ON THE NUMBER OF SIDES OF A PETRIE POLYGON

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Let $\{p, q, r\}$ be the regular 4-dimensional polytope for which each face is a $\{p, q\}$ and each vertex figure is a $\{q, r\}$, where $\{p, q\}$, for example, is the regular polyhedron with p-gonal faces, q at each vertex. A Petrie polygon of $\{p, q\}$ is a skew polygon made up of edges of $\{p, q\}$ such that every two consecutive sides belong to the same face, but no three consecutive sides do. Then a Petrie polygon of $\{p, q, r\}$ is defined by the property that every three consecutive sides belong to a Petrie polygon of a bounding $\{p, q\}$, but no four do. Let $h_{p,q,r}$ be the number of sides of such a polygon, and $g_{p,q,r}$ the order of the group of symmetries of $\{p, q, r\}$. Our purpose here is to prove the following formula:

$$\frac{h_{p,q,r}}{g_{p,q,r}} = \frac{1}{64}\left(12 - p - 2q - r + \frac{4}{p} + \frac{4}{r}\right). \tag{1}$$

We use the following result of Coxeter (**1**, p. 232; **2**):

$$\frac{h_{p,q,r}}{g_{p,q,r}} = \frac{1}{16}\left(\frac{6}{h_{p,q} + 2} + \frac{6}{h_{q,r} + 2} + \frac{1}{p} + \frac{1}{r} - 2\right), \tag{2}$$

where $h_{p,q}$, for example, denotes the number of sides of a Petrie polygon of $\{p, q\}$. Both proofs referred to depend on the fact that the number of hyperplanes of symmetry of $\{p, q, r\}$ is $2h_{p,q,r}$. This is proved in a more general form in (**3**). Clearly (1) is a consequence of (2) and the following result:

If h is the number of sides of a Petrie polygon of the polyhedron $\{p, q\}$, then

$$h + 2 = \frac{24}{10 - p - q}. \tag{3}$$

Proof of (3). The planes of symmetry of $\{p, q\}$ divide a concentric sphere into congruent spherical triangles each of which is a fundamental region for the group $\mathfrak{G}$ of symmetries of $\{p, q\}$ (**1**, p. 81). The number of triangles is thus g, the order of $\mathfrak{G}$. The vertices of one of these triangles can be labelled P, Q, R so that the corresponding angles are π/p, π/q, $\pi/2$. There are $g/2p$ images of P under $\mathfrak{G}$, since the subgroup leaving P fixed has order $2p$. At each of these points there are $p(p - 1)/2$ intersections of pairs of circles of symmetry. Counting intersections at the images of Q and R in a similar fashion, one gets for the total number of intersections of pairs of circles of symmetry the number

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$g(p+q-1)/4$. However, the number of such circles is $3h/2$ (**1**, p. 68), and every two intersect in two points. Hence

$$\frac{g(p+q-1)}{4}=\frac{3h}{2}\left(\frac{3h}{2}-1\right). \tag{4}$$

Dividing (4) by the relation $g = h(h+2)$ of Coxeter (**1**, p. 91), and solving for h, one obtains (3).

References

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