To express a Determinant of the *n*th Order in terms of Compound Determinants of the 2nd Order, and *vice-versa.*

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1. Let $\phi\,(a\,b'\,c'')$ denote the compound determinant,

$$\begin{vmatrix} (b'\,c''), & (a'\,c'') \\ (b\ \,c''), & (a\ \,c'') \end{vmatrix}, \text{ where } (b'\,c'') \text{ denotes } \begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix} \text{ etc.}$$

Then if A, B, etc., denote the co-factors of the elements a, b, etc. in the determinant $(a\,b'\,c'')$, we have

$$\phi\,(a\,b'\,c'') = \begin{vmatrix} \mathrm{A} & -\mathrm{B} \\ -\mathrm{A}' & \mathrm{B}' \end{vmatrix} = c''\,(a\,b'\,c'').$$

2. Again, denoting by $\phi\,(a\,b'\,c''\,d''')$ the compound determinant

$$\begin{vmatrix} \phi\,(b'\,c''\,d'''), & \phi\,(a'\,c''\,d''') \\ \phi\,(b\ \,c''\,d'''), & \phi\,(a\ \,c''\,a''') \end{vmatrix}, \text{ we have}$$

$$\phi\,(a\,b'\,c''\,d''') = \begin{vmatrix} d'''\,(b'\,c''\,d'''), & d'''\,(a'\,c''\,d''') \\ d'''\,(b\ \,c''\,d'''), & d'''\,(a\ \,c''\,d''') \end{vmatrix} = d'''^2 \begin{vmatrix} \mathrm{A} & -\mathrm{B} \\ -\mathrm{A}' & \mathrm{B}' \end{vmatrix}$$

$$= d'''^2\,.\,(c''\,d''')\,.\,(a\,b'\,c''\,d''').$$

Here A, B, etc., are the cofactors of a, b, etc., in the determinant $(a\,b'\,c''\,d''')$.

3. The general formula of which the two preceding are special cases is

$$\phi\,(a_1\,b_2\,c_3\ldots\ldots t_n) = (t_n)^{2^{n-3}}\,.\,(s_{n-1}t_n)^{2^{n-4}}\ldots\ldots(c_3\,d_4\ldots\ldots t_n)\,.\,(a_1\,b_2\,c_3\ldots\ldots t_n)$$

which can be established by mathematical induction without difficulty, observing that

$$\phi\,(a_1\,b_2\,c_3\ldots\ldots t_n u_{n+1}) \equiv \begin{vmatrix} \phi\,(b_2\,c_3\ldots\ldots u_{n+1}), & \phi\,(a_2\,c_3\ldots\ldots u_{n+1}) \\ \phi\,(b_1\,c_3\ldots\ldots u_{n+1}), & \phi\,(a_1\,c_3\ldots\ldots u_{n+1}) \end{vmatrix}$$

and using the well-known theorem that in the determinant $(a_1\,b_2\ldots t_n)$

$$\begin{vmatrix} \mathrm{A}_1 & \mathrm{B}_1 \\ \mathrm{A}_2 & \mathrm{B}_2 \end{vmatrix} = (c_3\,d_4\ldots\ldots t_n)\,.\,(a_1\,b_2\,c_3\ldots\ldots t_n)$$

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4. Now let us denote $\phi(a_1 b_2 c_3 \ldots\ldots t_n)$ by ϕ_1

$\phi(b_2 c_3 \quad \ldots\ldots t_n)$ by ϕ_2

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\phi(r_{n-2} s_{n-1}, t_n)$ by ϕ_{n-2}

$\phi(s_{n-1} t_n)$ by ϕ_{n-1}

Also denote $(a_1 b_2 \ldots\ldots t_n)$ by Δ_1

$(b_2 c_3. \ldots . t_n)$ by Δ_2

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$(s_{n-1} t_n)$ by Δ_{n-1}

t_n by Δ_n or ϕ_n

so that $$\phi_{n-1} = \Delta_{n-1} = \begin{vmatrix} s_{n-1}, & t_{n-1} \\ s_n, & t_n \end{vmatrix}$$

The formula of the preceding article can now be written

$$\phi_1 = \Delta_n^{2^{n-3}} \cdot \Delta_{n-1}^{2^{n-4}} \cdot \Delta_{n-2}^{2^{n-5}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \Delta_5^{2^2} \cdot \Delta_4^2 \cdot \Delta_3 \, \Delta_2^0 \, \Delta_1$$

Hence also

$$\phi_3 = \Delta_n^{2^{n-5}} \cdot \Delta_{n-1}^{2^{n-6}} \cdot \Delta_{n-2}^{2^{n-7}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \Delta_5 \, \Delta_4^0 \, \Delta_3$$

$$\phi_4 = \Delta_n^{2^{n-6}} \cdot \Delta_{n-1}^{2^{n-7}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Delta_6 \, \Delta_5^0 \, \Delta_4$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$$\phi_{n-3} = \Delta_n^2 \, \Delta_{n-1} \, \Delta_{n-2}^0 \, \Delta_{n-3}$$

$$\phi_{n-2} = \Delta_n \, \Delta_{n-1}^0 \, \Delta_{n-2}$$

$$\phi_{n-1} = \Delta_n^0 \, \Delta_{n-1}$$

$$\phi_n = \Delta_n$$

5. Hence we deduce

$$\phi_3 \, \phi_4^2 \, \phi_5^3 \ldots\ldots \phi_{n-1}^{n-3} \phi_n^{n-2} = \Delta_3 \cdot \Delta_4^2 \cdot \Delta_5^{2^2} \ldots\ldots \Delta_{n-1}^{2^{n-4}} \cdot \Delta_n^{2^{n-3}}.$$

To prove this we have to show that the exponent of Δ_{n-r} in the product, viz.,

$2^{n-r-5} + 2 . 2^{n-r-6} + 3 . 2^{n-r-7} + \ldots + \overline{(n-r-5)}2 + (n-r-4) + (n-r-2)$ is $= 2^{n-r-3}$.

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Now the former expression may be written

$$
\begin{aligned}
&2^{n-r-5} + 2^{n-r-6} + 2^{n-r-7} + \dots\dots + 2^2 + 2^1 + 1 + 1 \\
&\quad + 2^{n-r-6} + 2^{n-r-7} + \dots\dots + 2^2 + 2^1 + 1 + 1 \\
&\qquad + \dots\dots \\
&\qquad\qquad + 2^2 + 2^1 + 1 + 1 \\
&\qquad\qquad\quad + 2^1 + 1 + 1 \\
&\qquad\qquad\qquad + 1 + 1 + 2 \\
&= 2^{n-r-4} + 2^{n-r-3} + 2^{n-r-2} + \dots\dots + 2^2 + 2 + 2 \\
&= 2^{n-r-3}
\end{aligned}
$$

Comparing this result with the expression for ϕ_1 we find

$$
\begin{aligned}
\Delta_1 &= \phi_1 \div \left\{ \phi_3 \, \phi_4^2 \, \phi_5^3 \dots\dots \phi_{n-1}^{n-3} \, \phi_n^{n-2} \right\} \\
&= \phi_1^1 \, \phi_2^0 \, \phi_3^{-1} \, \phi_4^{-2} \, \phi_5^{-3} \dots\dots \phi_{n-1}^{-n+3} \, \phi_n^{-n+2} .
\end{aligned}
$$

Thus the general determinant of nth order is expressed in terms of compound determinants of the 2nd order.