Bull. Aust. Math. Soc. **93** (2016), 353–363
doi:10.1017/S0004972715001380

CAYLEY GRAPHS OVER A FINITE CHAIN RING AND GCD-GRAPHS

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(Received 28 July 2015; accepted 29 September 2015; first published online 22 January 2016)

Abstract

We extend spectral graph theory from the integral circulant graphs with prime power order to a Cayley graph over a finite chain ring and determine the spectrum and energy of such graphs. Moreover, we apply the results to obtain the energy of some gcd-graphs on a quotient ring of a unique factorisation domain.

2010 *Mathematics subject classification*: primary 05C50.

Keywords and phrases: energy, finite chain rings, gcd-graphs.

1. Introduction

The study of ring-theoretic graphs includes unitary Cayley graphs, integral circulant graphs, zero-divisor graphs and gcd-graphs. Mostly, this work involves determining the eigenvalues (which are real) and computing the energy (the sum of the absolute values of the eigenvalues) of the graph. The energy is a graph parameter introduced by Gutman (see [3]) arising from the Hückel molecular orbital approximation for the total π-electron energy.

Let D be a unique factorisation domain (UFD) and $c \in D$ a nonzero nonunit element. Assume that the commutative ring $D/(c)$ is finite. For a set C of proper divisors of c, we define the *gcd-graph*, $D_c(C)$, to be a graph whose vertex set is the quotient ring $D/(c)$ and whose edge set is

$$\{\{x + (c), y + (c)\} : x, y \in D \text{ and } \gcd(x - y, c) \in D^\times C\}.$$

This gcd-graph on a quotient ring of a unique factorisation domain introduced in [5] generalises a gcd-graph or an integral circulant graph (whose adjacency matrix is circulant and all eigenvalues are integers) defined over $\mathbb{Z}_n, n \geq 2$ (see [6, 11]). An integral circulant graph can also be considered as an extension of a unitary Cayley graph and has been widely studied (see, for example, [1, 3, 10]).

Since the number of divisors of $c = p_1^{s_1} \cdots p_k^{s_k}$ can be very large, the energy of gcd-graphs (over $D/(c)$ or $\mathbb{Z}_n$) is still not thoroughly studied. We shall give the energy

The corresponding author is Y. Meemark. The research was supported in part by the 90th Anniversary of Chulalongkorn University Rachadapisek Sompote Fund.

of gcd-graphs whose divisor set C consists of certain prime powers, by studying the energy of the Cayley graph over the finite ring $D/(p_i^{s_i})$. When $D = \mathbb{Z}$, this graph is the integral circulant graph with prime power order studied by Sander and Sander in [10]. They derived a closed formula for its energy and worked on minimal and maximal energies for a fixed prime power p^s and varying divisor sets. We extend their results to Cayley graphs over certain finite commutative rings, called finite chain rings, which have a simple ideal structure. The structure of these rings has been well studied (see [8, 9]). They are finite local rings which generalise the ring $D/(p^s)$ and the Galois ring $\mathbb{Z}_{p^s}[x]/(f(x))$, where $f(x)$ is a monic polynomial in $\mathbb{Z}_{p^s}[x]$ and the canonical reduction $\bar{f}(x)$ in $\mathbb{Z}_p[x]$ is irreducible.

We determine the spectrum and energy of a Cayley graph over a finite chain ring, extending the treatment of integral circulant graphs with prime power order where the energy is computed via a sum of Ramanujan sums [6, 10]. Our approach here is to examine all eigenvalues with multiplicities and then obtain the sum of their absolute values directly, similar to [5]. We also show that the graph defined over a finite chain ring is indeed an integral circulant graph. The final section presents some applications of the energy. We give further results for a gcd-graph over a quotient ring of a unique factorisation domain using a tensor product and a noncomplete extended p-sum.

2. Cayley graphs over a finite chain ring

We begin with some notation in algebraic graph theory and ring theory.

Let A be a symmetric matrix. The set of all eigenvalues of A is called the *spectrum* of A. If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A of respective multiplicities $m_1, \ldots, m_k$, we use the notation $\operatorname{Spec} A = \begin{pmatrix} \lambda_1 & \cdots & \lambda_k \\ m_1 & \cdots & m_k \end{pmatrix}$ to describe the spectrum of A. For a graph G, the eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$ and we write $\operatorname{Spec} G$ for the spectrum of $A(G)$. The sum of the absolute values of all the eigenvalues of a graph G is called the *energy* of G and denoted by $E(G)$.

For two graphs G and H, their *tensor product* $G \otimes H$ is the graph with vertices $V(G) \times V(H)$ and where (u, v) is adjacent to (u', v') if and only if u is adjacent to u' in G and v is adjacent to v' in H. The adjacency matrix of $G \otimes H$ is the Kronecker product of $A(G)$ and $A(H)$, that is, $A(G \otimes H) = A(G) \otimes A(H)$.

Proposition 2.1 [1, 12]. *Let G and H be graphs. Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G and $\mu_1, \ldots, \mu_m$ are the eigenvalues of H (repeated according to their multiplicities). Then the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$, where $1 \le i \le n$ and $1 \le j \le m$. Moreover, $E(G \otimes H) = E(G)E(H)$.*

The *complement of a graph* G, denoted by $\bar{G}$, is the graph with the same vertex set as G such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

Proposition 2.2 [2, 12]. *If a graph G with n vertices is k-regular, then G and $\bar{G}$ have the same eigenvectors. The eigenvalue associated with the n-vector $\vec{1}_n$, whose entries are all 1, is k for G and $n - k - 1$ for $\bar{G}$. If $\vec{x} \neq \vec{1}$ is an eigenvector of G for the eigenvalue λ, then its eigenvalue in $\bar{G}$ is $-1 - \lambda$.*

A *finite chain ring* is a finite local ring such that for any two ideals I_1 and I_2 of this ring, either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Let R be a finite chain ring with unique maximal ideal M and residue field of q elements. Let s be the nilpotency of R, that is, the least positive integer such that $M^s = \{0\}$. It can be shown that we have the chain of ideals

$$R = M^0 \supset M \supset M^2 \supset \cdots \supset M^s = \{0\}.$$

By [9, Lemma 2.4], we also have $|M^i| = q^{s-i}$ for all $0 \leq i \leq s$ and so

$$|M^i/M^{i+1}| = q$$

for all $0 \leq i < s$. Thus, $|R| = q^s$. Moreover, M is principal, generated by some $\theta \in M \backslash M^2$, and hence any element $x \in R$ can be written as

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$

where $v_i \in \mathcal{V} = \{e_0, e_1, \ldots, e_{p^t-1}\}$, a fixed set of representatives of cosets in R/M. Let

$$C = (M^{a_1} \backslash M^{a_1+1}) \cup (M^{a_2} \backslash M^{a_2+1}) \cup \cdots \cup (M^{a_r} \backslash M^{a_r+1}),$$

where $0 \leq a_1 < a_2 < \cdots < a_r \leq s - 1$.

Consider the Cayley graph $\mathrm{Cay}(R, C)$ whose vertex set is R and where $x, y \in R$ are adjacent if and only if $x - y \in C$. This graph generalises the gcd-graph defined over $\mathbb{Z}_{p^s}$ with the set $D = \{p^{a_1}, p^{a_2}, \ldots, p^{a_r}\}$ of proper divisors of p^s, where two vertices $a, b \in \mathbb{Z}_{p^s}$ are adjacent if and only if $\gcd(b - a, p^s) = p^{a_i}$ for some $i \in \{1, 2, \ldots, r\}$ [4, 5]. The adjacency condition can be stated in terms of ideals as $b - a$ belongs to the ideal $p^{a_i}\mathbb{Z}$ but not $p^{a_i+1}\mathbb{Z}$ for some $i \in \{1, 2, \ldots, r\}$.

Suppose that $x, y \in R$ have the form

$$\begin{aligned} x &= v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1}, \\ y &= u_0 + u_1\theta + v_2\theta^2 + \cdots + u_{s-1}\theta^{s-1} \end{aligned}$$

for some $v_i, u_j \in \mathcal{V}$. Then

$$x - y \in R \backslash M \Leftrightarrow v_0 \neq u_0.$$

Thus, the adjacency matrix for $\mathrm{Cay}(R, C)$ is

$$\begin{array}{cc} & \begin{array}{cccc} e_1 + M & e_2 + M & \cdots & e_q + M \end{array} \\ A_0 = & \begin{pmatrix} A_1 & B_1 & \cdots & B_1 \\ B_1 & A_1 & \cdots & B_1 \\ B_1 & B_1 & \cdots & B_1 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_1 & \cdots & A_1 \end{pmatrix}, \end{array}$$

where

$$B_1 = \begin{cases} J_{q^{s-1} \times q^{s-1}} & \text{if } R \backslash M \subseteq C, \\ \mathbf{0}_{q^{s-1} \times q^{s-1}} & \text{if } R \backslash M \nsubseteq C, \end{cases}$$

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and A_1 is a $q^{s-1} \times q^{s-1}$ submatrix depending on M^i, $i \geq 1$. If $B_1 = \mathbf{0}_{q^{s-1}\times q^{s-1}}$, we set

$$A_0 = I_q \otimes A_1 \qquad \text{(Process A)}$$

and, if $B_1 = J_{q^{s-1}\times q^{s-1}}$, we set

$$A_0 = \overline{(I_q \otimes \overline{A}_1)}. \qquad \text{(Process B)}$$

Here, $J_{n\times n}$ is the matrix all of whose entries are 1 and $\overline{X}$ for an adjacency matrix X of a graph G denotes the adjacency matrix $J - I - X$ of the complement graph of G.

Next, we consider $x, y \in M$ such that

$$x = v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$
$$y = u_1\theta + v_2\theta^2 + \cdots + u_{s-1}\theta^{s-1},$$

for some $v_i, u_j \in \mathcal{V}$. Then

$$x - y \in M\backslash M^2 \Leftrightarrow v_1 \neq u_1.$$

Similarly, we have submatrices

$$B_2 = \begin{cases} J_{q^{s-2}\times q^{s-2}} & \text{if } M\backslash M^2 \subseteq C, \\ \mathbf{0}_{q^{s-2}\times q^{s-2}} & \text{if } M\backslash M^2 \nsubseteq C, \end{cases}$$

and A_2, which is a $q^{s-2} \times q^{s-2}$ submatrix depending on M^i for $i \geq 2$ such that

$$A_1 = \begin{cases} I_q \otimes A_2 & \text{if } B_2 = \mathbf{0}_{q^{s-2}\times q^{s-2}}, \\ \overline{(I_q \otimes \overline{A}_2)} & \text{if } B_2 = J_{q^{s-2}\times q^{s-2}}. \end{cases}$$

Continuing this process yields the submatrices $\{A_1, \ldots, A_{s-1}\}$ and $\{B_1, \ldots, B_{s-1}\}$.

Lemma 2.3. *Let $i \in \{1, 2, \ldots, s-1\}$. Assume that* $\operatorname{Spec} A_i = \left(\begin{smallmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{smallmatrix}\right)$, *where λ_1 is the largest eigenvalue. Then*

$$\operatorname{Spec} \overline{(I_q \otimes \overline{A}_i)} = \begin{pmatrix} q^{s-i}(q-1) + \lambda_1 & \lambda_1 - q^{s-i} & \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ 1 & q-1 & q(m_1 - 1) & qm_2 & \cdots & qm_k \end{pmatrix}.$$

In particular, if $m_1 = 1$, then

$$\operatorname{Spec} \overline{(I_q \otimes \overline{A}_i)} = \begin{pmatrix} q^{s-i}(q-1) + \lambda_1 & \lambda_1 - q^{s-i} & \lambda_2 & \cdots & \lambda_k \\ 1 & q-1 & qm_2 & \cdots & qm_k \end{pmatrix}.$$

Proof. Observe that the size of A_i is $|M^i| = q^{s-i}$ and the graph associated with A_i is regular. Then

$$\operatorname{Spec} \overline{A}_i = \begin{pmatrix} q^{s-i} - \lambda_1 - 1 & -1 - \lambda_1 & -1 - \lambda_2 & \cdots & -1 - \lambda_k \\ 1 & m_1 - 1 & m_2 & \cdots & m_k \end{pmatrix},$$

which implies that

$$\operatorname{Spec}(I_q \otimes \overline{A}_i) = \begin{pmatrix} q^{s-i} - \lambda_1 - 1 & -1-\lambda_1 & -1-\lambda_2 & \cdots & -1-\lambda_k \\ q & q(m_1-1) & qm_2 & \cdots & qm_k \end{pmatrix}$$

and so

$$\begin{aligned} \operatorname{Spec}\overline{(I_q \otimes \overline{A}_i)} &= \begin{pmatrix} q^{s-i+1} - (q^{s-i} - \lambda_1 - 1) - 1 & -1 - (q^{s-i} - \lambda_1 - 1) \\ 1 & q-1 \end{pmatrix.} \\ &\qquad \begin{matrix} -1-(-1-\lambda_1) & -1-(-1-\lambda_2) & \cdots & -1-(-1-\lambda_k) \\ q(m_1-1) & qm_2 & \cdots & qm_k \end{matrix}\Bigg) \\ &= \begin{pmatrix} q^{s-i+1} - q^{s-i} + \lambda_1 & \lambda_1 - q^{s-i} & \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ 1 & q-1 & q(m_1-1) & qm_2 & \cdots & qm_k \end{pmatrix} \end{aligned}$$

by Propositions 2.1 and 2.2. □

Repeatedly applying (Process A), (Process B) and Lemma 2.3 yields the following two lemmas.

Lemma 2.4. *Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and nilpotency s. Let*

$$C = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1})$$

with $0 \le a_1 < a_2 < \cdots < a_r \le s-1$. If $a_r = s-1$, then $\operatorname{Cay}(R, C)$ has the eigenvalues:

(1) $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ *with multiplicity* q^{a_1};
(2) $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_{k-1}}(q-1)$ *for* $k = 2, \ldots, r$;
(3) $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_k - a_{k-1}-1} - q^{a_{k-1}+1}$ *for* $k = 2, \ldots, r$;
(4) -1 *with multiplicity* $q^{a_r}(q-1)$.

Proof. Since $a_r = s-1$, $A_{a_r} = A_{s-1}$ is the adjacency matrix of the complete graph on $|M^{a_r}| = |M^{s-1}| = q$ vertices and so

$$\operatorname{Spec} A_{a_r} = \operatorname{Spec} A_{s-1} = \begin{pmatrix} q-1 & -1 \\ 1 & q-1 \end{pmatrix}.$$

It follows from Proposition 2.1 and Lemma 2.3 that any eigenvalues of A_i except λ_1 (which is the degree of the regular graph) remain the same after (Process A) and (Process B). So, -1 is an eigenvalue of $\operatorname{Cay}(R, C)$ with multiplicity $q^{a_r}(q-1)$. Next, we consider the eigenvalue $q-1$ of A_{s-1}. We apply (Process A) until it reaches $a_{r-1}+1$, which makes its multiplicity $q^{a_r - a_{r-1}-1}$, and follow by (Process B). By Lemma 2.3, the eigenvalues of $A_{a_{r-1}}$ induced from $q-1$ are:

(1) $q^{s-a_{r-1}-1}(q-1) + (q-1) = q^{s-a_{r-1}-1}(q-1) + q^{s-a_r-1}(q-1)$ with multiplicity 1;
(2) $q - 1 - q^{s-a_{r-1}-1}$ with multiplicity $q-1$;
(3) $q-1$ with multiplicity $q(q^{a_r - a_{r-1}-1} - 1)$.

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By the same reasoning, $q - 1 - q^{s-a_{r-1}-1}$ and $q - 1$ are eigenvalues of $\mathrm{Cay}(R, C)$ with multiplicities $q^{a_{r-1}}(q-1)$ and $q^{a_{r-1}+1}(q^{a_r - a_{r-1}-1} - 1) = q^{a_r} - q^{a_{r-1}+1}$, respectively. Applying these processes to the eigenvalue $q^{s-a_{r-1}-1}(q-1) + (q-1)$ until it reaches a_{r-2} yields the eigenvalues:

(1) $q^{s-a_{r-2}-1}(q-1) + q^{s-a_{r-1}-1}(q-1) + (q-1)$ with multiplicity 1;
(2) $q^{s-a_{r-1}-1}(q-1) + (q-1) - q^{s-a_{r-2}-1}$ with multiplicity $q-1$;
(3) $q^{s-a_{r-1}-1}(q-1) + (q-1)$ with multiplicity $q(q^{a_{r-1}-a_{r-2}-1} - 1)$.

Continuing this argument, we obtain the eigenvalues of $\mathrm{Cay}(R, C)$ as follows:

(1) $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ with multiplicity a_1;
(2) $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_{k-1}}(q-1)$ for $k = 2, \ldots, r$;
(3) $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_k} - q^{a_{k-1}+1}$ for $k = 2, \ldots, r$;
(4) -1 with multiplicity $q^{a_r}(q-1)$.

This completes the proof of the lemma. □

Lemma 2.5. *Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and nilpotency s. Let*

$$C = (M^{a_1} \backslash M^{a_1+1}) \cup (M^{a_2} \backslash M^{a_2+1}) \cup \cdots \cup (M^{a_r} \backslash M^{a_r+1})$$

with $0 \le a_1 < a_2 < \cdots < a_r \le s-1$. If $a_r \ne s-1$, the eigenvalues of $\mathrm{Cay}(R, C)$ are:

(1) $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ *with multiplicity* q^{a_1}*;*
(2) $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_{k-1}}(q-1)$ *for* $k = 2, \ldots, r$*;*
(3) $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_k} - q^{a_{k-1}+1}$ *for* $k = 2, \ldots, r$*;*
(4) $-q^{s-a_r-1}$ *with multiplicity* $q^{a_r}(q-1)$*;*
(5) 0 *with multiplicity* $q^{a_r+1}(q^{s-a_r-1} - 1)$.

Proof. Since $a_r \ne s-1$, $A_{a_r+1} = \mathbf{0}$, so $\overline{A}_{a_r+1}$ is the adjacency matrix of the complete graph on $|M^{a_r+1}| = q^{s-a_r-1}$ vertices. Then

$$\operatorname{Spec} \overline{A}_{a_r+1} = \begin{pmatrix} q^{s-a_r-1} - 1 & -1 \\ 1 & q^{s-a_r-1} - 1 \end{pmatrix}$$

and hence

$$\operatorname{Spec} I_q \otimes \overline{A}_{a_r+1} = \begin{pmatrix} q^{s-a_r-1} - 1 & -1 \\ q & q(q^{s-a_r-1} - 1) \end{pmatrix}$$

and

$$\begin{aligned} \operatorname{Spec} A_{a_r} &= \operatorname{Spec} \overline{(I_q \otimes \overline{A}_{a_r+1})} \\ &= \begin{pmatrix} q^{s-a_r} - q^{s-a_r-1} & -q^{s-a_r-1} & 0 \\ 1 & q-1 & q(q^{s-a_r-1} - 1) \end{pmatrix}. \end{aligned}$$

By Lemma 2.3, $-q^{s-a_r-1}$ and 0 are eigenvalues of $\mathrm{Cay}(R, C)$ with respective multiplicities $q^{a_r}(q-1)$ and $q^{a_r+1}(q^{s-a_r-1} - 1)$. The eigenvalue $q^{s-a_r} - q^{s-a_r-1} = q^{s-a_r-1}(q-1)$ of A_{a_r} induces the eigenvalues of $A_{a_{r-1}}$ as follows:

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(1) $q^{s-a_{r-1}-1}(q-1)+q^{s-a_r-1}(q-1)$ with multiplicity 1;
(2) $q^{s-a_r-1}(q-1)-q^{s-a_{r-1}-1}$ with multiplicity $q-1$;
(3) $q^{s-a_r-1}(q-1)$ with multiplicity $q(q^{a_r-a_{r-1}-1}-1)$.

Similarly, $q^{s-a_r-1}(q-1)-q^{s-a_{r-1}-1}$ and $q^{s-a_r-1}(q-1)$ are eigenvalues of $\mathrm{Cay}(R,C)$ with multiplicities $q^{a_{r-1}}(q-1)$ and $q^{a_{r-1}+1}(q^{a_r-a_{r-1}-1}-1)$, respectively. Moreover, the eigenvalue $q^{s-a_{r-1}-1}(q-1)+q^{s-a_r-1}(q-1)$ of $A_{a_{r-1}}$ gives the following eigenvalues of $A_{a_{r-2}}$:

(1) $q^{s-a_{r-2}-1}(q-1)+q^{s-a_{r-1}-1}(q-1)+q^{s-a_r-1}(q-1)$ with multiplicity 1;
(2) $q^{s-a_{r-1}-1}(q-1)+q^{s-a_r-1}(q-1)-q^{s-a_{r-2}-1}$ with multiplicity $q-1$;
(3) $q^{s-a_{r-1}-1}(q-1)+q^{s-a_r-1}(q-1)$ with multiplicity $q(q^{a_{r-1}-a_{r-2}-1}-1)$.

Repeating this process, we finally obtain the eigenvalues of $\mathrm{Cay}(R,C)$:

(1) $(q-1)\sum_{i=1}^{r}q^{s-a_i-1}$ with multiplicity q^{a_1};
(2) $-q^{s-a_{k-1}-1}+(q-1)\sum_{i=k}^{r}q^{s-a_i-1}$ with multiplicity $q^{a_{k-1}}(q-1)$ for $k=2,\dots,r$;
(3) $(q-1)\sum_{i=k}^{r}q^{s-a_i-1}$ with multiplicity $q^{a_k}-q^{a_{k-1}+1}$ for $k=2,\dots,r$;
(4) $-q^{s-a_r-1}$ with multiplicity $q^{a_r}(q-1)$;
(5) 0 with multiplicity $q^{a_r+1}(q^{s-a_r-1}-1)$,

as desired. □

Finally, we compute the energy of the graph $\mathrm{Cay}(R,C)$.

THEOREM 2.6. *Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and nilpotency s. Let*

$$C=(M^{a_1}\backslash M^{a_1+1})\cup(M^{a_2}\backslash M^{a_2+1})\cup\cdots\cup(M^{a_r}\backslash M^{a_r+1})$$

with $0\le a_1<a_2<\cdots<a_r\le s-1$. *Then*

$$E(\mathrm{Cay}(R,C))=2(q-1)\Big(q^{s-1}r-(q-1)\sum_{k=1}^{r-1}\sum_{i=k+1}^{r}q^{s-a_i+a_k-1}\Big).$$

PROOF. Observe that the eigenvalues and multiplicities of items (1)–(3) in Lemmas 2.4 and 2.5 are identical. Moreover, the product of the eigenvalue and its multiplicity in item (4) of Lemmas 2.4 and 2.5 is $-q^{s-1}(q-1)$. Thus, both cases have the same energy, which can be obtained by a direct computation. □

REMARK 2.7. When $R=\mathbb{Z}_{p^s}$, this result is [10, Theorem 2.1].

We shall close this section by showing that our Cayley graph is indeed an integral circulant.

Let R be a finite chain ring R with unique maximal ideal M and residue field of $q=p^t$ elements. Assume that R is of nilpotency s and M is generated by $\theta\in M\backslash M^2$. Then, for each $x\in R$,

$$x=v_0+v_1\theta+v_2\theta^2+\cdots+v_{s-1}\theta^{s-1},$$

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where $v_i \in \mathcal{V} = \{e_0, e_1, \ldots, e_{p^t-1}\}$, a fixed set of representatives of cosets in R/M, and

$$C_1 = (M^{a_1} \backslash M^{a_1+1}) \cup (M^{a_2} \backslash M^{a_2+1}) \cup \cdots \cup (M^{a_r} \backslash M^{a_r+1})$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$. Note that the ring $\mathbb{Z}_{q^s} = \mathbb{Z}_{p^{ts}}$ is a finite chain ring with the chain

$$\mathbb{Z}_{p^{ts}} \supset p\mathbb{Z}_{p^{ts}} \supset p^2\mathbb{Z}_{p^{ts}} \supset \cdots \supset p^{ts-1}\mathbb{Z}_{p^{ts}} \supset p^{ts}\mathbb{Z}_{p^{ts}} = \{0\}$$

having

$$\mathbb{Z}_{p^{ts}} \supset p^t\mathbb{Z}_{p^{ts}} \supset p^{2t}\mathbb{Z}_{p^{ts}} \supset \cdots \supset p^{(s-1)t}\mathbb{Z}_{p^{ts}} \supset p^{st}\mathbb{Z}_{p^{ts}} = \{0\}$$

as a subchain. This observation implies that each $a \in \mathbb{Z}_{p^{st}}$ can be expressed as

$$a = c_0 + c_1 p^t + c_2 p^{2t} \cdots + c_{s-1} p^{(s-1)t},$$

where $c_i \in \{0, 1, \ldots, p^t - 1\}$. Let $g : e_i \mapsto i$ be a bijection from $\mathcal{V}$ onto $\{0, 1, \ldots, p^t - 1\}$. Let $C_2 = \{p^{a_1 t}, p^{a_1 t+1}, \ldots, p^{a_1 t+t-1}, \ldots, p^{a_r t}, p^{a_r t+1}, \ldots, p^{a_r t+t-1}\}$. We shall show that the graphs $\mathrm{Cay}(R, C_1)$ and $\mathrm{Cay}(\mathbb{Z}_{p^{st}})$ are isomorphic.

Define $f : \mathrm{Cay}(R, C_1) \to \mathrm{Cay}(\mathbb{Z}_{p^{st}}, C_2)$ by

$$f(v_0 + v_1\theta + \cdots + v_{s-1}\theta^{s-1}) = g(v_0) + g(v_1)p^t + g(v_2)p^{2t} + \cdots + g(v_{s-1})p^{(s-1)t}.$$

Then f is a well-defined bijection. To see that f is an isomorphism, we let

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1} \quad \text{and} \quad y = u_0 + u_1\theta + u_2\theta^2 + \cdots + u_{s-1}\theta^{s-1}.$$

Suppose that x and y are adjacent in $\mathrm{Cay}(R, C_1)$. Then $x - y \in M^{a_i} \backslash M^{a_i+1}$ for some a_i. This means that $v_i = u_i$ for $i < a_i$ and $v_{a_i} \neq u_{a_i}$. Thus, $g(v_i) = g(u_i)$ for $i < a_i$ and $g(v_{a_i}) \neq g(u_{a_i})$, so $f(x) - f(y) \in p^{a_i t}\mathbb{Z}_{p^{st}} \backslash p^{(a_i+1)t}\mathbb{Z}_{p^{st}}$. Then, as elements of $\mathbb{Z}$, $\gcd(f(x) - f(y), p^{st}) = p^j$, where $a_i t \leq j < (a_i + 1)t$ and thus $f(x)$ and $f(y)$ are adjacent in $\mathrm{Cay}(\mathbb{Z}_{p^{st}}, C_2)$. Conversely, assume that $f(x)$ and $f(y)$ are adjacent in $\mathrm{Cay}(\mathbb{Z}_{p^{st}}, C_2)$. Then, as elements of $\mathbb{Z}$, $\gcd(f(x) - f(y), p^{st}) = p^j$, where $a_i t \leq j < (a_i + 1)t$ for some a_i. It follows that for

$$f(x) = g(v_0) + g(v_1)p^t + g(v_2)p^{2t} + \cdots + g(v_{s-1})p^{(s-1)t} \quad \text{and}$$
$$f(y) = g(u_0) + g(u_1)p^t + g(u_2)p^{2t} + \cdots + g(u_{s-1})p^{(s-1)t},$$

we have $g(v_i) = g(u_i)$ for $i < a_i$ and $g(v_{a_i}) \neq g(u_{a_i})$. Thus, $x - y \in M^{a_i} \backslash M^{a_i+1}$ and hence x and y are adjacent in $\mathrm{Cay}(R, C_1)$. Hence, we have shown the following proposition.

Proposition 2.8. *Let R be a finite chain ring with unique maximal ideal M, residue field of $q = p^t$ elements and nilpotency s. Let*

$$C_1 = (M^{a_1} \backslash M^{a_1+1}) \cup (M^{a_2} \backslash M^{a_2+1}) \cup \cdots \cup (M^{a_r} \backslash M^{a_r+1})$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$. Then

$$\mathrm{Cay}(R, C_1) \cong \mathrm{Cay}(\mathbb{Z}_{p^{st}}, C_2),$$

where $C_2 = \{p^{a_1 t}, p^{a_1 t+1}, \ldots, p^{a_1 t+t-1}, \ldots, p^{a_r t}, p^{a_r t+1}, \ldots, p^{a_r t+t-1}\}$.

3. gcd-graphs over a unique factorisation domain

Let D be a unique factorisation domain (UFD) and $c \in D$ a nonzero nonunit element. Assume that the commutative ring $D/(c)$ is finite. Write $c = p_1^{s_1} \cdots p_k^{s_k}$ as a product of irreducible elements.

We now study the gcd-graph $D_c(C)$. Suppose that for each $i \in \{1, 2, \ldots, k\}$, there exists a set $C_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$ with $0 \leq a_{i1} < a_{i2} < \cdots < a_{ir_i} \leq s_i - 1$ so that

$$C = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} : t_i \in \{1, 2, \ldots, r_i\} \text{ for all } i \in \{1, 2, \ldots, k\}\}.$$

Then, for $x, y \in D/(c)$,

$$x \text{ is adjacent to } y \Leftrightarrow \gcd(x - y, c) \in D^\times C \Leftrightarrow \gcd(x - y, p_i^{s_i}) \in D^\times C_i \text{ for all } i.$$

This implies that

$$D_c(C) = \mathrm{Cay}(D/(p_1^{s_1}), C_1) \otimes \cdots \otimes \mathrm{Cay}(D/(p_k^{s_k}), C_k),$$

where each factor on the right is the Cayley graph over the finite chain ring $D/(p_i^{s_i})$ for which we have already computed the energy in Section 2. Recall from Proposition 2.1 that $E(G \otimes H) = E(G)E(H)$ for two graphs G and H. Therefore, we have the following theorem.

THEOREM 3.1. *Let D be a UFD and let $c = p_1^{s_1} \cdots p_k^{s_k}$ be a nonzero nonunit in D factored as a product of irreducible elements. Assume that $D/(c)$ is finite and, for each $i \in \{1, 2, \ldots, k\}$, there exists a set $C_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$ with $0 \leq a_{i1} < a_{i2} < \cdots < a_{ir_i} \leq s_i - 1$ such that*

$$C = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} : t_i \in \{1, 2, \ldots, r_i\} \textit{ for all } i \in \{1, 2, \ldots, k\}\}.$$

Then

$$E(D_c(C)) = E(D_{p_1^{s_1}}(C_1)) \cdots E(D_{p_k^{s_k}}(C_k)).$$

REMARK 3.2. Recall that if a matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of $A + I$ are $\lambda_1 + 1, \ldots, \lambda_n + 1$. Hence, one can obtain the energy of the gcd-graph in Theorem 3.1 when C_i contains $p_i^{s_i}$ using this fact and the eigenvalues computed in Lemma 2.4 or 2.5.

Now, we study the case where some $C_j = \{p_j^{s_j}\}$. To compute the energy in this case, we shall use a graph operation which is more general than the tensor product called a noncomplete extended p-sum [7] defined as follows.

Given a set $B \subseteq \{0, 1\}^k$ and graphs $G_1, \ldots, G_k$, the *NEPS (noncomplete extended p-sum)*, $G = \mathrm{NEPS}(G_1, \ldots, G_k; B)$, of these graphs with respect to the basis B has as its vertex set the Cartesian product of the vertex sets of the individual graphs, that is, $V(G) = V(G_1) \times \cdots \times V(G_k)$. Two distinct vertices $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are adjacent in G if and only if there exists some k-tuple $(\beta_1, \ldots, \beta_k) \in B$ such that $x_i = y_i$ whenever $\beta_i = 0$ and x_i, y_i are distinct and adjacent in G_i whenever $\beta_i = 1$. In particular, when $B = \{(1, 1, \ldots, 1)\}$,

$$\mathrm{NEPS}(G_1, \ldots, G_k; B) = G_1 \otimes G_2 \otimes \cdots \otimes G_k.$$

The eigenvalues of the graph $\mathrm{NEPS}(G_1, \ldots, G_k; B)$ are presented in the next theorem.

THEOREM 3.3 [1]. *Let $G_1, \ldots, G_k$ be graphs with $n_1, \ldots, n_k$ vertices, respectively, and, for $i \in \{1, \ldots, k\}$, let $\lambda_{i1}, \ldots, \lambda_{in_i}$ be the eigenvalues of G_i. Then the spectrum of the graph $G = \mathrm{NEPS}(G_1, \ldots, G_k; B)$ consists of all possible values*

$$\mu_{i_1,\ldots,i_k} = \sum_{(\beta_1,\ldots,\beta_k)\in B} \lambda_{1i_1}^{\beta_1} \cdots \lambda_{ki_k}^{\beta_k}$$

with $1 \le i_l \le n_l$ for $1 \le l \le k$.

Next, we consider $c = p_1^{s_1} \cdots p_k^{s_k}$ written as a product of irreducible elements. We suppose that $l \le k$ and that, for each $i \in \{1, 2, \ldots, l\}$, there exists a set $C_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$ with $0 \le a_{i1} < a_{i2} < \cdots < a_{ir_i} \le s_i - 1$ so that

$$C' = \{p_1^{a_{1t_1}} \cdots p_l^{a_{lt_l}} p_{l+1}^{s_{l+1}} \cdots p_k^{s_k} : t_i \in \{1, 2, \ldots, r_i\} \text{ for all } i \in \{1, 2, \ldots, l\}\}.$$

Then

$$D_c(C') = \mathrm{NEPS}(D_{p_1^{s_1}}(C_1), D_{p_2^{s_2}}(C_2), \ldots, D_{p_k^{s_k}}(C_k); \{(\underbrace{1, \ldots, 1}_{l}, \underbrace{0, \ldots, 0}_{k-l})\}),$$

where $C_j = \{p_j^{s_j}\}$ for $l < j \le k$. By Theorem 3.3, all eigenvalues of $D_c(C')$ are the eigenvalues of

$$\mathrm{Cay}(D/(p_1^{s_1}), C_1) \otimes \cdots \otimes \mathrm{Cay}(D/(p_l^{s_l}), C_l)$$

each repeated $\prod_{j=l+1}^{k} |D/(p_j^{s_j})|$ times. We deduce the following result from Theorem 3.1.

THEOREM 3.4. *Let D be a UFD and let $c = p_1^{s_1} \cdots p_k^{s_k} \in D$ a nonzero nonunit factored as a product of irreducible elements. Let $l \le k$. Assume that $D/(c)$ is finite and that, for each $i \in \{1, 2, \ldots, l\}$, there exists a set $C_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$ such that $0 \le a_{i1} < a_{i2} < \cdots < a_{ir_i} \le s_i - 1$ and*

$$C' = \{p_1^{a_{1t_1}} \cdots p_l^{a_{lt_l}} p_{l+1}^{s_{l+1}} \cdots p_k^{s_k} : t_i \in \{1, 2, \ldots, r_i\} \text{ for all } i \in \{1, 2, \ldots, l\}\}.$$

Then

$$E(D_c(C')) = E(D_{p_1^{s_1}}(C_1)) \cdots E(D_{p_l^{s_l}}(C_l)) \prod_{j=l+1}^{k} |D/(p_j^{s_j})|.$$

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