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CERTAIN INVARIANT SUBSPACES OF H^2 AND L^2 ON A BIDISC

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1. Introduction. We let T^2 be the torus that is the cartesian product of 2 unit circles in **C**. The usual Lebesgue spaces, with respect to the Haar measure m of T^2, are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_K f(z, w)\bar{z}^j\bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of (j, ℓ) is negative.

A closed subspace M of L^2 is said to be invariant if

$$zM \subset M \text{ and } wM \subset M.$$

Whenever this is the case, it follows that $fM \subset M$ for every f in H^∞. One can ask for a classification or an explicit description (in some sense) of all invariant subspaces of L^2, but this seems out of reach.

Ahern and Clark (cf. [**6**, 80]) described completely invariant subspaces of H^2, of finite codimension. It is easy to describe invariant subspaces M of L^2 in case $\bar{z}^jM \subset M$ for $j \geqq 0$ (cf. [**5**, 164-165]). Helson and Lowdenslager (cf. [**2**, 8]) described invariant subspaces M of L^2 in case $\bar{z}^jwM \subset M$ for $j \geqq 0$. Recently Curto, Muhly, Nakazi and Yamamoto [**1**] considered invariant subspaces M of L^2 in case $\bar{z}^jw^nM \subset M$ for $j \geqq 0$ when $n > 0$ is fixed. However these invariant subspaces do not have the form FH^2 for some unimodular function F.

In this paper we consider invariant subspaces of L^2 which have the form FH^2 for some unimodular function F. This is a direct generalization of a Beurling's theorem (cf. [**2**, 8]) in the case of one variable. It should be noted that there are many invariant subspaces M even in H^2 such that M does not have the form FH^2. Hence we wish to consider an invariant subspace of L^2 which has the form FN for some unimodular function F where N is an invariant subspace between H^2 and the L^2-closure of $\cup_{n \geqq 0} \bar{z}^n H^2$. In this paper, if M is an invariant subspace of H^2, of finite co-dimension, it is shown that M has the form FN for some unimodular function F.

Let $C(T^2)$ be the space of complex-valued continuous functions on T^2. We shall let $\mathscr{A}_j$, $\mathscr{B}_j$ and $\mathscr{C}_j$ denote the following subalgebras of $C(T^2)$ for $j = 1, 2$:

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(i) $\mathscr{A}_j$ is the uniform closure of the polynomials in z_j where $z_1 = z$ and $z_2 = w$;

(ii) $\mathscr{B}_j$ is the uniform closure of the polynomials in z_j, $\bar{z}_j$ and z_i where $j \neq i$;

(iii) $\mathscr{C}_j$ is the uniform closure of the polynomials in z_j and $\bar{z}_j$.

Definition. The closure in L^2 of $\mathscr{B}_j$, $[\mathscr{B}_j]_2$, will be denoted $\mathbf{H}_j$. We define $\mathscr{L}_j = [\mathscr{C}_j]_2$ and $\mathscr{H}_j = [\mathscr{A}_j]_2$.

Observe that:

(iv) $\mathscr{H}_j$ is isometrically isomorphic to the classical Hardy space on T;

(v) $\mathscr{L}_j$ is isometrically isomorphic to the Lebesgue space on T;

(vi) $\mathbf{H}_j$ is the tensor product of $\mathscr{L}_j$ and $\mathscr{H}_i$ where $j \neq i$, that is, $\mathbf{H}_j = \mathscr{L}_j \otimes \mathscr{H}_i$.

2. Invariant subspaces containing H^2. We are specially interested in invariant subspaces in H^2. But we have interest in invariant subspaces containing H^2 because they shed light on researches about invariant subspaces in H^2.

THEOREM 1. *If M is an invariant subspace which contains H^2 properly, then $M \ominus H^2$ is infinite dimensional.*

Proof. Suppose $M \ominus H^2$ is finite dimensional. Let P be the orthogonal projection in L^2 with range $M \ominus H^2$, and let the operator S_ϕ on $M \ominus H^2$ be defined by $S_\phi f = P(\phi f)$ where $\phi \in H^\infty$. S_ϕ is of finite rank and hence there exists an analytic polynomial p such that $p(S_\phi) = 0$. Therefore $S_{p(\phi)} = 0$ because $S_{\phi\psi} = S_\phi S_\psi$ for any ϕ and ψ in H^∞. If we choose $\phi = z$ then $S_{p(z)} = 0$ and hence $p(z)M \subset H^2$. By the inner outer factorization of $p(z) \in \mathscr{H}_1$ (cf. [**2**, 12]), there is a finite Blashke product $q_1 \in \mathscr{H}_1$ such that $q_1 M \subset H^2$. Similarly there is a finite Blashke product $q_2 \in \mathscr{H}_2$ such that $q_2 M \subset H^2$. Thus

$$\bar{q}_1 H^2 \cap \bar{q}_2 H^2 \supset M.$$

While $\bar{q}_1 H^2 \cap \bar{q}_2 H^2 = H^2$ because $\bar{q}_1 H^2 \subset \mathbf{H}_1$ and $\bar{q}_2 H^2 \subset \mathbf{H}_2$. This contradiction implies that $M \ominus H^2$ is infinite dimensional.

The theorem above is interesting because there are many invariant subspaces in H^2, of finite codimension. We shall study some special invariant subspaces containing H^2.

LEMMA 1. *Suppose M is an invariant subspace which contains H^2 and $K = M \ominus H^2$. If $wK \subset K$ then $K \subset \bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$.*

Proof. If there is a function $f \in K$ such that $\hat{f}(j, \ell) \neq 0$ for some $j \geqq 0$ and $\ell < 0$, then $\bar{w}^\ell f$ is not orthogonal to H^2. While $\bar{w}^\ell f \in K$ because $wK \subset K$ and $-\ell > 0$. This contradiction implies $\hat{f}(j, \ell) = 0$ if $j \geqq 0$ and $\ell < 0$, and hence $K \subset \bar{z}\bar{\mathbf{H}}_2 = \bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{L}_2$. Since

$$zK \subset H^2 \oplus K = \mathscr{H}_2 \oplus zH^2 \oplus K$$

and zK is orthogonal to zH^2, $zK \subset \mathscr{H}_2 \oplus K$. If $f \in K$ then

$$f = \sum_{j=1}^{\infty} u_j \bar{z}^j$$

where $u_j \in \mathscr{L}_2$ because $f \in \bar{z}\bar{\mathbf{H}}_2$. Hence

$$zf = u_1 + \sum_{j=2}^{\infty} u_j \bar{z}^{j-1},$$

$u_1 \in \mathscr{H}_2$ and $\sum_{j=2}^{\infty} u_j \bar{z}^{j-1}$ belongs to K because $zf \in \mathscr{H}_2 \oplus K$. Proceeding similarly $u_j \in \mathscr{H}_2$ for $j \geqq 2$ and hence $K \subset \bar{z}\bar{\mathscr{H}}_2 \otimes \mathscr{H}_2$.

THEOREM 2. *Let M be an invariant subspace of L^2. Suppose $M \supset H^2$ and $K = M \ominus H^2$.*

(1) *$wK \subset K$ if and only if $M \subset \mathbf{H}_1$.*

(2) *$zK \subset K$ and $wK \subset K$ if and only if $M = H^2$.*

Proof. (1) By Lemma 1 if $wK \subset K$ then $M \subset \mathbf{H}_1$. Conversely if $M \subset \mathbf{H}_1$ and $f \in K$ then

$$f = \sum_{j=1}^{\infty} f_j \bar{z}^j \quad \text{and} \quad f_j \in \mathscr{H}_2.$$

Since $wf \in \bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ and $wf \in M$, wf belongs to K. (2) If $zK \subset K$ and $wK \subset K$ then by (1) $M \subset \mathbf{H}_1 \cap \mathbf{H}_2 = H^2$.

3. The dimension of $M \ominus wM$. Let M be an invariant subspace of L^2. If $M = wM$ then

$$M = \chi_{E_1} F \mathbf{H}_2 + \chi_{E_2} L^2$$

where χ_{E_1} is a characteristic function of Borel set on T^2 in $\mathscr{L}_2$ and $\chi_{E_1} + \chi_{E_2} \leqq 1$ a.e. by [**1**] and [**5**, 164-165]. Thus, we are interested in M with $M \ominus wM \neq \{0\}$.

THEOREM 3. *Suppose M is an invariant subspace of L^2 such that if $gM \subset M$ then g belongs to H^∞. Then $M \ominus wM$ is infinite dimensional.*

Proof. Suppose $M \ominus wM$ is finite dimensional. Let P be the orthogonal projection in L^2 with range $M \ominus wM$, and let the operator S_ϕ on $M \ominus wM$ be defined by $S_\phi f = P(\phi f)$ where $\phi \in H^\infty$. As in the proof of Theorem 1, there is a finite Blashke product $q \in \mathscr{H}_1$ such that $qM \subset wM$ because S_z is of finite rank. Thus $q\bar{w}M \subset M$. This contradiction implies that $M \ominus wM$ is infinite dimensional.

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PROPOSITION 4. *Let M be an invariant subspace. If $M \ominus wM$ is one dimensional then*

$$\bigcap_{n \geqq 0} w^n M \neq \{0\}.$$

Proof. Suppose

$$\bigcap_{n \geqq 0} w^n M = \{0\}.$$

Then

$$M = \sum_{j=0}^{\infty} \oplus [f]_2 w^j$$

because $M \ominus wM$ is one dimensional. Hence

$$zf = \sum_{j=0}^{\infty} c_j f w^j$$

and z is in the closure of $\mathscr{A}_2$ in $L^2(|f|^2 dm)$. This contradiction implies that

$$\bigcap_{n \geqq 0} w^n M \neq \{0\}.$$

Set $M = F(q\mathscr{H}_2 \oplus z\mathbf{H}_2)$ where F and q are unimodular functions, and $q \in \mathscr{H}_2$. This is an example of an invariant subspace such that $M \ominus wM$ is one dimensional.

4. Beurling type. Beurling (cf. [**2**, 8]) showed that any nonzero invariant subspaces in the usual Hardy space $H^2(T)$ on T has the form $FH^2(T)$ for some inner function F, that is, some unimodular function in $H^2(T)$. This was generalized to invariant subspaces M in the usual Lebesgue space $L^2(T)$ on T such that $M \ominus wM \neq \{0\}$ by [**3**]. We wish to generalize this well known Beurling's theorem to invariant subspaces in $L^2(T^2)$.

THEOREM 5. *Let M be an invariant subspace of L^2 and $M \ominus wM = S \neq \{0\}$.*

(1) *$zS = S$ if and only if*

$$M = \chi_{E_1} F\mathbf{H}_1 \oplus \chi_{E_2} L^2$$

where χ_{E_1} is in $\mathscr{L}_1$, $\chi_{E_1} + \chi_{E_2} \leqq 1$ a.e. and F is a unimodular function.

(2) *$zS \subsetneq S$ if and only if $M = FH^2$ for some unimodular function F.*

Proof. (1) The 'if' part is obvious. If $zS = S$ then $zM = M$. Hence by the remark above Theorem 3, the 'only if' part follows.

(2) The 'if' part is obvious. Suppose $zS \subsetneq S$ and set $S_0 = S \ominus zS$.

Suppose nonzero $f \in S_0$. Since $z^j f \in S$ for any $j \geqq 0$, $z^j f$ is orthogonal to $w^\ell f$ if $\ell \geqq 1$ and $j \geqq 0$. Hence

$$\int |f|^2 z^j \bar{w}^\ell dm = 0 \quad \text{if } \ell \geqq 1 \text{ and } j \geqq 0.$$

Since $z^j w^\ell f \in wM$ for any $j \geqq 0$ and any $\ell \geqq 1$, f is orthogonal to $z^j w^\ell f$ if $\ell \geqq 1$ and $j \geqq 0$. Hence

$$\int |f|^2 \bar{z}^j \bar{w}^\ell dm = 0 \quad \text{if } \ell \geqq 1 \text{ and } j \geqq 0.$$

Moreover f is orthogonal to $z^j f$ if $j \geqq 1$ and hence

$$\int |f|^2 \bar{z}^j dm = 0 \quad \text{if } j \geqq 1.$$

Thus, for any non zero $f \in S_0$, $|f|^2$ is constant a.e. If $f, g \in S_0$ and $|f| = |g| = 1$ a.e. then $|1 + \bar{f}g| = 2$ and hence $\arg f = \arg g$. Thus $f = g$ because $|f| = |g|$, and $S_0 = [F]_2$ for some unimodular function F.

Set

$$M_1 \bigcap_{j=0}^{\infty} w^j M \quad \text{and} \quad S_1 = \bigcap_{j=0}^{\infty} z^j S$$

then

$$M = \left(\sum_{j=0}^{\infty} \oplus S w^j \right) \oplus M_1 \quad \text{and} \quad S = \left(\sum_{j=0}^{\infty} \oplus S_0 z^j \right) \oplus S_1.$$

Since $S_0 = [F]_2$ and $|F| = 1$, $S = F\mathscr{H}_1 \oplus S_1$ and $zS_1 = S_1$. Hence

$$M = FH^2 \oplus \left(\sum_{j=0}^{\infty} \oplus S_1 w^j \right) \oplus M_1.$$

Since $wM_1 = M_1$,

$$M_1 = \chi_{E_1} F_2 \mathbf{H}_2 \oplus \chi_{E_2} L^2$$

where $\chi_{E_1} \in \mathscr{L}_2$, $\chi_{E_1} + \chi_{E_2} \leqq 1$ a.e. and $|F_2| = 1$ a.e. Since $zS_1 = S_1$, by (1)

$$\sum_{j=0}^{\infty} \oplus S_1 w^j = \chi_{G_1} F_1 \mathbf{H}_1$$

where $\chi_{G_1} \in \mathscr{L}_1$ and $|F_1| = 1$ a.e. Hence

$$M = FH^2 \oplus \chi_{G_1} F_1 \mathbf{H}_1 \oplus \chi_{E_1} F_2 \mathbf{H}_2 \oplus \chi_{E_2} L^2.$$

Since FH^2 is orthogonal to $\chi_{E_2} L^2$, $\chi_{E_2} L^2 = \{0\}$. Thus if we set

$$K = \chi_{G_1} \bar{F} F_1 \mathbf{H}_1 \oplus \chi_{E_1} \bar{F} F_2 \mathbf{H}_2 \quad \text{and} \quad N = H^2 \oplus K,$$

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then $M = FN$ and N is an invariant subspace containing H^2, $zK \subset K$ and $wK \subset K$. By (2) of Theorem 2 $N = H^2$ and $M = FH^2$.

If M is an invariant proper subspace of H^2, of finite codimension and $M = FH^2$ then $\bar{F}M$ contains H^2 properly and $\bar{F}M \ominus H^2$ is finite dimensional. This contradicts Theorem 1. Hence we wish to consider invariant subspaces which don't have the form FH^2.

THEOREM 6. *Let M be an invariant subspace of L^2 and $M \ominus wM = S$.*

(1) *There exists f in S such that $z^j f$ belongs to S for any j and $|f| > 0$ a.e. if and only if $M = F\mathbf{H}_1$ for some unimodular function F.*

(2) *There exists f in S such that $z^j f$ belongs to S for any $j \geqq 0$ and $z^\ell f$ is not in S for some $\ell < 0$, if and only if $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in $\mathbf{H}_1$, and F is a unimodular function.*

Proof. Putting $M_1 = \cap_{j \geqq 0} w^j M$,

$$M = \left(\sum_{j \geqq 0} \oplus Sw^j\right) \oplus M_1.$$

Let S' be the largest closed subspace of S with $zS' \subset S'$. If we let

$$S_3 = S \ominus S', \quad S_2 = \bigcap_{j \geqq 0} z^j S' \quad \text{and} \quad S_1 = S' \ominus S_2,$$

then

$$M = \left(\sum_{j \geqq 0} \oplus S_1 w^j\right) \oplus \left(\sum_{j \geqq 0} \oplus S_2 w^j\right) \oplus \left(\sum_{j \geqq 0} \oplus S_3 w^j\right) \oplus M_1.$$

(1) Since $\sum_{j \geqq 0} \oplus S_2 w^j$ has no reducing subspaces under the multiplication by w and $zS_2 = S_2$, by (1) in Theorem 5

$$\sum_{j \geqq 0} \oplus S_2 w^j = \chi_{E_1} F \mathbf{H}_1$$

where $\chi_{E_1} \in \mathcal{L}_1$ and $|F| = 1$ a.e. If there exists f in S such that $z^j f \in S$ for any j and $|f| > 0$ a.e., then $f \in S_2$ and so $\chi_{E_1} = 1$ a.e. Thus

$$\bar{F}M \supset \mathbf{H}_1 \quad \text{and} \quad w(\bar{F}M \ominus \mathbf{H}_1) \subset \bar{F}M \ominus \mathbf{H}_1.$$

This implies $M = F\mathbf{H}_1$. The converse is obvious.

(2) If there exists $f \in S$ such that $z^j f \in S$ for any $j \geqq 0$ and $z^\ell f \notin S$ for some $\ell < 0$ then $S_1 \neq \{0\}$. By (2) in Theorem 5,

$$\sum_{j \geqq 0} \oplus S_1 w^j = FH^2$$

for some unimodular function F. Thus

$$\bar{F}M \supset H^2 \quad \text{and} \quad w(\bar{F}M \ominus H^2) \subset \bar{F}M \ominus H^2.$$

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By (1) in Theorem 2 $\bar{F}M \subset \mathbf{H}_1$. Set $N = \bar{F}M$ then N is the desired invariant subspace. Conversely if $M = FN$ and $H^2 \subset N \subsetneq \mathbf{H}_1$ then $wK \subset K$ by (1) in Theorem 2, where $K = N \ominus H^2$. Hence

$$M \ominus wM = F\mathscr{H}_1 \oplus F(K \ominus wK).$$

This implies that $z^j F \in M \ominus wM$ for any $j \geqq 0$. If $z^j F \in M \ominus wM$ for any j then

$$\mathscr{L}_1 \subset \mathscr{H}_1 \oplus (K \ominus wK)$$

and hence $\bar{z}\bar{\mathscr{H}}_1 \subset K \ominus wK$. Thus $K = \bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ and $N = \mathbf{H}_1$. This contradiction implies that $z^\ell F \notin M \ominus wM$ for some $\ell < 0$.

COROLLARY 1. *Let M be an invariant subspace of H^2 and $M \ominus wM = S$.*

(1) *$zS \subset S$ if and only if $M = FH^2$ for some inner function F.*

(2) *There exists f in S such that $z^j f$ belongs to S for any $j \geqq 0$ if and only if $M = FN$ where N is an invariant subspace containing H^2 and is contained properly in $\mathbf{H}_1$, and F is an inner function.*

5. Invariant subspaces between H^2 and $\mathbf{H}_1$. In Theorems 2 and 6 invariant subspaces between H^2 and $\mathbf{H}_1$ were important. In this section we shall study invariant subspaces N between H^2 and $\mathbf{H}_1$. Suppose $K = N \ominus H^2$. Let Q be the orthogonal projection from L^2 to $\bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$. The operator T_ϕ on $\bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ is defined by $T_\phi f = Q(\phi f)$ where $\phi \in H^\infty$. $H^2 \oplus K$ is an invariant subspace in $\mathbf{H}_1$ if and only if $wK \subset K$ and $T_z K \subset K$.

Definition. For a closed subspace K in $\bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ we say K has the property (*) when $wK \subset K$ and $T_z K \subset K$.

In order to study invariant subspaces between H^2 and $\mathbf{H}_1$ it is sufficient to study closed subspaces K in $\bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ that have the property (*).

PROPOSITION 7. *Let K be a closed subspace in $\bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ with the property (*). If T_z has finite rank n on K then $K \ominus wK$ is finite dimensional $\ell \leqq n$.*

Proof. As in the proof of Theorems 1 and 3 there is a finite Blashke product $q \in \mathscr{H}_1$ of degree $\ell' \leqq n$ such that $T_q K = \{0\}$. Hence

$$K \subset \bar{q}(H^2 \ominus qH^2) = \sum_{j=0}^{\infty} \oplus\, \bar{q}(\mathscr{H}_1 \ominus q\mathscr{H}_1)w^j$$

$$= \sum_{j=0}^{\infty} \oplus\, (\bar{z}\bar{\mathscr{H}}_1 \ominus \bar{z}\bar{q}\bar{\mathscr{H}}_1)w^j.$$

By Corollary 1 in [4] $K \ominus wK$ is finite dimensional $\ell \leqq \ell'$.

PROPOSITION 8. *Let K be a closed subspace in $\bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ with the property* (*). *T_z^n is zero on K for some $n > 0$ if and only if*

$$K = [w; f_1, \ldots, f_{n+1}]_2$$

where $[w; f_1, \ldots, f_{n+1}]_2$ is a closed invariant subspace under the multiplication by w that is generated by $f_1, \ldots, f_{n+1}$. Here $f_1, \ldots, f_{n+1}$ satisfy the following conditions:

$$(1) \quad f_j = \sum_{\ell=1}^{n} f_{j\ell}\bar{z}^{\ell}$$

where $f_{j\ell}$ is in $\mathscr{H}_2$ and $|f_{j\ell}| \leqq 1$ a.e. *for $1 \leqq \ell \leqq n$*;

$$(2) \quad \sum_{\ell=1}^{n} f_{j\ell}\bar{f}_{i\ell} = \delta_{ji};$$

$$(3) \quad \sum_{\ell=t}^{n} f_{j\ell}\bar{z}^{\ell-t} \text{ is in } K \text{ for any } t \leqq n.$$

Proof. Suppose $T_z^n = 0$ on K. By the proof of Proposition 7

$$K \subset \sum_{j=0}^{\infty} \oplus (\bar{z}\bar{\mathscr{H}}_1 \ominus \bar{z}^{n+1}\bar{\mathscr{H}}_1)w^j.$$

By a theorem of Lax (cf. [**2**, 61-64]), K has the form $[w; f_1, \ldots, f_{n+1}]_2$ that satisfies (1) and (2). (3) follows from $T_zK \subset K$. The converse is obvious.

By Proposition 8, T_z is zero on K if and only if $K = \bar{z}q\mathscr{H}_2$ for some inner function q in $\mathscr{H}_2$.

THEOREM 9. *Let K be a closed subspace in $\bar{z}\bar{\mathscr{H}}_1 \otimes \mathscr{H}_2$ with the property* (*).

(1) *T_z is rank one on K if and only if*

$$K = \bar{z}(1 - a\bar{z})^{-1}q\mathscr{H}_2$$

where $|a| < 1$, $a \neq 0$ and q is an inner function in $\mathscr{H}_2$.

(2) *$K \ominus wK$ is one dimensional if and only if*

$$K = \sum_{j=0}^{\infty} \oplus [f]_2 w^j$$

where $f = u(1 - v\bar{z})^{-1}$, u is in $\mathscr{H}_2$ and v is in the closure of $\mathscr{A}_2$ in $L^2(|f|^2dm)$.

Proof. (1) If T_z is rank one on K then by the proof of Proposition 7 for some nonzero a with $|a| < 1$

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$$K \subset \sum_{j=0}^{\infty} \oplus \left(\bar{z}\bar{\mathscr{H}}_1 \ominus \bar{z}\frac{1 - \bar{a}z}{z - a}\bar{\mathscr{H}}_1 \right) w^j$$

and hence $K \subset \bar{z}(1 - a\bar{z})^{-1}\mathscr{H}_2$. Thus by Beurling's theorem

$$K = \bar{z}(1 - a\bar{z})^{-1}q\mathscr{H}_2$$

for some inner function q in $\mathscr{H}_2$. For the converse, since

$$T_z(\bar{z}(1 - a\bar{z})^{-1}) = a\bar{z}(1 - a\bar{z})^{-1},$$

K has the property (*).

(2) If $K \ominus wK$ is one dimensional then

$$K = \sum_{j=0}^{\infty} \oplus [f]_2 w^j = [f\mathscr{A}_2]_2$$

$$= f \times \{\text{the closure of } \mathscr{A}_2 \text{ in } L^2(|f|^2dm)\}.$$

We can write

$$f = \sum_{\ell=1}^{\infty} f_\ell \bar{z}^\ell$$

where $f_\ell \in \mathscr{H}_2$. Then $f = \bar{z}f_1 + \bar{z}fv$ for some v in the closure of $\mathscr{A}_2$ in $L^2(|f|^2dm)$ because

$$T_z f = \sum_{\ell=2}^{\infty} f_\ell \bar{z}^{\ell-1}$$

is in K. Set $u = f_1$ then

$$f = u(1 - v\bar{z})^{-1}.$$

Conversely if $f = u(1 - v\bar{z})^{-1}$ then

$$zf = u + vf \quad \text{and} \quad T_z f = vf \in K.$$

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