generated by

$$\{E_S(B); S \in \mathcal{S}, B \text{ a Borel subset of } \mathbb{C}\}.$$

By the general spectral theorem (see [**5**] or [**7**]) there exists a unique resolution of the identity on the set $\mathcal{B}(\Delta)$ of all Borel sets of the maximal ideal space Δ of $\mathfrak{B}$ such that for all operators $A \in \mathfrak{B}$ we have

$$A = \int_\Delta \hat{A}(M)\, dE(M),$$

where $\hat{A}$ denotes the Gelfand transform of A. If $f : \Delta \to \mathbb{C}$ is a Borel measurable function, we denote by $\Psi(f)$ the closed linear operator in $\mathcal{K}$ given by

$$\langle \Psi(f)x, y\rangle := \int_\Delta f(M)\, d\langle E(M)x, y\rangle,$$

for all

$$x \in D(f) := D(\Psi(f)) := \left\{u \in \mathcal{K}; \int_\Delta |f(M)|^2\, d\langle E(M)x, x\rangle < \infty\right\}.$$

(See for example [**7**, Theorem 13.24].) For all $S \in \mathcal{S}$ and all bounded Borel sets $B \subset \mathbb{C}$, the bounded linear operator $N_S E_S(B)$ is in $\mathfrak{B}$. For $S \in \mathcal{S}$ we define a function $\varphi_S : \Delta \to \mathbb{C}$ by

$$\varphi_S(M) := \begin{cases} N_S E_S(B))^\wedge(M), & \text{if there exists a bounded Borel set} B \subset \mathbb{C} \\ & \quad \text{with } \widehat{E_S(B)}(M) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $M \in \Delta$ and if B_1, B_2 are bounded Borel sets in $\mathbb{C}$ satisfying $\widehat{E_S(B_j)}(M) \neq 0$ (and hence $\widehat{E_S(B_j)}(M) = 1$) for $j = 1, 2$, then

$$\begin{aligned}(N_S E_S(B_1))^\wedge(M) &= (N_S E_S(B_1))^\wedge(M)\widehat{E_S(B_2)}(M)\\ &= (N_S E_S(B_1 \cap B_2))^\wedge(M)\\ &= (N_S E_S(B_2))^\wedge(M).\end{aligned}$$

This shows that φ_S is indeed a well defined function. It follows from the definition that it is continuous on the open set

$$\Omega_S := \bigcup_B \widehat{E_S(B)}^{-1}(\mathbb{C}\backslash\{0\}),$$

where the union is taken over all bounded Borel sets $B \subset \mathbb{C}$, and that φ_S vanishes on the compact set $\Delta\backslash\Omega_S$. Hence, φ_S is Borel measurable. Using the fact that

$$\begin{aligned}\int_B z\, dE_S(z)x = N_S E_S(B)x &= \int_\Delta (N_S E_S(B))^\wedge(M)\, dE(M)x\\ &= \int_\Delta \widehat{E_S(B)}(M)\varphi_S(M)\, dE(M)x,\end{aligned}$$

for all $x \in \mathcal{K}$ and all compact $B \subset \mathbb{C}$, a straightforward computation shows that $N_S = \Psi(\varphi_S)$ holds for all $S \in \mathcal{S}$. In particular, it follows that $\Psi(f)\mathcal{D} \subset \mathcal{D}$, for all f in the algebra $\mathcal{A}_S$ of functions generated by $\{\varphi_S; S \in \mathcal{S}\}$. Let us fix some non null vector $x_0 \in \mathcal{D}$, denote the scalar Borel measure $\langle E(\cdot)x_0, x_0\rangle$ by μ, and let $\mathcal{H}_0$ be the closure of $\Psi(\mathcal{A}_S)x_0$. This is a non null closed quasi-invariant subspace for $\mathcal{S}$. If $\mathcal{H}_0 \neq \mathcal{H}$, we are done. Hence we assume now that $\mathcal{H}_0 = \mathcal{H}$. The isometry $f \mapsto \Psi(f)x_0$ then extends to an isometry J from $\mathcal{A}^2_S\,(\mu)$ onto $\mathcal{H}$ satisfying $SJ = JM_{\varphi_S}$, for all $S \in \mathcal{S}$, where M_{φ_S} denotes the operator of multiplication with φ_S in $\mathcal{A}^2_S\,(\mu)$. As $\mathcal{A}_S \subset L^\omega(\mu)$, by Lemma 2, we see from the previous theorem that there exists a quasi-invariant subspace for $\{\Psi(f); f \in \mathcal{A}^\omega(\mu)\}$. $\square$

5. Existence of invariant subspaces. We fix a positive measure μ on $\mathbb{C}^n$ such that $\mathcal{P}_a \subset L^2(\mu)$ (and hence $\mathcal{P}_a \subset L^\omega(\mu)$, by Lemma 2). The n-tuple $N = (N_1, \ldots, N_n)$, consisting of commuting normal operators in $L^2(\mu)$, is defined by $(N_j f)(z) := z_j f(z)$, for all $f \in D(N_j) := \{h \in L^2(\mu); z \mapsto z_j h(z) f \in L^2(\mu)\}$. If $K \subset \mathbb{C}^n$ is a compact subset, we put $L^2(K, \mu) := \{f|K; f \in L^2(\mu)\}$, whose norm will be denoted by $\|\cdot\|_{2,K}$. Let $\mathcal{P}^2_a(K, \mu)$ be the closure of $\mathcal{P}_a$ in $L^2(K, \mu)$. We also define

$$\tilde{\mathcal{P}}^2_a(\mu) := \big\{f \in L^2(\mu); f|K \in \mathcal{P}^2_a(K, \mu), K \subset \mathbb{C}^n, K \text{ compact}\big\}.$$

We clearly have $\tilde{\mathcal{P}}^2_a(\mu) \supset \mathcal{P}^2_a(\mu)$.

LEMMA 12. *The space $\tilde{\mathcal{P}}^2_a(\mu)$ is closed in $L^2(\mu)$ and invariant under N_j, $(j = 1, \ldots, n)$.*

Proof. If $f = \lim_m f_m$ in $L^2(\mu)$ with $f_m \in \tilde{\mathcal{P}}^2_a(\mu)$, then $f_m|K \in \mathcal{P}^2_a(K, \mu)$, for all $m \geq 1$ and all compact subsets $K \subset \mathbb{C}^n$. Therefore, $f|K = \lim_k f_m|K \in \mathcal{P}^2_a(K, \mu)$, showing that $f \in \tilde{\mathcal{P}}^2_a(\mu)$.

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

Now, assume that $f \in D(N_j) \cap \tilde{\mathcal{P}}^2_a(\mu)$. If $K \subset \mathbb{C}^n$ is a fixed compact set, we can find a sequence $(P_m)_{m \geq 1}$ in $\mathcal{P}_a$ such that $\|f - P_m\|_{2,K} \to 0$ as $m \to \infty$. But

$$\|N_j f - N_j P_m\|_{2,K} \leq \sup\{|z_j|; z \in K\} \|f - P_m\|_{2,K} \to 0,$$

showing that $N_j f | K \in \mathcal{P}^2_a(K, \mu)$, for all compact subsets K of $\mathbb{C}^n$. □

Lemma 12 shows that we may define the multiplication operator $\tilde{M}_j$ induced by N_j, in $\tilde{\mathcal{P}}^2_a(\mu)$, with the domain $D(\tilde{M}_j) = \tilde{\mathcal{P}}^2_a(\mu) \cap D(N_j)$, for all $j = 1, \ldots, n$. Now set $\tilde{M} := (\tilde{M}_1, \ldots, \tilde{M}_n)$. We have the following result.

THEOREM 13. *Suppose that the measure μ has the following property:*

$(*)$ for every compact subset $K \subset \mathbb{C}^n$ such that $\mathcal{P}^2_a(K, \mu) \neq L^2(K, \mu)$ there exists a point $w \in \mathbb{C}^n$ such that the map $\mathcal{F}_w \ni P \to P(w) \in \mathbb{C}$ is continuous in the norm of $L^2(K, \mu)$.

If the support of μ has at least two points, the subnormal tuple $\tilde{M} = (\tilde{M}_1, \ldots, \tilde{M}_n)$ has a proper invariant subspace.

Proof. If $\tilde{\mathcal{P}}^2_a(\mu) = L^2(\mu)$, then $\tilde{M} = N$ and, for a bounded Borel set B with $\mu(B) \neq 0$, $\mu(\mathbb{C}^n \setminus B) \neq 0$ the space $\chi_B L^2(\mu)$ is a proper invariant subspace for $\tilde{M}$, where χ_B is the characteristic function of B.

Assume now that $L^2(\mu) \neq \tilde{\mathcal{P}}^2_a(\mu)$. Then there exists a compact set K such that $L^2(K, \mu) \neq \mathcal{P}^2_a(K, \mu)$. In this case, in virtue of property $(*)$, there exists a point $w \in \mathbb{C}^n$ such that the map $\mathcal{F}_w \ni P \to P(w) \in \mathbb{C}$ is continuous in the norm of $L^2(K, \mu)$. Let h_w be a Hahn–Banach extension of this map to $\mathcal{P}^2_a(K, \mu)$ and let ϵ_w be the map $\epsilon_w(f) = h_w(f|K), f \in \tilde{\mathcal{P}}^2_a(\mu)$.

If $\prod_{j=1}^n (z_j - w_j) = 0$ μ-a.e., we choose $J \subset \{1, \ldots, n\}$ maximal with the property $\prod_{j \in J}(z_j - w_j) \neq 0$ and $(z_k - w_k) \prod_{j \in J}(z_j - w_j) = 0$ μ-a.e., for all $k \notin J$. If $J \neq \emptyset$, then $\mathcal{M} := \bigcap_{j \notin J} \ker(M_k - w_k)$ is a non null closed subspace invariant under $\tilde{M}$. If $J = \emptyset$, then the support of the measure μ is the set $\{w\}$, which contradicts the hypothesis.

If $\prod_{j=1}^n (z_j - w_j) \neq 0$, we shall fix a compact set $L \supset K$ such that $\prod_{j=1}^n (z_j - w_j) | L \neq 0$. Let $\mathcal{N}_{0,L} := \{\prod_{j=1}^n (z_j - w_j) P | L; P \in \mathcal{P}_a\}$, which is non null, and let $\mathcal{N}_L$ be the closure of $\mathcal{N}_{0,L}$ in $\mathcal{P}^2_a(L, \mu)$. Let also $\mathcal{N}$ be the space $\{f \in \tilde{\mathcal{P}}^2_a(\mu) : f|L \in \mathcal{N}_L\} \subset \ker \epsilon_w$. We shall show that $\mathcal{N}$, which is clearly a proper closed subspace, is invariant under $\tilde{M}$. Indeed, if $f \in \mathcal{N} \cap D(\tilde{M}_j)$ for a fixed j, we can find a sequence $(P_m)_{m \geq 1}$ in $\mathcal{N}_{0,L}$ convergent to $f|L$. Note that $z_j P_m \to z_j f | L (m \to \infty)$ in $\mathcal{N}_L$. Therefore $z_j f$ is in $\mathcal{N}$, which completes the proof. □

REMARK . It follows from a theorem of Thomson [**11**] (see also [**4**, Theorem VIII.4.3]), that property $(*)$ is automatically fulfilled if $n = 1$. We do not know whether or not this property is automatically fulfilled when $n > 1$. A related property is provided by Lemma 6.

We shall present in the following another type of invariant subspace, related to Hardy spaces on tubes. Unlike in the preceding cases, we work here with the Lebesgue measure in $\mathbb{R}^n$, which is not a finite measure.

We shall use in the following some results from [**8**]. (See especially Chapter III.)

If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ are arbitrary points, we denote by $z \cdot s$ the complex number $\sum_{j=1}^n z_j s_j$. In particular, $\|s\| = (s \cdot s)^{1/2}$ is precisely the norm of s.

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press

For every $f \in L^2(\mathbb{R}^n)$ we denote by $\hat{f}$ its *Fourier transform*. The map $L^2(\mathbb{R}^n) \ni f \to \hat{f} \in L^2(\mathbb{R}^n)$ is a unitary operator, via the Plancherel Theorem.

Set $\Gamma := (0, \infty) \times \cdots \times (0, \infty) \subset \mathbb{R}^n$, and $\Omega := \mathbb{R}^n + i\Gamma \subset \mathbb{C}^n$. The space $H^2(\Omega)$ consists of those holomorphic functions $F : \Omega \to \mathbb{C}$ such that

$$\|F\|^2 := \sup_{t \in \Gamma} \int_{\mathbb{R}^n} |F(s + it)|^2 \, ds < \infty.$$

The space $H^2(\Omega)$, endowed with the norm $\|F\|$ defined above, becomes a Hilbert space.

If $F \in H^2(\Omega)$, then the limit

$$F^*(s) = \lim_{t \to 0} F(s + it)$$

exists almost everywhere in $\mathbb{R}^n$ and $F^* \in L^2(\mathbb{R}^n)$. Moreover, this limit also exists in the $L^2(\mathbb{R}^n)$-norm, and the map

$$H^2(\Omega) \ni F \to F^* \in L^2(\mathbb{R}^n)$$

is a linear isometry. This allows us to identify the space $H^2(\Omega)$ with a closed subspace of $L^2(\mathbb{R}^n)$. In fact, we have the following result.

LEMMA 14. *The image of the map $H^2(\Omega) \ni F \to F^* \in L^2(\mathbb{R}^n)$ is the closed subspace*

$$H^2_\Gamma = \{f \in L^2(\mathbb{R}^n); \operatorname{supp}(\hat{f}) \subset \bar{\Gamma}\}.$$

Proof. As this result is not explicitly stated in **[8]** (although all ingredients are present), for the convenience of the reader we shall give a short proof.

If $g \in L^2(\mathbb{R}^n)$ and $\operatorname{supp}(g) \subset \bar{\Gamma}$, we set

$$F(z) = \int_{\bar{\Gamma}} e^{2\pi i z \cdot u} g(u) \, du, \, (z \in \Omega). \tag{2}$$

Then $F \in H^2(\Omega)$ and $\|F\| = \|g\|$, by Theorem III.3.1 of **[8]**. Moreover, $\|F^*\| = \|F\|$, by virtue of Theorem III.5.1 and Corollary III.3.4 from **[8]**.

It is also clear that $F(s + it)$ is the inverse Fourier transform of the function $e^{-2\pi t \cdot u} g(u)$, and the latter converges to $g(u)$ as $t \to 0$ in $L^2(\mathbb{R}^n)$. Since $F(s + it) \to F^*(s)$ as $t \to 0$ in $L^2(\mathbb{R}^n)$, it follows that g is the Fourier transform of F^*.

Conversely, every function $F \in H^2(\Omega)$ is of the form (2), by Theorem III.3.1 from **[8]**, implying the desired equality. □

Let $\mathcal{P}$ be the algebra of all polynomials on $\mathbb{R}^n$ with complex coefficients. We define on $L^2(\mathbb{R}^n)$ the operators

$$M_p f(s) = p(s) f(s), \quad (s \in \mathbb{R}^n, p \in \mathcal{P}, f \in D(M_p)),$$

with $D(M_p) = \{f \in L^2(\mathbb{R}^n); pf \in L^2(\mathbb{R}^n)\}$. The operator is (unbounded) normal **[5]** in $L^2(\mathbb{R}^n)$, for each $p \in \mathcal{P}$.

We are interested in the action of the operators M_p in the space H^2_Γ. We have the following result.

THEOREM 15. *Let $C \subset \Gamma$ be a closed set whose boundary is a λ_n-null set, and let*

$$H^2_\Gamma(C) := \{f \in H^2_\Gamma; \operatorname{supp}\hat{f} \subset C\}.$$

Then $H^2_\Gamma(C)$ is invariant under M_p, for each $p \in \mathcal{P}$.

If C has a nonempty interior G, then the subspace $H^2_\Gamma(C)$ is proper.

Proof. The subspace $H^2_\Gamma(C)$ is clearly closed.

If all involved functions are regarded as distributions, for $f \in L^2(\mathbb{R}^n)$ and $p \in \mathcal{P}$, the Fourier transform of pf is the distribution $p(\hat{D})\hat{f}$, where $\hat{D}_j = (2\pi i)^{-1}\partial/\partial s_j$ and $\hat{D} = (\hat{D}_1, \dots \hat{D}_n)$. In particular, the support of the distribution $p(\hat{D})\hat{f}$ must be contained in the support of the function $\hat{f}$.

Let $f \in H^2_\Gamma(C)$ be such that $pf \in L^2(\mathbb{R}^n)$. The preceding remark shows that the support of the Fourier transform of pf as a distribution must be in C. But the Fourier transform of a function in $L^2(\mathbb{R}^n)$ coincides with its Fourier transform as a distribution (see [**8**, Section I.3]). Therefore, if $f \in H^2_\Gamma$ and $pf \in L^2(\mathbb{R}^n)$, then $pf \in H^2_\Gamma$.

If $g \in C_0^\infty(\Gamma)$, and if $F(z) = \int_{\bar{\Gamma}} e^{2\pi i z\cdot u} g(u)\,du$ we infer that

$$z^\alpha F(z) = (-1)^{|\alpha|}\int_{\bar{\Gamma}} e^{2\pi i z\cdot u}\hat{D}^\alpha g(u)\,du, \quad (z \in \Omega),$$

for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$, where $\hat{D}^\alpha = \hat{D}_1^{\alpha_1}\cdots\hat{D}_n^{\alpha_n}$. This shows that

$$\{f \in H^2_\Gamma; \hat{f} \in C_0^\infty(G)\} \subset \bigcap_{p\in\mathcal{P}} D(M_p).$$

The density of the set $C_0^\infty(G)$ in $L^2(C)$ implies the density of $D(M_p) \cap H^2_\Gamma(C)$ in $H^2_\Gamma(C)$, for all p.

As $C = \bar{C} \subset \Gamma$, we can choose a closed set $C_1 \subset \Gamma\backslash C$ with non empty interior. The subspace $H^2_\Gamma(C_1)$ is obviously non null. Moreover, since the Fourier transform is a unitary operator, the subspaces $H^2_\Gamma(C)$ and $H^2_\Gamma(C_1)$ are orthogonal in H^2_Γ. In particular, $H^2_\Gamma(C)$ is a proper subspace of H^2_Γ. □

Note, that the point evaluations $\mathcal{E}_z : F \mapsto F(z)$, are continuous on $H^2(\Omega)$ for all $z \in \Omega$. Indeed, for all $F \in H^2(\Omega)$, $z \in \Omega$, we have by means of the Cauchy–Schwarz inequality,

$$|F(z)| = \left|\int_{\bar{\Gamma}} F^*(u)e^{2\pi i z\cdot u}\,du\right| \le \|F^*\|\left(\int_{\bar{\Gamma}} e^{-4\pi \operatorname{Im} z\cdot u}\,du\right)^{1/2} \le \|F\|\cdot C(z)$$

with a positive constant $C(z)$ only depending on z. By means of this we show now that the algebra of all operators of multiplication by polynomials on $H^2(\Omega)$ has indeed a very rich invariant subspace structure.

THEOREM 16. *Let $\mathcal{M}$ be a closed subspace of $H^2(\Omega)$ that is invariant, for all M_p, $p \in \mathcal{P}$, such that $\mathcal{M} \cap \bigcap_{p\in\mathcal{P}} D(M_p) \neq \{0\}$. Then $\mathcal{M}$ contains a nontrivial closed subspace which is invariant, for all M_p, $p \in \mathcal{P}$.*

Proof. We fix some $0 \neq G \in \mathcal{M} \cap \bigcap_{p\in\mathcal{P}} D(M_p)$ and some $w \in \Omega$ with $G(w) \neq 0$. By multiplying G by $1/G$ we may assume in the following that $G(w) = 1$. Now $\mathcal{M}_w := \mathcal{M} \cap \ker(\mathcal{E}_w)$ is a closed subspace of $\mathcal{M}$ not containing G. Let q be a non-constant polynomial such that $q(w) = 1$. In particular, M_q has empty point spectrum. Therefore

$0 \neq M_q G - G \in \mathcal{M}_w$ and $\mathcal{M}_w \neq \{0\}$. To show that $\mathcal{M}_w$ is invariant under all M_p, $p \in \mathcal{P}$, we fix an arbitrary $p \in \mathcal{P}$. Since $\mathcal{M}$ is invariant for M_p, we have that $D(M_p) \cap \mathcal{M}$ is dense in $\mathcal{M}$. Let now F be an arbitrary element of $\mathcal{M}_w$. Then there is a sequence $(F_n)_{n=1}^{\infty}$ in $D(M_p) \cap \mathcal{M}$ converging to F for $n \to \infty$. By the continuity of $\mathcal{E}_w$ we also have $F_n(w) \to F(w)$ as $n \to \infty$. Then $(F_n - F_n(w)G)_{n=1}^{\infty}$ is a sequence in $D(M_p) \cap \mathcal{M}_w$ converging to F. Since obviously $M_p(D(M_p) \cap \mathcal{M}_w) \subset \mathcal{M}_w$ we see that $\mathcal{M}_w$ is indeed an invariant subspace for M_p. □

REFERENCES

1. R. Arens, The space L^{ω} and convex topological rings, *Bull. Amer. Math. Soc.* **52** (1946), 931–935.

2. J. E. Brennan, Invariant subspaces and rational approximation, *J. Funct. Anal.* **7** (1971), 285–310.

3. S. W. Brown, Some invariant subspaces for subnormal operators, *Integral Equations and Operator Theory* **1** (1978), 310–333.

4. J. B. Conway, *The theory of subnormal operators*, Mathematical surveys and monographs, Vol. 36 (AMS, Providence, Rhode Island, 1991).

5. N. Dunford and J. T. Schwartz, *Linear Operators, Part II* (Interscience Publishers, 1963).

6. K. Floret and J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume*, Lecture Notes in Math. No. 56 (Springer-Verlag, 1968).

7. W. Rudin, *Functional analysis* (McGraw-Hill, 1973).

8. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces* (Princeton University Press, Princeton, New Jersey, 1971).

9. J. Stochel and F. H. Szafraniec, On normal extensions of unbounded operators, *J. Operator Theory* **14** (1985), 31–55.

10. J. E. Thomson, Invariant subspaces for algebras of subnormal operators, *Proc. Amer. Math. Soc.* **96** (1986), 462–464.

11. J. E. Thomson, Approximation in the mean by polynomials, *Ann. of Math. (2)* **133** (1991), 477–507.

12. T. T. Trent, Invariant subspaces for operators in subalgebras of $L^{\infty}(\mu)$, *Proc. Amer. Math. Soc.* **99** (1987), 268–272.

13. Yan Ke Ren, Invariant subspaces for joint subnormal systems, *Chinese Ann. Math. Ser. A* **9** (1988), No. 5, 561–566.

https://doi.org/10.1017/S0017089502008984 Published online by Cambridge University Press