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NON-PARALLELIZABILITY OF GRASSMANN MANIFOLDS

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ABSTRACT. The fact that no real Grassmann manifolds $G_k(R^n)$ are parallelizable (or even stably parallelizable) except for the obvious cases $G_1(R^2) \cong S^1$, $G_1(R^4) \cong G_3(R^4) \cong RP^3$, and $G_1(R^8) \cong G_7(R^8) \cong RP^7$ was first noted by Hiller and Stong. Their work in turn depends on induction and the work of Oproiu, who examined detailed calculations of Stiefel-Whitney classes for $k = 2, 3$. In this note we give a short proof of this result, using elementary results from K-theory, that also covers the complex and quaternionic Grassmann manifolds.

THEOREM. *The only Grassmann manifolds that are stably parallelizable* (*as real manifolds*) *are* $G_1(\mathbb{F}^2)$, $G_1(\mathbb{R}^4) \cong G_3(\mathbb{R}^4)$, *and* $G_1(\mathbb{R}^8) \cong G_7(\mathbb{R}^8)$, *where* $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$.

The proof uses the known characterization of the tangent bundle of $G_k(\mathbb{F}^n)$ (cf. [5]) and the composite inclusion

$$j: \mathbb{F}P^{n-k} \cong G_1(\mathbb{F}^{n-k+1}) \xrightarrow{i} \cdots \xrightarrow{i} G_{k-1}(\mathbb{F}^{n-1}) \xrightarrow{i} G_k(\mathbb{F}^n),$$

where i is the usual inclusion map arising from $\mathbb{F}^n \approx \mathbb{F}^{n-1} \oplus \mathbb{F}$. One shows $j^*(r\tau^F_{n,k}) \sim (n-2k+2) r\xi^F_{n-k} \in \tilde{K}_{\mathbb{R}}(\mathbb{F}P^{n-k})$, where r is the realification of an $\mathbb{F}$-vector bundle, $\tau^F_{n,k}$ the $\mathbb{F}$-tangent bundle of $G_k(\mathbb{F}^n)$, "$\sim$" is stable equivalence, and ξ^F_m the canonical line bundle over $\mathbb{F}P^m$. The proof is completed by comparing with the known order of $r\xi^F_{n-k}$ ([1], [2], and [4] for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ respectively).

This theorem generalizes work of Hiller–Stong [3] and Oproiu [6], who obtained the above result for $\mathbb{F} = \mathbb{R}$ (in addition to many other results on immersions) by examining the Stiefel–Whitney classes.

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