MENGER AND CONSONANT SETS IN THE SACKS MODEL

VALENTIN HABERL, PIOTR SZEWCZAK, AND LYUBOMYR ZDOMSKYY

Abstract. Using iterated Sacks forcing and topological games, we prove that the existence of a totally imperfect Menger set in the Cantor cube with cardinality continuum is independent from ZFC. We also analyze the structure of Hurewicz and consonant subsets of the Cantor cube in the Sacks model.

§1. Introduction. By *space* we mean an infinite topological Tychonoff space. A space X is *Menger* if for any sequence $\mathcal{U}_0, \mathcal{U}_1, \dots$ of open covers of X, there are finite families $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \dots$ such that the family $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X. Every σ-compact space is Menger and every Menger space is Lindelöf. The Menger conjecture asserts that every subset of the real line with the Menger property is σ-compact. By a result of Fremlin and Miller [11, Theorem 4], this conjecture is false in ZFC. This opened a wide stream of investigations in the realm of special subsets of the real line. The Menger property is a subject of research in the combinatorial covering properties theory but also appears in other branches of mathematics as local properties of function spaces [14], forcing theory [8, 9] or additive Ramsey theory in algebra [23, 26].

The Menger property is closely related to infinite combinatorics. Let $a, b \in \omega^\omega$. We write $a \leq^* b$ if the set $\{ n : a(n) > b(n) \}$ is finite. In such a case we say that the function a is dominated by the function b. A set $D \subseteq \omega^\omega$ is *dominating* if any function in ω^ω is dominated by some function from D. Let $\mathfrak{d}$ be the minimal cardinality of a dominating subset of ω^ω. The Menger property can be characterized in terms of continuous images, as follows: a set $X \subseteq 2^\omega$ is Menger if and only if no continuous image of X into ω^ω is dominating. This characterization was proved by Hurewicz [12, Section 5] and then much later but independently by Recław [18, Proposition 3], so we call it the *Hurewicz–Recław characterization of the Menger property*. It follows that any subset of 2^ω with cardinality smaller than $\mathfrak{d}$ is Menger and there is a non-Menger set of cardinality $\mathfrak{d}$.

The above mentioned result of Fremlin and Miller is dichotomic, i.e., it splits ZFC into two cases using undecidable statements. Bartoszyński and Tsaban provided in [1] a uniform ZFC counterexample to the Menger conjecture. By *set* with a given topological property we mean a space homeomorphic with a subspace of 2^ω. A set is *totally imperfect* if it does not contain a homeomorphic copy of 2^ω.

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THEOREM 1.1 (Bartoszyński, Tsaban [1, Theorem 16]). *There is a totally imperfect Menger set of cardinality* $\mathfrak{d}$.

In the first part of the paper we consider the following problem.

PROBLEM 1.2. *Is there a totally imperfect Menger set of cardinality* $\mathfrak{c}$?

By Theorem 1.1, it suffices to consider the case $\mathfrak{d} < \mathfrak{c}$. In Section 2, we introduce a game characterization of the Menger property, which is one of the main tools needed in Section 3. We show that adding iteratively ω_2 Sacks reals to a ground model satisfying CH, we get a model where $\mathfrak{d} < \mathfrak{c}$ and the answer to Problem 1.2 is negative. In Section 7 we also prove that $\mathfrak{d} < \mathfrak{c}$ is consistent with the existence of a totally imperfect Menger set of cardinality $\mathfrak{c}$.

In the second part of the paper we analyze the structure of *Hurewicz* and *consonant* subsets of 2^ω. Recall that a space X is *Hurewicz* if for any sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X, there are finite families $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots$ such that the family $\{\bigcup \mathcal{F}_n : n \in \omega\}$ is a γ-*cover* of X, i.e., the sets $\{ n : x \in \bigcup \mathcal{F}_n \}$ are cofinite for all $x \in X$. Obviously, every σ-compact space is Hurewicz and every Hurewicz space is Menger. Similarly to the Menger property, the Hurewicz property can be characterized in terms of continuous images, as follows. A set $A \subseteq \omega^\omega$ is *bounded* if there is a function $b \in \omega^\omega$ such that $a \leq^* b$ for all $a \in A$. A set $X \subseteq 2^\omega$ is Hurewicz if and only if every continuous image of X into ω^ω is bounded. This characterization was proved independently by Hurewicz [12, Section 5] and Recław [18, Proposition 1], so we again call it the *Hurewicz–Recław characterization of the Hurewicz property*.

Consonant spaces were introduced by Dolecki, Greco, and Lechicki [10] and characterized by Jordan [13, Corollary 11] in the following way. Let $X \subseteq 2^\omega$. A cover of X is a k-*cover* if any compact subset of X is contained in some set from the cover. A game $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ played on X is a game for two players, ALICE and BOB. For a natural number n, in round n: ALICE picks an open k-cover $\mathcal{U}_n$ of X and BOB picks a set $U_n \in \mathcal{U}_n$. BOB wins the game if the family $\{ U_n : n \in \omega \}$ is a cover of X, and ALICE wins otherwise. A set $Y \subseteq 2^\omega$ is consonant if and only if ALICE has no winning strategy in the game $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ played on $2^\omega \setminus Y$. We treat here this characterization as a definition of consonant sets.

Consonant spaces have close connections to combinatorial covering properties. Let $Y \subseteq 2^\omega$. It follows from the game characterization of the Menger property given below that if the set Y is consonant, then the set $2^\omega \setminus Y$ is Menger. A space X is *Rothberger* if for any sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X, there are sets $U_0 \in \mathcal{U}_0, U_1 \in \mathcal{U}_1, \ldots$ such that the family $\{ U_n : n \in \omega \}$ is a cover of X. Using a game characterization of the Rothberger property given by Pawlikowski [16], if the set $2^\omega \setminus Y$ is totally imperfect, then Y is consonant if and only if the set $2^\omega \setminus Y$ is Rothberger. Indeed, if $2^\omega \setminus Y$ is Rothberger, then Alice does not even have a winning strategy in the game $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ on $2^\omega \setminus Y$, hence Y is consonant. Assuming now that Y is consonant and $2^\omega \setminus Y$ is totally imperfect, we shall show that $2^\omega \setminus Y$ is Rothberger. Let $\mathcal{U}_{n,m}$ be an open cover of $2^\omega \setminus Y$ for all $\langle n, m \rangle \in \omega \times \omega$. Set $\mathcal{W}_n = \{\bigcup_{m \in \omega} U_{n,m} : U_{n,m} \in \mathcal{U}_{n,m}$ for all $m \in \omega\}$. Since each compact subset of $2^\omega \setminus Y$ is countable, each $\mathcal{W}_n$ is a k-cover of $2^\omega \setminus Y$. Since Alice has no winning strategy in the game $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ on $2^\omega \setminus Y$, we conclude that $2^\omega \setminus Y$ is $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$,

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and hence there are $W_n \in \mathcal{W}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} W_n = 2^\omega \setminus Y$. For each n, m let $U_{n,m} \in \mathcal{U}_{n,m}$ be such that $W_n = \bigcup_{m\in\omega} U_{n,m}$. Then $2^\omega \setminus Y = \bigcup_{n,m\in\omega} U_{n,m}$, which shows that $2^\omega \setminus Y$ is Rothberger.[1]

In Section 5 we introduce a topological game which allows us to analyze the structure of Hurewicz and consonant spaces in 2^ω. We show that in the Sacks model, mentioned above, each consonant (Hurewicz) set $X \subseteq 2^\omega$ and its complement $2^\omega \setminus X$ are unions of ω_1 compact sets. This approach allows us also to capture a result of Miller [15, Section 5] that in this model, every *perfectly meager* subset of 2^ω, i.e., a set which is meager in any perfect subset of 2^ω, has size at most ω_1.

The main tools used in our investigations are game characterizations of the considered properties. Let X be a space. The *Menger game* played on X is a game for two players, Alice and Bob. For a natural number n, in round n: Alice picks an open cover $\mathcal{U}_n$ of X and Bob picks a finite family $\mathcal{F}_n \subseteq \mathcal{U}_n$. Bob wins the game if the family $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X, and Alice wins otherwise. For more details about this game, we refer to the works of Scheepers [21, Theorem 13] or Tsaban and the second named author [24]. Similarly to the Menger property, the Hurewicz property can also be characterized using a topological game. The *Hurewicz game* played on X is a game for two players, Alice and Bob. For a natural number n, in round n: Alice picks an open cover $\mathcal{U}_n$ of X and Bob picks a finite family $\mathcal{F}_n \subseteq \mathcal{U}_n$. Bob wins the game if the family $\{ \bigcup \mathcal{F}_n : n \in \omega \}$ is a γ-cover of X, and Alice wins otherwise.

Remark 1.3. In the definition of the Menger and Hurewicz games we could assume, in addition, that none of the $\mathcal{U}_n$'s contains a finite subcover, and get an equivalent definition. Indeed, if X is compact, then Alice has no legal moves, and we standardly adopt the convention that the player having no moves loses. Thus, Bob has a "trivial" winning strategy. On the other hand, if X is not compact, then it has a cover $\mathcal{U}_*$ without finite subcovers, and there is no loss of generality in assuming that Alice always plays refinements of $\mathcal{U}_*$. Similar comments apply to the definitions of the Menger and Hurewicz properties.

Theorem 1.4 (Hurewicz). *A set $X \subseteq 2^\omega$ is Menger if and only if* Alice *has no winning strategy in the Menger game played on X.*

Theorem 1.5 (Scheepers [21, Theorem 27]). *A set $X \subseteq 2^\omega$ is Hurewicz if and only if* Alice *has no winning strategy in the Hurewicz game played on X.*

§2. Menger game and perfect sets. Now we shall address some specific instances of perfect spaces and families of their clopen subsets. Suppose that $\langle F_n : n \in \omega \rangle$ is a non-decreasing sequence of finite subsets of some well-ordered set $\langle S, < \rangle$ such that $S = \bigcup_{n\in\omega} F_n$, and $\langle k_n : n \in \omega \rangle$ is a strictly increasing sequence of natural numbers. For each $n \in \omega$ let $\Sigma_n \subseteq (2^{k_n})^{F_n}$. Fix natural numbers n, m with $n < m$. For $\nu \in \Sigma_n$ and $\sigma \in \Sigma_m$, we write $\nu \prec \sigma$ if ν is *extended* by σ, i.e., $\sigma(\beta) \restriction k_n = \nu(\beta)$ for all $\beta \in F_n$. Let $C \subseteq \Sigma_n$. A map $e\colon C \to \Sigma_m$ such that $\nu \prec e(\nu)$ for all $\nu \in C$, is *coherent*, if for any $\nu, \nu' \in C$, letting $\beta \in F_n$ be the minimal element of F_n with $\nu(\beta) \neq \nu'(\beta)$, we have $e(\nu) \restriction (F_m \cap \beta) = e(\nu') \restriction (F_m \cap \beta)$. In what follows we shall assume that

[1] We have learned this argument from Paul Gartside.

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(e_f) For every $C \subset \Sigma_n$, coherent $e_0 : C \to \Sigma_m$, and $v \in \Sigma_n \setminus C$, there exist two different coherent maps $e, e' \colon (C \cup \{v\}) \to \Sigma_m$ such that $e \restriction C = e' \restriction C = e_0$.

Note that (e_f) applied to $C = \emptyset$ yields that for every $v \in \Sigma_n$ there exists at least two $\sigma \in \Sigma_m$ such that $v \prec \sigma$. Moreover, using (e_f) iteratively for all natural numbers n, m with $n < m$, set $C \subseteq \Sigma_n$, elements $v \in C$ and $\sigma \in \Sigma_m$ with $v \prec \sigma$, there is a coherent map $e \colon C \to \Sigma_m$ with $e(v) = \sigma$.

The objects defined above give rise to the perfect subset $K \subseteq (2^\omega)^S$ consisting of those x such that for every n there exists $v \in \Sigma_n$ such that $x \in [v]$, where

$$[v] := \{\, x \in (2^\omega)^S : x(\beta) \restriction k_n = v(\beta) \text{ for all } \beta \in F_n \,\}.$$

For every $\beta \leq S$, let $\mathrm{pr}_\beta \colon K \to (2^\omega)^\beta$ be the projection map. For a set $C \subseteq \Sigma_n$, a map $E \colon C \to K$ is a *coherent selection* if for every $v \in C$ we have $E(v) \in [v]$ and for every $v, v' \in C$ and $\beta \in F_n$, which is the minimal element in F_n with $v(\beta) \neq v'(\beta)$, we have $\mathrm{pr}_\beta(E(v)) = \mathrm{pr}_\beta(E(v'))$.

LEMMA 2.1. *In the notation above, if a set $X \subseteq K$ is totally imperfect, then for every n there exists a coherent selection $E \colon \Sigma_n \to K$ such that $E[\Sigma_n] \subseteq K \setminus X$.*

PROOF. Fix n and enumerate Σ_n injectively as $\{v_0, \dots, v_N\}$. Since $[v_0] \cap K$ is perfect by (e_f), we can pick $E_0(v_0) \in ([v_0] \cap K) \setminus X$. Fix a number $k < N$ and put $C := \{v_i : i \leq k\}$. Assume that we have already defined a coherent map $E_k : C \to K \setminus X$. Then for every $m > n$ and $v_i \in C$, there is $e_0^m(v_i) \in \Sigma_m$ such that $E_k(v_i) \in [e_0^m(v_i)]$. In that way we define a map $e_0^m \colon C \to \Sigma_m$. It is clear that this map is coherent. By (e_f) there are $\mu_{\langle 0 \rangle} \neq \mu_{\langle 1 \rangle} \in \Sigma_{n+1}$ such that both $e_{\langle 0 \rangle}^{n+1} = e_0^{n+1} \cup \{\langle v_{k+1}, \mu_{\langle 0 \rangle} \rangle\}$ and $e_{\langle 1 \rangle}^{n+1} = e_0^{n+1} \cup \{\langle v_{k+1}, \mu_{\langle 1 \rangle} \rangle\}$ are coherent as maps from $C \cup \{v_{k+1}\}$ to Σ_{n+1}.

Suppose that for some $m > n$ we have defined mutually different $\{\mu_s : s \in 2^{m-n}\} \subseteq \Sigma_m \setminus e_0^m[C]$ such that $e_s^m = e_0^m \cup \{\langle v_{k+1}, \mu_s \rangle\}$ is coherent as a map from $C \cup \{v_{k+1}\}$ to Σ_m for all $s \in 2^{m-n}$. By (e_f) for every $s \in 2^{m-n}$ there are $\mu_{s^\frown 0} \neq \mu_{s^\frown 1} \in \Sigma_{m+1}$ such that

$$\{\langle e_0^m(v_i), e_0^{m+1}(v_i) \rangle : i \leq k\} \cup \{\langle \mu_s, \mu_{s^\frown j} \rangle\}$$

is coherent as a map from Σ_m to Σ_{m+1} for all $j \in 2$. It follows that

$$e_{s^\frown j}^{m+1} := e_0^{m+1} \cup \{\langle v_{k+1}, \mu_{s^\frown j} \rangle\}$$

is coherent as a map from $C \cup \{v_{k+1}\} \to \Sigma_{m+1}$ for all $s \in 2^{m-n}$ and $j \in s$, which completes our construction of the maps e_s^m and elements μ_s as above for *all* $m > n$ and $s \in 2^{m-n}$.

For every $t \in 2^\omega$ let $z_t \in K$ be the unique element in $\bigcap_{m>n} [\mu_{t \restriction (m-n)}]$ and note that $z_t \neq z_{t'}$ for any $t \neq t'$ in 2^ω. Since the set $\{\, z_t : t \in 2^\omega \,\}$ is perfect, there exists t with $z_t \notin X$. It suffices to observe that $E_{k+1} := E_k \cup \{\langle v_{k+1}, z_t \rangle\}$ is a coherent map whose range is disjoint from X, which allows us to complete our proof by induction on $k < N$. ⊣

LEMMA 2.2. *In the notation used above, let $X \subseteq K$ be a totally imperfect Menger set. Then there exists a sequence $\langle\, \langle i_n, j_n, C_n \rangle : n \in \omega \,\rangle$ such that*

(1) *$\langle\, i_n : n \in \omega \,\rangle$ is a strictly increasing sequence of natural numbers;*
(2) *$C_n \subseteq \Sigma_{i_n}$;*

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(3) *for every* $n \in \omega$, *we have* $j_n \in [i_n, i_{n+1})$, *and for every* $v \in C_n$ *there exists* $e_n(v) \in \Sigma_{j_n}$ *extending* v, *such that*

$$C_{n+1} = \bigcup_{v \in C_n} \{\sigma \in \Sigma_{i_{n+1}} \ : \ \sigma \succ e_n(v)\};$$

(4) *the maps* $e_n \colon C_n \to \Sigma_{j_n}$ *are coherent*; *and*
(5) $\bigcap_{n\in\omega} \bigcup_{v \in C_n} [v] \cap X = \emptyset$.

PROOF. We shall describe a strategy § for ALICE in the Menger game played on X such that each play lost by ALICE gives rise to the objects whose existence we need to establish. For every $n \in \omega$, let $E_n \colon \Sigma_n \to K$ be a coherent selection from Lemma 2.1. Put

$$i_0 := 0, \quad C_0 := \Sigma_{i_0}, \quad Z_0 := E_{i_0}[\Sigma_{i_0}] \subseteq K \setminus X.$$

For every $j \geq i_0$ and $v \in \Sigma_{i_0}$, let $\sigma_{0,j}(v)$ be the unique element of Σ_j such that $E_{i_0}(v) \in [\sigma_{0,j}(v)]$. Then,

$$\Big\{ \bigcup_{v \in \Sigma_{i_0}} [\sigma_{0,j}(v)] : j \geq i_0 \Big\}$$

is a decreasing family of clopen sets in $(2^\omega)^S$, whose intersection is equal to the set Z_0. Since $Z_0 \subseteq K \setminus X$, the family $\mathcal{U}_0$ of all sets

$$U^0_j := K \setminus \bigcup_{v \in \Sigma_{i_0}} [\sigma_{0,j}(v)],$$

where $j \geq i_0$, is an increasing open cover of X. Now, § instructs ALICE to start the play with $\mathcal{U}_0$.

Suppose that BOB chooses $U^0_{j_0}$ for some $j_0 \geq i_0$. For each $v \in \Sigma_{i_0}$, let $e_0(v) := \sigma_{0,j_0}(v)$, an element of Σ_{j_0}. Since E_{i_0} is a coherent selection, the map $e_0 \colon C_0 \to \Sigma_{j_0}$ is coherent.

Then we put

$$i_1 := j_0 + 1, \quad C_1 := \bigcup_{v \in C_0} \{\, \sigma \in \Sigma_{i_1} : \sigma \succ e_0(v) \,\}.$$

Suppose that a natural number i_n and a set $C_n \subseteq \Sigma_{i_n}$ have already been defined for some $n > 0$. Let $Z_n := E_{i_n}[C_n] \subseteq K \setminus X$. For every $j \geq i_n$ and $v \in C_n$, let $\sigma_{n,j}(v)$ be the unique element of Σ_j such that $E_{i_n}(v) \in [\sigma_{n,j}(v)]$. Then,

$$\Big\{ \bigcup_{v \in C_n} [\sigma_{n,j}(v)] : j \geq i_n \Big\}$$

is a decreasing family of clopen sets in $(2^\omega)^S$, whose intersection is equal to the set Z_n. Since $Z_n \subseteq K \setminus X$, the family $\mathcal{U}_0$ of all sets

$$U^n_j := K \setminus \bigcup_{v \in C_n} [\sigma_{n,j}(v)],$$

where $j \geq i_n$, is an increasing open cover of X. Now, § instructs ALICE to play the family $\mathcal{U}_n$.

Suppose that Bob chooses $U^n_{j_n}$ for some $j_n \geq i_n$. Since E_{i_n} is a coherent selection, the map $e_n\colon C_n \to \Sigma_{j_n}$ is coherent. Then we put

$$i_{n+1} := j_n + 1, \quad C_{n+1} := \bigcup_{v \in C_n} \{\, \sigma \in \Sigma_{i_{n+1}} : \sigma \succ e_n(v) \,\}.$$

This completes our definition of the strategy § for Alice such that each infinite play in the Menger game on X in which Alice uses § gives rise to a sequence

$$\langle\, \langle i_n, j_n, C_n, e_n, \mathcal{U}_n \rangle : n \in \omega \,\rangle$$

as described above. In particular, conditions (1)–(4) are satisfied by the construction. By Theorem 1.4, there is a play, where Alice uses the strategy § and the play is won by Bob. Then $X \subseteq \bigcup_{n\in\omega} U^n_{j_n}$, i.e.,

$$\emptyset = X \cap \bigcap_{n\in\omega} \bigcup_{v \in C_n} [\sigma_{n,j_n}(v)].$$

For each n, we have

$$\bigcup_{v\in C_n} [\sigma_{n,j_n}(v)] = \bigcup_{v\in C_n} [e_n(v)] = \bigcup_{v\in C_n} \bigcup \{\, [\sigma] : \sigma \in \Sigma_{i_{n+1}}, \sigma \succ e_n(v) \,\} = \bigcup_{\sigma \in C_{n+1}} [\sigma],$$

and thus

$$\emptyset = X \cap \bigcap_{n\in\omega} \bigcup_{v \in C_n} [\sigma_{n,j_n}(v)] = X \cap \bigcap_{n\in\omega} \bigcup_{v \in C_{n+1}} [v].$$

It follows that condition (5) is also satisfied. ⊣

§3. Combinatorics of conditions in the iterated Sacks forcing. Here we deal with countable support iterations of the forcing notion introduced by Sacks [19]. We do not prove any essentially new results about these iterations in this section, but rather "tailor" several results established in [3, 15] and perhaps somewhere else for the purposes we have in Sections 4 and 6. We try to follow notations used in [3].

Let $2^{<\omega} := \bigcup_{n\in\omega} 2^n$. For elements $s, t \in 2^{<\omega}$, we write $s \subseteq t$ if the sequence s is an initial segment of the sequence t, i.e., $s(i) = t(i)$ for all $i \in \mathrm{dom}(s)$. A *Sacks tree* is a set $p \subseteq 2^{<\omega}$ such that for every $s \in p$ and a natural number n, we have $s \restriction n \in p$ and there are elements $t, u \in p$ with $s \subseteq t$, $s \subseteq u$, $t \not\subseteq u$ and $t \not\subseteq u$. For Sacks trees p and q, a condition q is stronger than p which we write $q \geq p$ if $q \subseteq p$. The *Sacks poset* $\mathbb{S}$ is the set of all Sacks trees ordered by $\geq$. For $p, q \in \mathbb{S}$ and natural numbers $m > n$, we write $(q, m) \geq (p, n)$ if $q \subseteq p$ and for every $s \in p \cap 2^n$, there are different elements $t, u \in q \cap 2^m$ such that $s \subseteq t$ and $s \subseteq u$. For $p \in \mathbb{S}$ and $s \in p$, let $p_s := \{\, t \in p : t \subseteq s \text{ or } s \subseteq t \,\}$.

Let α be an ordinal number and $\mathbb{S}_\alpha$ be an iterated forcing of length α with countable support, where each iterand is a Sacks poset. For $p, q \in \mathbb{S}_\alpha$, let $q \geq p$ if $\mathrm{supp}(q) \supseteq \mathrm{supp}(p)$ and for every $\beta \in \mathrm{supp}(p)$, we have $q \restriction \beta \Vdash_\beta q(\beta) \geq p(\beta)$.

Let $p \in \mathbb{S}_\alpha$, $F \subseteq \alpha$ be a finite set, n a natural number and $\sigma\colon F \to 2^n$ a map. If $F = \emptyset$, then the map σ is *consistent* with p and $p|\sigma := p$. Assume that β is the greatest element in F, the map $\sigma \restriction \beta$ is consistent with p and the condition $p|(\sigma \restriction \beta)$

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has already been defined. If

$$\big(p|(\sigma \restriction \beta)\big) \restriction \beta \Vdash_\beta \sigma(\beta) \in p(\beta),$$

then the map σ is *consistent* with p and we define $p|\sigma(\gamma)$ to be the following $\mathbb{S}_\gamma$-name τ:

- $p|(\sigma \restriction \beta)(\gamma)$, if $\gamma < \beta$;
- If $\gamma = \beta$, then $\big(p|(\sigma \restriction \beta)\big) \restriction \beta \Vdash_\beta \tau = p(\beta)_{\sigma(\beta)}$ and $r \Vdash_\beta \tau = p(\beta)$ for $r \in \mathbb{S}_\beta$ incompatible with $\big(p|(\sigma \restriction \beta)\big) \restriction \beta$; and
- $p(\gamma)$, otherwise.

The following fact can be established by induction on $|F|$ in a rather straightforward way.

OBSERVATION 3.1. *In the notation above, if σ is consistent with p, then $\sigma \restriction \beta$ is consistent with both $p \restriction \beta$ and p, and $\big(p|(\sigma \restriction \beta)\big) \restriction \beta = (p \restriction \beta)|(\sigma \restriction \beta)$.*

A condition $p \in \mathbb{S}_\alpha$ is (F, n)-*determined*, where $F \subseteq \alpha$ and $n \in \omega$, if every map $\sigma \colon F \to 2^n$ is either consistent with p, or there is $\beta \in F$ such that $\sigma \restriction \beta$ is consistent with p and $(p \restriction \beta)|(\sigma \restriction \beta) \Vdash_\beta \sigma(\beta) \notin p(\beta)$. For $q \in \mathbb{S}_\alpha$ and a natural number $m > n$, we write $(q, m) \geq_F (p, n)$ if $q \geq p$ and for every $\beta \in F$, we have $q \restriction \beta \Vdash_\beta (q(\beta), m) \geq (p(\beta), n)$.

Let p be an (F, n)-determined condition. We write $(q, n) \geq_F (p, n)$ if $q \geq p$ and every map $\sigma \colon F \to 2^n$ consistent with p, is also consistent with q. Next, we collect rather straightforward facts about the notions introduced above, these are used in nearly all works investigating iterations of the Sacks forcing or similar posets consisting of trees.

OBSERVATION 3.2. *Let $p \in \mathbb{S}_\alpha$ be an (F, n)-determined condition, Σ the set of all maps $v \colon F \to 2^n$ consistent with p and $\beta < \alpha$. Then the following assertions hold.*

1. *If $(q, n) \geq_F (p, n)$, then q is also (F, n)-determined;*
2. *If $\sigma \in \Sigma$, then $p|(\sigma \restriction \beta)$ is $(F \setminus \beta, n)$-determined, and $v \in (2^n)^{F \setminus \beta}$ is consistent with $p|(\sigma \restriction \beta)$ iff $(\sigma \restriction \beta) \cup v \in \Sigma$;*
3. *The set $\{\, p|\sigma : \sigma \in \Sigma \,\}$ is a maximal antichain above p;*
4. *p is $(F \cap \beta, n)$-determined and $\{\sigma \restriction (F \cap \beta) : \sigma \in \Sigma\}$ is the family of all functions from $F \cap \beta$ to 2^n consistent with p;*
5. *If $\sigma \in \Sigma$ and $r \geq p|\sigma$, then there exists $q \in \mathbb{S}_\alpha$ with $(q, n) \geq_F (p, n)$ and $q|\sigma = r$;*
6. *If $D \subseteq \mathbb{S}_\alpha$ is open and dense, then there exists $q \in \mathbb{S}_\alpha$ with $(q, n) \geq_F (p, n)$ and $q|\sigma \in D$ for all $\sigma \in \Sigma$; and*
7. *If τ is an $\mathbb{S}_\alpha$-name for a real, then for each natural number l there is a condition $q \in \mathbb{S}_\alpha$ and a family $\{y_\sigma : \sigma \in \Sigma\} \subseteq 2^l$ such that $(q, n) \geq_F (p, n)$ and $q|\sigma \Vdash \tau \restriction l = y_\sigma$ for all $\sigma \in \Sigma$.*

The last three items of Observation 3.2 imply the following easy fact.

LEMMA 3.3 (Miller [15, Lemma 2]). *Let $p \in \mathbb{S}_\alpha$ be an (F, n)-determined condition and τ an $\mathbb{S}_\alpha$-name for a real such that $p \Vdash \tau \in 2^\omega \setminus V$. Then for each finite set*

$Y \subseteq 2^\omega \cap V$, there is a finite set $X \subseteq 2^\omega \cap V$ disjoint from Y such that $|X| \le 2^{n\cdot|F|}$, and for each natural number l, there is a condition $q \in \mathbb{S}_\alpha$ such that $(q,n) \ge_F (p,n)$ and

$$q \Vdash \big(\exists\, x \in X\big)\big(\tau \restriction l = x \restriction l\big).$$

Let $\dot{G}_\alpha$ be an $\mathbb{S}_\alpha$-name for a $\mathbb{S}_\alpha$-generic filter, and for $\beta < \alpha$ let $\mathbb{S}_{\beta,\alpha}$ be an $\mathbb{S}_\beta$-name for the iteration from (including) β to α, so that $\mathbb{S}_\alpha$ is forcing equivalent to $\mathbb{S}_\beta * \mathbb{S}_{\beta,\alpha}$. Whenever we work in the forcing extension $V[G_\beta]$ for some $\mathbb{S}_\beta$-generic filter G_β, we denote by $\dot{G}_{\beta,\alpha}$ a $\mathbb{S}_{\beta,\alpha}^{G_\beta}$-name for a $\mathbb{S}_{\beta,\alpha}^{G_\beta}$-generic filter over $V[G_\beta]$. We shall need the following easy observation, we use the notation from the above.

Observation 3.4. *Suppose that p is (F,n)-determined, $\beta \le \alpha$, and $p \restriction \beta \in G_\beta$. Then in $V[G_\beta]$, $p \restriction [\beta,\alpha)^{G_\beta} \in \mathbb{S}_{\beta,\alpha}^{G_\beta}$ is $(F \setminus \beta, n)$-determined.*

Moreover, if $\sigma \in (2^n)^F$ is consistent with p and $(p|(\sigma \restriction \beta)) \restriction \beta \in G_\beta$, then $\sigma \restriction (F \setminus \beta)$ is consistent with $p \restriction [\beta,\alpha)^{G_\beta}$ in $V[G_\beta]$; and if $(p|(\sigma \restriction \beta)) \restriction \beta \in G_\beta$ and $v \in (2^n)^{F\setminus\beta}$ is consistent with $p \restriction [\beta,\alpha)^{G_\beta}$ in $V[G_\beta]$, then $(\sigma \restriction \beta) \cup v \in (2^n)^F$ is consistent with p.

Lemma 3.5 (Miller [15, Lemma 5]). *Let $p \in \mathbb{S}_\alpha$ be an (F,n)-determined condition and τ be an $\mathbb{S}_\alpha$-name for a real such that*

$$p \Vdash \tau \in \big(2^\omega \cap V[\dot{G}_\alpha]\big) \setminus \bigcup_{\beta<\alpha} \big(2^\omega \cap V[\dot{G}_\beta]\big).$$

Then for any $k \in \omega$ there exist a condition $q \in \mathbb{S}_\alpha$, a natural number $l > k$, and elements $y_\sigma \in 2^l$, for all maps $\sigma\colon F \to 2^n$ consistent with p, with the following properties:

(1) $(q,n) \ge_F (p,n)$,
(2) $q|\sigma \Vdash \tau \restriction l = y_\sigma$,
(3) *the maps y_σ are pairwise different.*

Proof. Let $\xi = \min(F)$ and note that the fact that p is (F,n)-determined yields $N \le 2^n$ and $\{s_i : i < N\} \subseteq 2^n$ such that $p \restriction \xi$ forces $p(\xi) \cap 2^n = \{s_i : i < N\}$. For every $i < N$ let μ_i be the map $\{\langle \xi, s_i\rangle\}$ and note that μ_i is consistent with p.

By induction on $i < N$, using Lemma 3.3 and Observation 3.2(2), we can find mutually disjoint finite sets $X_i \subseteq 2^\omega \cap V$, $i < N$, such that $|X_i| \le 2^{n\cdot(|F|-1)}$, and for each natural number l there is a condition $u_i^l \in \mathbb{S}_\alpha$ with $u_j^l \restriction \xi \ge u_i^l \restriction \xi$ for all $i < j \le N$, $u_0^l \restriction \xi \ge p \restriction \xi$, $(u_i^l, n) \ge_{F\setminus\{\xi\}} (p|\mu_i, n)$ and

$$u_i^l \Vdash \big(\exists\, x \in X_i\big)\big(\tau \restriction l = x \restriction l\big).$$

Pick a natural number $l_* > k$ such that $x \restriction l_* \ne x' \restriction l_*$ for any distinct x, x' in $\bigcup_{i<N} X_i$. As a result, the elements of the family $\{\, X_i \restriction l_* : i < N \,\}$ are mutually disjoint.

Now we proceed by induction on the cardinality of F. If $F = \{\xi\}$, then $|X_i| = 1$ for all $i < N$ (because $2^{n\cdot(|F|-1)} = 1$), i.e., $X_i = \{x_i\}$ for some $x_i \in 2^\omega$. Put $y_{\mu_i} := x_i \restriction l_*$ for all $i < N$ and let $r \in \mathbb{S}_\alpha$ be a condition such that $r \restriction \xi = u_{N-1}^{l_*} \restriction \xi$, $r \restriction \xi$ forces $r(\xi)$ to be $\bigcup\{\, u_i^{l_*}(\xi) : i < N \,\}$, and $(r|\mu_i) \restriction \beta$ forces $r(\beta) = u_i^{l_*}(\beta)$ for all $\beta > \xi$.

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It follows from the above that $r|\mu_i \geq u_i^{l_*}$ and hence $r|\mu_i$ forces $\tau \upharpoonright l_* = x_i \upharpoonright l_* = y_{\mu_i}$, hence $q := r$ and $l := l_*$ are as required.

Now assume that $|F| > 1$ and the statement holds for each set of cardinality smaller than $|F|$. Let ξ, l_*, r be as above and note that by the construction we have

$$(r|\mu_i, n) \geq_{F\setminus\{\xi\}} (u_i^{l_*}, n) \geq_{F\setminus\{\xi\}} (p|\mu_i, n)$$

for all $i < N$, and hence $(r, n) \geq_F (p, n)$. Fix $i < N$ and let G be an $\mathbb{S}_{\xi+1}$-generic filter containing $(r|\mu_i) \upharpoonright (\xi + 1)$. Work in $V[G]$. Then

$$r_i' := \big((r|\mu_i) \upharpoonright [\xi + 1, \alpha)\big)^G \in \mathbb{S}^G_{\xi+1,\alpha}$$

is $(F \setminus \{\xi\}, n)$-determined by Observation 3.2(1) because

$$(r_i', n) \geq_{F\setminus\{\xi\}} \Big(\big(u_i^{l_*} \upharpoonright [\xi + 1, \alpha)\big)^G, n\Big)$$

by the construction, and $\big(u_i^{l_*} \upharpoonright [\xi + 1, \alpha)\big)^G$ is $(F \setminus \{\xi\}, n)$-determined by Observation 3.4. Note that

$$r_i' \Vdash \tau \in (2^\omega \cap V[\dot{G}_{\xi+1,\alpha}]) \setminus \bigcup_{\xi<\beta<\alpha} (2^\omega \cap V[\dot{G}_{\xi+1,\beta}]).$$

because $r_i' \geq p \upharpoonright [\xi + 1, \alpha)^G$ and $p \upharpoonright (\xi + 1) \in G$. By the inductive assumption, there exist a condition $r_i'' \in \mathbb{S}^G_{\xi+1,\alpha}$, a natural number $l_i > l_*$ and pairwise different elements $t_{\sigma'} \in 2^{l_i}$ for all maps $\sigma' \colon F \setminus \{\xi\} \to 2^n$ consistent with r_i'', such that $(r_i'', n) \geq_{F\setminus\{\xi\}} (r_i', n)$ and $r_i''|\sigma' \Vdash \tau \upharpoonright l_i = t_{\sigma'}$. Let Σ_i' be the set of all maps $\sigma' \colon F \setminus \{\xi\} \to 2^n$ consistent with r_i''.

Now we work in V. Let Σ be the set of all maps $\sigma \colon F \to 2^n$ consistent with p. Let $\underset{\sim}{r}_i''$, $\underset{\sim}{\Sigma}_i'$, $\underset{\sim}{l}_i$ and $\underset{\sim}{t}_{\sigma'}$ be $\mathbb{S}_{\xi+1}$-names for the condition r_i'', the set Σ_i', natural number l_i and finite sequences $t_{\sigma'}$, respectively. Note that

$$(r|\mu_i) \upharpoonright (\xi + 1) \Vdash \underset{\sim}{\Sigma}_i' = \{\, \sigma \upharpoonright F \setminus \{\xi\} : \sigma \in \Sigma \text{ and } \sigma(\xi) = s_i \,\}$$

by the second part of Observation 3.4. By induction on $i < N$ pick a condition $r_i \in \mathbb{S}_{\xi+1}$ stronger than $(r|\mu_i) \upharpoonright (\xi + 1)$ and such that $r_j \upharpoonright \xi \geq r_i \upharpoonright \xi$ for all $i < j < N$, $r_0 \upharpoonright \xi \geq r \upharpoonright \xi$, and which forces all the above properties, and also decides all the names mentioned in the previous sentences. More precisely, there exist $l_i > l_*$, $t^i_\sigma \in 2^{l_i}$ for all $\sigma \in \Sigma$ with $\sigma(\xi) = s_i$, and $r_i'' \in \mathbb{S}_{\xi+1,\alpha}$ such that r_i forces that $\underset{\sim}{r}_i'' = r_i''$, $\underset{\sim}{l}_i = l_i$ and $\underset{\sim}{t}_{\sigma\upharpoonright F\setminus\{\xi\}} = t^i_\sigma$ for all maps $\sigma \in \Sigma$ with $\sigma(\xi) = s_i$. Thus the elements t^i_σ are pairwise different for all maps $\sigma \in \Sigma$ with $\sigma(\xi) = s_i$.

Let $w \in \mathbb{S}_\alpha$ be a condition such that $w \upharpoonright \xi = r_{N-1} \upharpoonright \xi$,

$$w(\xi) = \bigcup\{\, r_i(\xi) : i < N \,\},$$

and for every ordinal number β with $\xi < \beta < \alpha$ and natural number $i < N$, we have

$$(w|\mu_i) \upharpoonright \beta \Vdash w(\beta) = r_i''(\beta).$$

It follows that

$$w|\sigma \Vdash \tau \upharpoonright l_i = t^i_\sigma$$

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for all $\sigma \in \Sigma$ with $\sigma(\xi) = s_i$ because

$$w|\sigma \geq r_i \cup r_i''|(\sigma \upharpoonright (F \setminus \{\xi\})).$$

Let $\sigma, \nu \in \Sigma$ be different maps. Assume that $\sigma(\xi) \neq \nu(\xi)$. Then there are different natural numbers i, j such that $\sigma(\xi) = s_i$ and $\nu(\xi) = s_j$. Since $w|\sigma \geq w|\mu_i \geq u_i^{l_*}$ and $w|\nu \geq w|\mu_j \geq u_j^{l_*}$, we have

$$w|\sigma \Vdash \tau \upharpoonright l_i = t^i_\sigma, \quad \tau \upharpoonright l_* \in \{\, x \upharpoonright l_* : x \in X_i \,\}$$

and

$$w|\nu \Vdash \tau \upharpoonright l_j = t^j_\nu \quad \tau \upharpoonright l_* \in \{\, x \upharpoonright l_* : x \in X_j \,\}.$$

The sets $\{\, x \upharpoonright l_* : x \in X_i \,\}$ and $\{\, x \upharpoonright l_* : x \in X_j \,\}$ are disjoint, and thus $t^i_\sigma \upharpoonright l_* \neq t^j_\nu \upharpoonright l_*$. Now assume that there is a natural number i such that $\sigma(\xi) = \nu(\xi) = s_i$. Then $\sigma \upharpoonright F \setminus \{\xi\} \neq \nu \upharpoonright F \setminus \{\xi\}$ and the condition $(w|\mu_i) \upharpoonright (\xi + 1)$ forces that $\underset{\sim}{t}_{\sigma \upharpoonright F \setminus \{\xi\}} = t^i_\sigma$ and $\underset{\sim}{t}_{\nu \upharpoonright F \setminus \{0\}} = t^i_\nu$ because $(w|\mu_i) \upharpoonright (\xi + 1) \geq r_i$, while $t^i_\sigma \neq t^i_\nu$. Summarizing, if $\nu(\xi) = \sigma(\xi) = s_i$, then $t^i_\sigma \neq t^i_\nu$, and if $\nu(\xi) = s_j \neq s_i = \sigma(\xi)$, then $t^i_\sigma \upharpoonright l_* \neq t^j_\nu \upharpoonright l_*$.

Finally, applying Observation 3.2(7) to $l = \max_{i<N} l_i$ and w which is (F, n)-determined (because $(w, n) \geq_F (p, n)$ by the construction), we get a condition q such that $(q, n) \geq_F (w, n)$, and for every $\sigma \in \Sigma$ a sequence $y_\sigma \in 2^l$ such that $q|\sigma \Vdash \tau \upharpoonright l = y_\sigma$. Since $q|\sigma \geq w|\sigma$, we have $y_\sigma \upharpoonright l_i = t^i_\sigma$ for all $\sigma \in \Sigma$ with $\sigma(\xi) = s_i$. It follows from the above that the y_σ's are mutually different: if $\nu(\xi) = \sigma(\xi) = s_i$, then $y_\sigma \upharpoonright l_i = t^i_\sigma \neq t^i_\nu = y_\nu \upharpoonright l_i$; and if $\nu(\xi) = s_j \neq s_i = \sigma(\xi)$, then $y_\sigma \upharpoonright l_* = t^i_\sigma \upharpoonright l_* \neq t^j_\nu \upharpoonright l_* = y_\nu \upharpoonright l_*$. ⊣

The following fact is reminiscent of [3, Lemma 2.3(i)], and we use a rather similar approach to the proof, which we present for the sake of completeness.

LEMMA 3.6. *Let α be an ordinal, $p \in \mathbb{S}_\alpha$, $n \in \omega$ and $F \subseteq \alpha$ a nonempty finite set. Then there are a natural number $k > n$ and an (F, k)-determined condition $q \in \mathbb{S}_\alpha$ such that $(q, k) \geq_F (p, n)$.*

PROOF. We proceed by induction on $|F|$. Suppose that $F = \{\beta\}$ for some $\beta < \alpha$ and pick $r \in \mathbb{S}_\beta$, $r \geq p \upharpoonright \beta$ which decides $k > n$ and $p(\beta) \cap 2^k$ (and hence also decides $p(\beta) \cap 2^n$), so that each element $s \in p(\beta) \cap 2^n$ has at least 2 extensions in $p(\beta) \cap 2^k$. Then $q := r \cup p \upharpoonright [\beta, \alpha)$ is as required.

Now assume that $|F| > 1$ and let $\beta = \max(F)$. By the inductive assumption there exists $r \in \mathbb{S}_\beta$ and $k_0 > n$ such that r is $(F \cap \beta, k_0)$-determined and $(r, k_0) \geq_{F \cap \beta} (p \upharpoonright \beta, n)$. Let Σ be the family of all $\sigma : F \cap \beta \to 2^{k_0}$ consistent with r. Let $N = |\Sigma|$ and write Σ in the form $\{\sigma_i : i < N\}$. By induction on $i < N$ let us construct a sequence

$$(r^0_{N-1}, k_0) \geq_{F \cap \beta} \cdots \geq_{F \cap \beta} (r^0_0, k_0) \geq_{F \cap \beta} (r^0_{-1}, k_0),$$

where $r^0_{-1} = r$ and $r^0_i \in \mathbb{S}_\beta$ for all $i < N$, as follows: Given r^0_{i-1} for some $i < N$, let $\mathbb{S}_\beta \ni u^0_i \geq r^0_{i-1}|\sigma_i$ be a condition such that there exists $T^0_i \subseteq 2^{l_i}$ for some $l_i \geq k_0$ such that u^0_i forces $p(\beta) \cap 2^{l_i} = T^0_i$, and for every $s \in T^0_i \upharpoonright n$ there exists at least two $t \in T^0_i$ extending s. Now, let $r^0_i \in \mathbb{S}_\beta$ be such that $(r^0_i, k_0) \geq_{F \cap \beta} (r^0_{i-1}, k_0)$ and $r^0_i|\sigma_i = u^0_i$, by Observation 3.2(5).

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Let $k = \max_{i<N} l_i$ and set $r^1_{-1} = r^0_{N-1}$. By induction on $i < N$ let us construct a sequence

$$(r^1_{n-1}, k_0) \geq_{F\cap\beta} \cdots \geq_{F\cap\beta} (r^1_0, k_0) \geq_{F\cap\beta} (r^1_{-1}, k_0),$$

where $r^1_i \in \mathbb{S}_\beta$ for all $i < N$, as follows: Given r^1_{i-1} for some $i < N$, let $\mathbb{S}_\beta \ni u^1_i \geq r^1_{i-1}|\sigma_i$ be a condition such that there exists $T^1_i \subseteq 2^k$ for which u^1_i forces $p(\beta) \cap 2^k = T^1_i$. Now, let $r^1_i \in \mathbb{S}_\beta$ be such that $(r^1_i, k_0) \geq_{F\cap\beta} (r^1_{i-1}, k_0)$ and $r^1_i|\sigma_i = u^1_i$. Note that $T^0_i = \{t \restriction l_i : t \in T^1_i\}$.

Set $r^2_{-1} = r^1_{N-1}$. By induction on $i < N$ let us construct a sequence

$$(r^2_{n-1}, k_0) \geq_{F\cap\beta} \cdots \geq_{F\cap\beta} (r^2_0, k_0) \geq_{F\cap\beta} (r^2_{-1}, k_0),$$

where $r^2_i \in \mathbb{S}_\beta$ for all $i < N$, as follows: Given r^2_{i-1} for some $i < N$, let $\mathbb{S}_\beta \ni u^2_i \geq r^2_{i-1}|\sigma_i$ be a condition such that for every $\gamma \in F \cap \beta$ there exists $v_i(\gamma) \in 2^k$ such that $\sigma_i(\gamma) = v_i(\gamma) \restriction k_0$ and $u^2_i \restriction \gamma$ forces that $v_i(\gamma)$ is an initial segment of the stem of $u^2_i(\gamma)$. Such an u^2_i can be constructed recursively over $\gamma \in F \cap \beta$, moving from the bigger to smaller elements. Now, let $r^2_i \in \mathbb{S}_\beta$ be such that $(r^2_i, k_0) \geq_{F\cap\beta} (r^2_{i-1}, k_0)$ and $r^1_i|\sigma_i = u^2_i$.

We claim that $q = r^2_{N-1} \cup p \restriction [\beta, \alpha)$ and k are as required. Indeed, we have that $(q \restriction \beta, k) \geq_{F\cap\beta} (p \restriction \beta, n)$ because $q \restriction \beta = r^2_{N-1}$, $k \geq k_0$, and $(r^2_{N-1}, k_0) \geq_{F\cap\beta} (r, k_0) \geq_{F\cap\beta} (p \restriction \beta, n)$. Moreover, since $r^2_{N-1}|\sigma_i \geq u^0_i$, we have that $r^2_{N-1}|\sigma_i$ decides $p(\beta) \cap 2^k$ as T^1_i, which has the property that any $s \in T^1_i \restriction n$ has at least two extensions in T^1_i (because $T^0_i = T^1_i \restriction l_i$ has this property). It follows that $r^2_{N-1}|\sigma_i$ forces $q(\beta) = p(\beta)$ and $(p(\beta), k) \geq (p(\beta), n)$. Since $\{r^2_{N-1}|\sigma_i : i < N\}$ is dense above r^2_{N-1}, we conclude that $r^2_{N-1} = q \restriction \beta$ forces $(q(\beta), k) \geq (p(\beta), n)$, and therefore $(q, k) \geq (p, n)$.

Finally, by the construction of r^2_{N-1} we have that q is (F, k)-determined, with

$$\{v_i \cup \{\langle \beta, t\rangle\} \;:\; i < N, t \in T^1_i\}$$

being the family of those $v : F \to 2^k$ which are consistent with q. ⊣

LEMMA 3.7 (Miller [15, Lemma 6]). *Let α be an ordinal number, $p_0 \in \mathbb{S}_\alpha$, and τ an $\mathbb{S}_\alpha$-name for a real such that*

$$p_0 \Vdash \tau \in \big(2^\omega \cap V[\dot{G}_\alpha]\big) \setminus \bigcup_{\beta<\alpha} \big(2^\omega \cap V[\dot{G}_\beta]\big).$$

Then there exist a condition $p \geq p_0$, an increasing sequence $\langle F_n : n \in \omega \rangle$ of finite subsets of α, increasing sequences of natural numbers $\langle k_n : n \in \omega \rangle$, $\langle l_n : n \in \omega \rangle$, and elements $y_\sigma \in 2^{l_n}$ for all maps $\sigma \colon F_n \to 2^{k_n}$ consistent with p, with the following properties:

(1) $\bigcup_{n\in\omega} F_n = \mathrm{supp}(p)$,
(2) *p is (F_n, k_n)-determined,*
(3) $(p, k_{n+1}) \geq_{F_n} (p, k_n)$,
(4) *$p|\sigma \Vdash \tau \restriction l_n = y_\sigma$ for all $\sigma \in (2^{k_n})^{F_n}$ consistent with p, and*
(5) *the maps y_σ, where σ is as above, are mutually different.*

PROOF. The choice of the F_n's is standard and thus will not be specified, except that we set $F_0 = \{0\}$. Set also $k_0 = 0$. Trivially, p_0 is (F_0, k_0)-determined since the

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unique map $\{\langle 0,\emptyset\rangle\}$ in $(2^{k_0})^{F_0}$ is consistent with p_0. By Lemma 3.5, there are a condition $q_0 \in \mathbb{S}_\alpha$, a natural number l_0 and pairwise different elements $y_\sigma \in 2^{l_0}$ for all maps[2] $\sigma\colon F_0 \to 2^{k_0}$ consistent with p_0 such that $(q_0,k_0) \geq_{F_0} (p_0,k_0)$ and $q_0|\sigma \Vdash \tau \restriction l_0 = y_\sigma$.

Let $k_1 > k_0$ be a natural number and $p_1 \in \mathbb{S}_\alpha$ a condition from Lemma 3.6, applied to the set F_1 and the condition q_0. Then p_1 is (F_1,k_1)-determined and $(p_1,k_1) \geq_{F_0} (p_0,k_0)$.

Fix a natural number $n \geq 1$ and assume that a set F_n, natural number k_n, and an (F_n,k_n)-determined condition $p_n \in \mathbb{S}_\alpha$ with $(p_n,k_n) \geq_{F_{n-1}} (q_{n-1},k_{n-1})$ have already been defined. By Lemma 3.5, there are a condition $q_n \in \mathbb{S}_\alpha$, a natural number $l_n > l_{n-1}$ and pairwise different elements $y_\sigma \in 2^{l_n}$ for all maps $\sigma\colon F_n \to 2^{k_n}$ consistent with p_n such that $(q_n,k_n) \geq_{F_n} (p_n,k_n)$ and $q_n|\sigma \Vdash \tau \restriction l_n = y_\sigma$.

Let p be the fusion of the sequence $\langle\, (p_n,k_n,F_n) : n \in \omega \,\rangle$ [3, Lemma 1.2] and note that it is as required. $\dashv$

Let F be a subset of H, $n \leq m$ be natural numbers, and $\nu\colon F \to 2^n$, $\sigma\colon H \to 2^m$ be maps. Following our convention at the beginning of Section 2, the map σ is an *extension* of ν (we denote this by $\nu \prec \sigma$) if $\nu(\beta) = \sigma(\beta) \restriction n$ for all $\beta \in F$. The next fact is standard.

Observation 3.8. *In the notation above, if $F, H \subseteq \alpha$, $p \in \mathbb{S}_\alpha$, ν,σ are consistent with p and $\nu \prec \sigma$, then $p|\sigma \geq p|\nu$.*

Lemma 3.9. *Let $p \in \mathbb{S}_\alpha$ be an (F,n)-determined condition and Σ_n be the set of all maps $\nu\colon F \to 2^n$ consistent with p. Then for every $G \subseteq F$ and $k \leq n$, if p is (G,k)-determined and $\mu : G \to 2^k$ is consistent with p, then there exists $\nu \in \Sigma_n$ extending μ.*

Moreover, if $H \subseteq \alpha$ is a finite set with $F \subseteq H$, $\beta \in F$, $m > n$ is a natural number, p is also (H,m)-determined, Σ_m is the set of all maps $\sigma\colon H \to 2^m$ consistent with p, and $(p,m) \geq_F (p,n)$, then the following assertions hold.

1. *For every $\nu \in \Sigma_n$ and $\sigma \in \Sigma_m$ such that $\sigma \restriction (H \cap \beta)$ extends $\nu \restriction (F \cap \beta)$, there are $\sigma_1,\sigma_2 \in \Sigma_m$ extending ν with $\sigma_1 \restriction (H\cap\beta) = \sigma_2 \restriction (H\cap\beta) = \sigma \restriction (H\cap\beta)$, and such that $\sigma_1(\beta),\sigma_2(\beta)$ are distinct extensions of $\nu(\beta)$.*
2. *For every $\nu \in \Sigma_n$ and $\rho : H\cap\beta \to 2^m$ consistent with p, if ρ extends $\nu \restriction (F \cap \beta)$, then there are $\sigma_1,\sigma_2 \in \Sigma_m$ extending ν with $\sigma_1 \restriction (H\cap\beta) = \sigma_2 \restriction (H\cap\beta) = \rho$ and such that $\sigma_1(\beta),\sigma_2(\beta)$ are distinct extensions of $\nu(\beta)$.*
3. *For every $C \subset \Sigma_n$, coherent $e_0 : C \to \Sigma_m$, and $\nu_* \in \Sigma_n \setminus C$, there exist two different coherent maps $e, e'\colon (C \cup \{\nu_*\}) \to \Sigma_m$ such that $e \restriction C = e' \restriction C = e_0$.*

Proof. We start with proving the first part. Proceed by induction on $|G|$. If $G = \emptyset$, then there is nothing to prove. Let $\gamma := \max G$ and assume that the statement holds for $G' := G \setminus \{\gamma\}$. Fix a map $\mu\colon G \to 2^k$ consistent with p and let $\mu' := \mu \restriction G'$. By Observation 3.2(4), p is both $(F\cap\gamma,n)$- and $(G\cap\gamma,k)$-determined, and hence there exists $\nu' : (F\cap\gamma) \to 2^n$ consistent with p such that $\nu' \succ \mu'$. Since $(p\restriction\gamma)|\mu' \Vdash \mu(\gamma) \in p(\gamma)$, the condition $(p\restriction\gamma)|\nu'$ also forces this because $(p\restriction\gamma)|\nu' \geq (p\restriction\gamma)|\mu'$ by Observation 3.8. Strengthening the latter condition to some r if necessary, we may find $t \in 2^n$ extending $\mu(\gamma)$ such that $r \Vdash t \in p(\gamma)$. Since p is (F,n)-determined, we

[2] As we noted above, there is just one map like that.

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conclude that $(p \restriction \gamma)|\nu' \Vdash t \in p(\gamma)$. It follows that $\nu' \cup \{(\gamma, t)\} \in (2^n)^F$ is consistent with p and extends μ.

The second part of the lemma is rather straightforward. Item (2) is an equivalent reformulation of (1), by Observation 3.2(2). We shall present the proof of (3), because we find it the least obvious one.

Proceed by induction on $|F|$. Assume that $F = \{\beta\}$. Let $\nu \in C$ and $\sigma := e_0(\nu)$. Choose some distinct $\mu, \mu' \in \Sigma_m$ such that $\mu, \mu' \succ \nu_*$ and $\mu \restriction (H \cap \beta) = \sigma \restriction (H \cap \beta) = \mu' \restriction (H \cap \beta)$, which is possible by (1). Then the maps $e = e_0 \cup \{\langle \nu_*, \mu\rangle\}$ and $e' = e_0 \cup \{\langle \nu_*, \mu'\rangle\}$ are easily seen to be as required.

Now assume that $|F| > 1$ and (3) holds for any finite subset of the support of p of cardinality smaller than $|F|$. Set $\beta = \max F$, $C^- = \{\nu \restriction (F \cap \beta) : \nu \in C\}$, $e_0^-(\nu \restriction (F \cap \beta)) = e_0(\nu) \restriction (H \cap \beta)$ for all $\nu \in C$, and $\nu_*^- = \nu_* \restriction (F \cap \beta)$. Let also Σ_m^- be the family of all maps $\sigma : H \cap \beta \to 2^m$ consistent with p. By the inductive assumption applied to the objects defined above we can get a coherent $e^- : C^- \cup \{\nu_*^-\} \to \Sigma_m^-$ such that $e^- \restriction C^- = e_0^-$. (Actually, we could get even two different such e^- if $\nu_*^- \notin C^-$, but thus irrelevant here.)

By (2) applied to ν_* and $\rho = e^-(\nu_*^-)$ we can find distinct $\sigma, \sigma' \in \Sigma_m$ such that

$$\sigma \restriction (H \cap \beta) = \rho = \sigma' \restriction (H \cap \beta).$$

It suffices to show that $e = e_0 \cup \{\langle \nu_*, \sigma\rangle\}$ and $e' = e_0 \cup \{\langle \nu_*, \sigma'\rangle\}$ are both coherent. We shall check this for e, the case of e' is analogous. Pick $\nu \in C$ and let $\gamma \in F$ be the minimal element with $\nu(\gamma) \neq \nu_*(\gamma)$. Such an ordinal γ must exist because $\nu \neq \nu_*$ as $\nu \in C \not\ni \nu_*$. Fix $\beta \geq \gamma$. In the assertions from the cases below we use the fact that

$$e^-(\nu_* \restriction (F \cap \beta)) = e^-(\nu_*^-) = \rho = \sigma \restriction (H \cap \beta)$$

and $e(\nu_*) = \sigma$.

If $\gamma < \beta$, then

$$\begin{aligned} e(\nu) \restriction (H \cap \gamma) &= \big(e(\nu) \restriction (H \cap \beta)\big) \restriction (H \cap \gamma) = \big(e^-(\nu \restriction (F \cap \beta)\big) \restriction (H \cap \gamma) \\ &= \big(e^-(\nu_* \restriction (F \cap \beta)\big) \restriction (H \cap \gamma) = \big(e(\nu_*) \restriction (H \cap \beta)\big) \restriction (H \cap \gamma) \\ &= e(\nu_*) \restriction (H \cap \gamma). \end{aligned}$$

If $\gamma = \beta$, then

$$e(\nu) \restriction (H \cap \beta) = e^-(\nu \restriction (F \cap \beta)) = e^-(\nu_* \restriction (F \cap \beta)) = e(\nu_*) \restriction (H \cap \beta).$$

⊣

Lemma 3.10. *We use notation from the formulation of Lemma 3.7. Set* $S = \operatorname{supp}(p)$, $\Sigma_n = \{\nu \in (2^{k_n})^{F_n} : \nu$ *is consistent with* $p\}$, *and suppose that* $\langle \langle i_n, j_n, C_n\rangle : n \in \omega \rangle$ *is a sequence such that items* (1)–(4) *of Lemma* 2.2 *are satisfied. Then there exists* $q \geq p$ *such that* $\{p|\sigma : \sigma \in C_n\}$ *is predense above* q *for all* $n \in \omega$.

Proof. For each natural number n, let $q_n \geq p$ be such that for every $\beta < \alpha$ and $\nu \in C_n$ we have

$$\begin{aligned} &(q_n \restriction \beta)|(\nu \restriction (F_{i_n} \cap \beta)) \Vdash_\beta q_n(\beta) \\ &\quad = \bigcup \big\{ (p|\nu')(\beta) : \nu' \in C_n, \nu' \restriction (F_{i_n} \cap \beta) = \nu \restriction (F_{i_n} \cap \beta) \big\}. \end{aligned}$$

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The correctness of this definition formally requires to prove by recursion over $\beta \leq \alpha$, along with defining q_n, that each $\nu \restriction (F_{i_n} \cap \beta)$ is consistent with $q_n \restriction \beta$, where $\nu \in C_n$, and the set

$$\{ (q_n \restriction \beta)|(\nu \restriction (F_{i_n} \cap \beta)) : \nu \in C_n \}$$

is a maximal antichain above $q_n \restriction \beta$, but this is rather easy and standard. As a result, the set $\{ q_n|\nu : \nu \in C_n \}$ is a maximal antichain above q_n.

It remains to show that

$$(q_{n+1}, k_{i_{n+1}}) \geq_{F_{i_n}} (q_n, k_{i_n})$$

for all $n \in \omega$, and then let q be the fusion of the q_n's. Suppose that for some $\beta \in F_{i_n}$ we have already shown that

$$(q_{n+1} \restriction \beta, k_{i_{n+1}}) \geq_{F_{i_n} \cap \beta} (q_n \restriction \beta, k_{i_n}),$$

and we will prove that

$$q_{n+1} \restriction \beta \Vdash (q_{n+1}(\beta), k_{i_{n+1}}) \geq (q_n(\beta), k_{i_n}).$$

This boils down to proving that for every $\sigma \in C_{n+1}$, if $\nu(\gamma) := \sigma(\gamma) \restriction k_n$ for all $\gamma \in F_{i_n}$, then

$$\begin{aligned} &(q_{n+1} \restriction \beta)|(\sigma \restriction (F_{i_{n+1}} \cap \beta)) \Vdash_\beta \\ &\quad \Big(\bigcup \{ (p|\sigma')(\beta) : \sigma' \in C_{n+1}, \sigma' \restriction (F_{i_{n+1}} \cap \beta) = \sigma \restriction (F_{i_{n+1}} \cap \beta) \}, k_{n+1}\Big) \\ &\qquad \geq \Big(\bigcup \{ (p|\nu')(\beta) : \nu' \in C_n, \nu' \restriction (F_{i_n} \cap \beta) = \nu \restriction (F_{i_n} \cap \beta) \}, k_n\Big). \end{aligned}$$

Fix $s \in 2^{k_n}$ such that there exists $\nu' \in C_n$ with $\nu'(\beta) = s$ and $\nu' \restriction (F_{i_n} \cap \beta) = \nu \restriction (F_{i_n} \cap \beta)$. By Lemma 2.2(4), we have $e(\nu) \restriction (F_{j_n} \cap \beta) = e(\nu') \restriction (F_{j_n} \cap \beta)$. By Lemma 3.9(1), there are $\sigma_1, \sigma_2 \in \Sigma_{i_{n+1}}$, both extending $e(\nu')$ (and hence $\sigma_1, \sigma_2 \in C_{n+1}$) such that $\sigma_1(\beta) \neq \sigma_2(\beta)$ and

$$\sigma_1 \restriction (F_{i_{n+1}} \cap \beta) = \sigma_2 \restriction (F_{i_{n+1}} \cap \beta) = \sigma \restriction (F_{i_{n+1}} \cap \beta).$$

It follows that

$$(q_{n+1} \restriction \beta)|(\sigma \restriction (F_{i_{n+1}} \cap \beta)) \Vdash_\beta \sigma_1(\beta), \sigma_2(\beta) \in q_{n+1}(\beta) \cap 2^{k_{i_{n+1}}}$$

and $\sigma_1(\beta), \sigma_2(\beta)$ are distinct extensions of $e(\nu')(\beta)$, which in its turn extends $\nu'(\beta) = s$. ⊣

§4. Totally imperfect Menger sets and Sacks forcing. By V we mean a ground model of ZFC and G_{ω_2} is an $\mathbb{S}_{\omega_2}$-generic filter over V.

THEOREM 4.1. *In $V[G_{\omega_2}]$, every totally imperfect Menger set $X \subseteq 2^\omega$ has size at most ω_1.*

For the proof of Theorem 4.1, we need the following auxiliary result, whose proof is rather standard (see, e.g., [5, Lemma 5.10] for a similar argument) and is left to the reader.

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LEMMA 4.2. *In $V[G_{\omega_2}]$, let $X \subseteq 2^\omega$. Then there exists $\alpha < \omega_2$ of cofinality ω_1 such that*

(1) *$X \cap V[G_\alpha] \in V[G_\alpha]$ and if $K, K' \subseteq 2^\omega$ are closed crowded sets and coded in $V[G_\alpha]$, and $K' \subseteq K \setminus (X \cap V[G_\alpha])$, then $K' \subseteq K \setminus X$.*

Moreover, if X is a totally imperfect Menger set in $V[G_{\omega_2}]$, then

(2) *$X \cap V[G_\alpha]$ is a totally imperfect Menger set in $V[G_\alpha]$.*

In what follows we use the same notation for a Borel (typically closed) subset of 2^ω in the ground model as well as for its reinterpretation in the forcing extensions we consider, it will be always clear in which set-theoretic universe we work.

PROOF OF THEOREM 4.1. Let α be such as in Lemma 4.2. Working in $V[G_{\omega_2}]$, we claim that $X \subseteq V[G_\alpha]$. Since in $V[G_\alpha]$ the remainder $\mathbb{S}_{\alpha,\omega_2}$ is order-isomorphic to $\mathbb{S}_{\omega_2}$, there is no loss of generality in assuming that $\alpha = 0$, i.e., that $V = V[G_\alpha]$.

Let us pick $z \in X \setminus V$ and let γ be the minimal ordinal with $z \in V[G_\gamma]$. Next, we work in V. Let $p_0 \in G_\gamma$ and $\tau \in V^{\mathbb{S}_\gamma}$ be such that $\tau^{G_\gamma} = z$ and

$$p_0 \Vdash_\gamma \tau \in (2^\omega \cap V[\dot{G}_\gamma]) \setminus \bigcup_{\beta<\gamma} V[\dot{G}_\beta].$$

We shall find $q \geq p_0$, $q \in \mathbb{S}_\gamma$ such that $q \Vdash_{\omega_2} \tau \notin X$. This would accomplish the proof: The genericity of G_γ implies that there is q as above which lies in $G_\gamma \subseteq G_{\omega_2}$, which would yield $z = \tau^{G_\gamma} = \tau^{G_{\omega_2}} \notin X$.

Take p and $F_n, k_n, l_n, y_{\sigma_n}$ from Lemma 3.7, applied to p_0 and τ. Note also that Lemma 3.9(3) ensures that $S = \operatorname{supp}(p)$ and the sequences $\langle k_n : n \in \omega \rangle$, $\langle F_n : n \in \omega \rangle$, $\langle \Sigma_n : n \in \omega \rangle$ satisfy (e_f) from the first paragraph of Section 2. Let K and $[\sigma]$ for $\sigma \in \bigcup_{n\in\omega} \Sigma_n$ be defined in the same way as in the first paragraph of Section 2.

Fix an element $x \in K$ and let $\sigma_n \in \Sigma_n$ be such that $\{x\} = \bigcap_{n\in\omega}[\sigma_n]$. Fix a natural number n. We have $\sigma_n(\beta) \subseteq \sigma_{n+1}(\beta)$ for all $\beta \in F_n$, i.e., $\sigma_n \prec \sigma_{n+1}$. Then $p|\sigma_{n+1} \geq p|\sigma_n$ and it follows from Lemma 3.7(4) that

$$p|\sigma_{n+1} \Vdash \tau \restriction l_n = y_{\sigma_n} \text{ and } p|\sigma_{n+1} \Vdash \tau \restriction l_{n+1} = y_{\sigma_{n+1}},$$

which gives $y_{\sigma_n} \subseteq y_{\sigma_{n+1}}$. Thus, the map $h\colon K \to 2^\omega$ such that

$$h(x) := \bigcup_{n\in\omega} y_{\sigma_n},$$

for all $x \in K$, is well defined. By Lemma 3.7(5), the map h is a continuous injection. Consequently, the map $h\colon K \to h[K]$ is a homeomorphism, and hence $h[K]$ is perfect.

Fix a natural number n. By Lemma 3.7(2) and Observation 3.2(3), the set $\{ p|\sigma : \sigma \in \Sigma_n \}$ is a maximal antichain above p. Applying Lemma 3.7(4), we have that

$$p \Vdash \tau \restriction l_n \in \{ y_\sigma : \sigma \in \Sigma_n \}.$$

It follows from the above and from the definition of the function h that $p \Vdash \tau \in h[K]$.

By our assumption on $\alpha = 0$, the set $h[K] \cap X \cap V$ is an element of V and it is totally imperfect and Menger in V. Let $\langle \langle i_n, j_n, C_n \rangle : n \in \omega \rangle$ be a sequence from

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Lemma 2.2, applied to $S := \mathrm{supp}(p)$ and $h^{-1}[X \cap V] \subseteq K$. Lemma 2.2(5) yields

$$K' := \bigcap_{n\in\omega} \bigcup_{v\in C_n} [v] \subseteq K \setminus h^{-1}[X \cap V],$$

and therefore, $h[K'] \subseteq h[K] \setminus (X \cap V)$. Since K, K', h are all coded in V, we conclude that $h[K'] \subseteq h[K] \setminus X$ holds in $V[G_{\omega_2}]$ by Lemma 4.2(2).

Let $q \geq p$ be a condition given by Lemma 3.10. Since the sets $\{p|v : v \in C_n\}$ are predense above q for all $n \in \omega$, we have $q \Vdash \tau \restriction l_{i_n} \in \{ y_v : v \in C_n \}$, and thus $q \Vdash \tau \in h[K']$. We conclude that $q \Vdash \tau \notin X$. ⊣

§5. A modification of the Menger game and consonant spaces. Let $X \subseteq 2^\omega$. We introduce a modification of the Menger game played on X, which we call *grouped Menger game* played on X.

Round 0: Alice selects a natural number $l_0 > 0$, and then the players play the usual Menger game l_0 subrounds, thus constructing a partial play

$$(l_0, \mathcal{U}_0, \mathcal{F}_0, \dots, \mathcal{U}_{l_0-1}, \mathcal{F}_{l_0-1}),$$

where $\mathcal{F}_i$ is a finite subfamily of $\mathcal{U}_i$ for all natural numbers $i < l_0$.

Round 1: Alice selects a natural number $l_1 > 0$, and then the players play the usual Menger game additional l_1 subrounds, thus constructing a partial play

$$(l_0, \mathcal{U}_0, \dots, \mathcal{U}_{l_0-1}, \mathcal{F}_{l_0-1};\ l_1, \mathcal{U}_{l_0}, \mathcal{F}_{l_0}, \dots, \mathcal{U}_{l_0+l_1-1}, \mathcal{F}_{l_0+l_1-1}),$$

where $\mathcal{F}_i$ is a finite subfamily of $\mathcal{U}_i$ for all natural numbers $i < l_0 + l_1$.

Fix a natural number $n > 0$ and assume that natural numbers $l_0, \dots, l_{n-1} > 0$ and $n-1$ rounds of the game have been defined.

Round n: Alice selects a natural number $l_n > 0$, and then the players play the usual Menger game additional l_n subrounds, thus constructing a partial play

$$\begin{aligned}(l_0, \mathcal{U}_0, \mathcal{F}_0, \dots, \mathcal{U}_{l_0-1}, \mathcal{F}_{l_0-1};\ l_1, \mathcal{U}_{l_0}, \mathcal{F}_{l_0}, \dots, \mathcal{U}_{l_0+l_1-1}, \mathcal{F}_{l_0+l_1-1}; \dots;\\ l_n, \mathcal{U}_{l_0+l_1+\cdots+l_{n-1}}, \mathcal{F}_{l_0+l_1+\cdots+l_{n-1}}, \dots, \mathcal{U}_{l_0+l_1+\cdots+l_n-1}, \mathcal{F}_{l_0+l_1+\cdots+l_n-1}),\end{aligned}$$

where $\mathcal{F}_i$ is a finite subfamily of $\mathcal{U}_i$ for all natural numbers $i < l_0 + \cdots + l_n$.

Let $L_0 := 0$ and $L_{n+1} := l_0 + l_1 + \cdots + l_n$ for all natural numbers n. Bob wins the game if

$$X = \bigcup_{n\in\omega} \bigcap_{i\in[L_n, L_{n+1})} \bigcup \mathcal{F}_n,$$

and Alice wins otherwise.

Remark 5.1. In the grouped Menger game played on a set $X \subseteq 2^\omega$, there is no loss of generality if we assume that covers given by Alice in each step are countable and increasing and the families $\mathcal{F}_n$ chosen by Bob are singletons.

We are interested in sets $X \subseteq 2^\omega$ for which Alice has no winning strategy in the grouped Menger game played on X. They include two important classes of subspaces of the Cantor space: Hurewicz spaces and those ones whose complement is consonant.

The following observation is an immediate consequence of Theorem 1.5.

Observation 5.2. *If $X \subseteq 2^\omega$ is Hurewicz, then* Alice *has no winning strategy in the grouped Menger game played on X.*

Proposition 5.3. *If $Y \subseteq 2^\omega$ is consonant, then* Alice *has no winning strategy in the grouped Menger game played on $2^\omega \setminus Y$.*

Proof. Let σ be a strategy for Alice in the grouped Menger game played on $X := 2^\omega \setminus Y$. By Remark 5.1, we may assume that each family given by Alice according to the strategy σ is countable and increasing and Bob chooses one set from the families given by Alice. We shall define a strategy § for Alice in the game $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ played on X such that each play in $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ played according to § and lost by Alice, gives rise to a play in the grouped Menger game on X in which Alice uses σ and loses.

Suppose that σ instructs Alice to start round 0 by selecting a natural number $l_0 > 0$. Let $L_0 := 0$ and $L_1 := l_0$. Then § instructs Alice to play the family of all sets

$$\bigcap_{i\in[L_0,L_1)} U_i,$$

where

$$(l_0, \mathcal{U}_0, U_0, \dots, \mathcal{U}_{l_0-1}, U_{l_0-1})$$

is a play, where Alice uses the strategy σ. Note that the family of all the intersections as above is indeed an open k-cover of X. Suppose that Bob replies in $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ by selecting $\bigcap_{i\in[L_0,L_1)} U_i$ for some sequence $(\mathcal{U}_0, U_0, \dots, \mathcal{U}_{l_0-1}, U_{l_0-1})$ as above. The strategy σ instructs Alice to proceed in round 1 by selecting a natural number $l_1 > 0$. Let $L_2 := l_0 + l_1$. Then § instructs Alice to play the family of all sets

$$\bigcap_{i\in[L_1,L_2)} U_i,$$

where

$$(l_0, \mathcal{U}_0, U_0, \dots, \mathcal{U}_{l_0-1}, U_{l_0-1};\ l_1, \mathcal{U}_{l_0}, U_{l_0}, \dots, \mathcal{U}_{l_0+l_1-1}, U_{l_0+l_1-1})$$

is a play in which Alice uses σ, an open k-cover of X. Suppose that Bob replies in $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ by selecting $\bigcap_{i\in[L_1,L_2)} U_i$ for some sequence $(\mathcal{U}_0, \mathcal{F}_0, \dots, \mathcal{U}_{l_0-1}, U_{l_0-1}; \mathcal{U}_{l_0}, U_{l_0}, \dots, \mathcal{U}_{l_0+l_1-1}, U_{l_0+l_1-1})$ as above.

In general, let σ instruct Alice to start round n by selecting a natural number $l_n > 0$. Let $L_{n+1} := l_0 + l_1 + \dots + l_n$. Then the next move of Alice in $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ according to § is, by the definition, the family of all sets

$$\bigcap_{i\in[L_n,L_{n+1})} U_i,$$

where

$$\begin{gathered}(l_0, \mathcal{U}_0, U_0, \dots, \mathcal{U}_{l_0-1}, U_{l_0-1};\ l_1, \mathcal{U}_{l_0}, U_{l_0}, \dots, \mathcal{U}_{l_0+l_1-1}, U_{l_0+l_1-1}; \dots;\\ l_n, \mathcal{U}_{l_0+l_1+\dots+l_{n-1}}, U_{l_0+l_1+\dots+l_{n-1}}, \dots, \mathcal{U}_{l_0+l_1+\dots+l_{n-1}+l_n-1}, U_{l_0+l_1+\dots+l_{n-1}+l_n-1})\end{gathered}$$

is a play in which Alice uses σ, an open k-cover of X.

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Since the strategy § is not winning, there is a play in $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ in which ALICE uses § and loses, which gives rise to an infinite play

$$(l_0, \mathcal{U}_0, U_0, \dots, \mathcal{U}_{l_0-1}, U_{l_0-1};\ l_1, \mathcal{U}_{l_0}, U_{l_0}, \dots, \mathcal{U}_{l_0+l_1-1}, U_{l_0+l_1-1}; \dots;$$
$$l_n, \mathcal{U}_{l_0+l_1+\dots+l_{n-1}}, U_{l_0+l_1+\dots+l_{n-1}}, \dots, \mathcal{U}_{l_0+l_1+\dots+l_{n-1}+l_n-1}, U_{l_0+l_1+\dots+l_{n-1}+l_n-1}; l_{n+1}, \dots)$$

in which ALICE uses σ and

$$X = \bigcup_{n\in\omega} \bigcap_{i\in[L_n, L_{n+1})} U_i.$$

This means that the strategy σ is not winning as well. ⊣

5.1. Menger game versus the grouped Menger game. Let $\mathcal{GM}$ be the class of all subspaces X of 2^ω such that ALICE has no winning strategy in the grouped Menger game played on X. Obviously, $\mathcal{GM}$ is contained in the class of all Menger subspaces of 2^ω. As we established in Section 5, $\mathcal{GM}$ includes Hurewicz subspaces and subspaces with consonant complement, and hence also all Rothberger subspaces of 2^ω (see the discussion at the end of Section 1 in the work of Jordan [13]). Our next result gives a consistent example of a Menger space which does not belong to $\mathcal{GM}$.

PROPOSITION 5.4. *The class $\mathcal{GM}$ contains no ultrafilters.*

PROOF. Given an ultrafilter X on ω, we shall describe a winning strategy σ for ALICE in the grouped Menger game played on X. For natural numbers $n < k$, let

$$U_{[n,k)} := \{\, a \subseteq \omega : a \cap [n, k) \neq \emptyset \,\}.$$

Then the families $\mathcal{U}_n := \{\, U_{[n,k)} : k > n \,\}$ are increasing open covers of X for all $n \in \omega$. Playing according to the strategy σ, ALICE chooses $l_n = 2$ for all $n \in \omega$. The strategy σ instructs ALICE to play some cover $\mathcal{U}_m$. Then if the set chosen by BOB is of the form $U_{[m,k)}$, then ALICE plays the family $\mathcal{U}_k$. Each play where ALICE uses σ has the following form

$$(2, \mathcal{U}_{i_0}, U_{[i_0,i_1)}, \mathcal{U}_{i_1}, U_{[i_1,i_2)}; 2, \mathcal{U}_{i_2}, U_{[i_2,i_3)}, \mathcal{U}_{i_3}, U_{[i_3,i_4)}; \dots),$$

where $\langle\, i_n : i \in \omega \,\rangle$ is an increasing sequence of natural numbers with $i_0 = 0$. Since the sets

$$a := \bigcup_{k\in\omega} [i_{2k}, i_{2k+1}), \quad b := \bigcup_{k\in\omega} [i_{2k+1}, i_{2k+2})$$

are disjoint and $a \cup b = \omega$, exactly one of them is a member of X. Assume that $a \in X$. Since $a \notin \bigcup_{k\in\omega} U_{[i_{2k+1}, i_{2k+2})}$, we have

$$X \not\subseteq \bigcup_{k\in\omega} (U_{[i_{2k}, i_{2k+1})} \cap U_{[i_{2k+1}, i_{2k+2})}).$$

⊣

Combining [9, Theorem 1.1] and [6, Theorem 10], we conclude that $\mathfrak{d} = \mathfrak{c}$ implies the existence of a Menger ultrafilter.

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COROLLARY 5.5. *Assume that* $\mathfrak{d} = \mathfrak{c}$. *There exists a Menger subspace* X *of* 2^ω *such that* ALICE *has a winning strategy in the grouped Menger game on* X, *i.e.,* $X \notin \mathcal{GM}$.

Let us note that there are models of ZFC without Menger ultrafilters. Indeed, every Menger ultrafilter is a P-point, see, e.g., [4, Observation 3.4]. According to a result of Shelah published in [27] (see also [7]), consistently there are no P-points.

Next, we introduce a variation of the grouped Menger game which is "harder" for ALICE since her choices are more restricted, i.e., are not arbitrary open covers of the space in question. Let $K \subseteq 2^\omega$ be a perfect set. For $y \in K$, let $\mathcal{U}_y$ be a countable increasing family of clopen sets in K such that $\bigcup \mathcal{U}_y = K \setminus \{y\}$. Let $X \subseteq K$ be a set such that $K \setminus X$ is dense in K. The *weak grouped Menger game played on* X *in* K ($\mathrm{wgM}(K, X)$ in short) is played as follows: In round 0 ALICE selects a natural number $l_0 > 0$, and BOB selects a closed nowhere dense subset K_0 of K. Let $L_0 := 0$ and $L_1 := l_0$. Then the players play the usual Menger game l_0 subrounds, with the following restrictions. In each subround $i \in [L_0, L_1)$, ALICE chooses $y_i \in K \setminus (X \cup K_0)$ and plays the family $\mathcal{U}_i := \mathcal{U}_{y_i}$. Then BOB replies by choosing a set $U_i \in \mathcal{U}_i$ with $K_0 \subseteq U_i$.

Afterwards, in round 1 ALICE selects a natural number $l_1 > 0$, and BOB selects a closed nowhere dense set $K_1 \subseteq K$. Let $L_2 := l_0 + l_1$. Then the players play the Menger game further l_1 subrounds with the restriction given above, i.e., in each subround $i \in [L_1, L_2)$ ALICE chooses $y_i \in K \setminus (X \cup K_1)$ and plays the family $\mathcal{U}_i := \mathcal{U}_{y_i}$. Then BOB replies by choosing a set $U_i \in \mathcal{U}_i$ with $K_1 \subseteq U_i$.

In round n ALICE selects a natural number $l_n > 0$, and BOB selects a closed nowhere dense set $K_n \subseteq K$. Let $L_{n+1} := l_0 + l_1 + \cdots + l_n$. Then the players play the Menger game further l_n subrounds such that in each subround $i \in [L_n, L_{n+1})$ ALICE chooses $y_i \in K \setminus (X \cup K_n)$ and plays the family $\mathcal{U}_i := \mathcal{U}_{y_i}$. Then BOB replies by choosing a set $U_i \in \mathcal{U}_i$ with $K_n \subseteq U_i$.

BOB wins the game if

$$X = \bigcup_{n \in \omega} \bigcap_{i \in [L_n, L_{n+1})} U_i,$$

and ALICE wins otherwise.

REMARK 5.8. Let X be a subset of a perfect set $K \subseteq 2^\omega$ such that $K \setminus X$ is dense in K. If ALICE has a winning strategy in the weak grouped Menger game played on X in K, then ALICE has a winning strategy in the grouped Menger game played on X.

A set $X \subseteq 2^\omega$ is *perfectly meager* if for any perfect set $K \subseteq 2^\omega$, the intersection $X \cap K$ is meager in K.

PROPOSITION 5.9. *Let* $K \subseteq 2^\omega$ *be a perfect set and* $X \subseteq 2^\omega$ *be a perfectly meager set. Then* BOB *has a winning strategy in the weak grouped Menger game played on* $X \cap K$ *in* K.

PROOF. For each natural number n let $K_n \subseteq K$ be a closed nowhere dense subset of K such that $X \cap K \subseteq \bigcup_{n \in \omega} K_n \subseteq K$. Then any strategy for BOB in the weak grouped Menger game played on $X \cap K$ in K, where in each round n, BOB plays the set K_n, is a winning strategy. ⊣

Let $K \subseteq 2^\omega$ be a perfect set and $X \subseteq 2^\omega$. The diagram below presents the relations between the considered properties. By $\mathrm{A} \not\uparrow \mathrm{wgM}(K, X \cap K)$ we mean that ALICE has no winning strategy and by $\mathrm{B} \uparrow \mathrm{wgM}(K, X \cap K)$ we mean that BOB has a winning strategy in the game $\mathrm{wgM}(K, X \cap K)$. Note that co-consonant spaces are preserved by closed subspaces.

$$\begin{tikzcd}
\mathrm{B} \uparrow \mathrm{wgM}(K, X \cap K) \arrow[rr] & & \mathrm{A} \not\uparrow \mathrm{wgM}(K, X \cap K) & \\
X \text{ is perfectly meager} \arrow[u] & 2^\omega \setminus X \text{ is consonant} \arrow[ur] \arrow[dr] & & X \text{ is Hurewicz} \arrow[ul] \arrow[dl] \\
& X \text{ is Rothberger} \arrow[u] & X \text{ is Menger} &
\end{tikzcd}$$

The next fact is similar to Lemma 2.2.

LEMMA 5.10. *We use the notation and objects described in Section* 2. *Suppose that X is a subset of the perfect set $K \subseteq (2^\omega)^S$ such that $K \setminus X$ is dense in K and* ALICE *has no winning strategy in* $\mathrm{wgM}(K, X)$.

Then there exists a sequence $\langle \langle i_n, j_n, C_n \rangle : n \in \omega \rangle$ such that

(1) *$\langle i_n : n \in \omega \rangle$ is a strictly increasing sequence of natural numbers*;
(2) $C_n \subseteq \Sigma_{i_n}$;
(3) *for every $n \in \omega$, we have $j_n \in [i_n, i_{n+1})$, and for every $v \in C_n$ there exists $e_n(v) \in \Sigma_{j_n}$ extending v, such that*

$$C_{n+1} = \bigcup_{v \in C_n} \{\sigma \in \Sigma_{i_{n+1}} : \sigma \succ e_n(v)\};$$

(4) *the maps $e_n \colon C_n \to \Sigma_{j_n}$ are coherent*; *and*
(5) $\bigcap_{n \in \omega} \bigcup_{v \in C_n} [v] \cap X = \emptyset$.

PROOF. We shall describe a strategy § for ALICE in the weak grouped Menger game played on X in K such that each play lost ALICE gives rise to the objects whose existence we need to establish.

Round 0. Let $C_0 := \Sigma_0$ and $i_0 := 0$. ALICE declares that the 0th group will have length $l_0 := |C_0|$. Let $\{v_j : j < l_0\} \subseteq \Sigma_0$ be an enumeration of C_0. Suppose that BOB plays a closed nowhere dense $K_0 \subseteq K$.

Subround $\langle 0, 0 \rangle$. By density of $K \setminus X$, ALICE picks $y_{\langle 0,0 \rangle} \in [v_0] \setminus (X \cup K_0)$ and she plays the family $\mathcal{U}_{\langle 0,0 \rangle} := \mathcal{U}_{y_{\langle 0,0 \rangle}}$. Suppose that BOB replies by choosing $U_{\langle 0,0 \rangle} \in \mathcal{U}_{\langle 0,0 \rangle}$ with $K_0 \subseteq U_{\langle 0,0 \rangle}$. Take $j_{\langle 0,0 \rangle} > i_0$ and $v_{\langle 0,0 \rangle} \in \Sigma_{j_{\langle 0,0 \rangle}}$ with $v_{\langle 0,0 \rangle} \succ v_0$ such that $y_{\langle 0,0 \rangle} \in [v_{\langle 0,0 \rangle}]$ and $[v_{\langle 0,0 \rangle}] \cap U_{\langle 0,0 \rangle} = \emptyset$. Let $e_{\langle 0,0 \rangle} \colon C_0 \to \Sigma_{j_{\langle 0,0 \rangle}}$ be a coherent map such that $e_{\langle 0,0 \rangle}(v_0) = v_{\langle 0,0 \rangle}$. Fix a natural number a with $0 < a < l_0$ and assume that the players have already defined the following sequences:

- $\langle y_{\langle 0,b \rangle} : 0 \leq b < a \rangle$ of elements $y_{\langle 0,b \rangle} \in K \setminus (X \cup K_0)$,
- $\langle \mathcal{U}_{\langle 0,b \rangle} : 0 \leq b < a \rangle$ of covers of X by clopen subsets of K such that $\mathcal{U}_{\langle 0,b \rangle} = \mathcal{U}_{y_{\langle 0,b \rangle}}$,
- $\langle U_{\langle 0,b \rangle} : 0 \leq b < a \rangle$ of sets $U_{\langle 0,b \rangle} \in \mathcal{U}_{\langle 0,b \rangle}$ with $K_0 \subseteq U_{\langle 0,b \rangle}$,

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- $\langle j_{\langle 0,b\rangle} : 0 \le b < a \rangle \in \omega^{\uparrow a}$ with $i_0 < j_{\langle 0,0\rangle}$,
- $\langle v_{\langle 0,b\rangle} : 0 \le b < a \rangle$ of maps $v_{\langle 0,b\rangle} \in \Sigma_{j_{\langle 0,b\rangle}}$ such that $y_{\langle 0,b\rangle} \in [v_{\langle 0,b\rangle}]$, $v_b \prec v_{\langle 0,b\rangle}$ and $[v_{\langle 0,b\rangle}] \cap U_{\langle 0,b\rangle} = \emptyset$,
- $\langle e_{\langle 0,b\rangle} : 0 \le b < a \rangle$ of coherent maps $e_{\langle 0,b\rangle} \colon C_0 \to \Sigma_{j_{\langle 0,b\rangle}}$ with $e_{\langle 0,b\rangle}(v_b) = v_{\langle 0,b\rangle}$ and for every $0 \le b_0 < b_1 < a$ and $j < l_0$ we have $e_{\langle 0,b_0\rangle}(v_j) \prec e_{\langle 0,b_1\rangle}(v_j)$.

Subround $\langle 0, a\rangle$: Alice picks $y_{\langle 0,a\rangle} \in [e_{\langle 0,a-1\rangle}(v_a)] \setminus (X \cup K_0)$ and she plays the family $\mathcal{U}_{\langle 0,a\rangle} := \mathcal{U}_{y_{\langle 0,a\rangle}}$. Suppose that Bob replies by choosing $U_{\langle 0,a\rangle} \in \mathcal{U}_{\langle 0,a\rangle}$ with $K_0 \subseteq U_{\langle 0,a\rangle}$. Take $j_{\langle 0,a\rangle} > j_{\langle 0,a-1\rangle}$ and $v_{\langle 0,a\rangle} \in \Sigma_{j_{\langle 0,a\rangle}}$ with $v_{\langle 0,a\rangle} \succ e_{\langle 0,a-1\rangle}(v_a)$ such that $y_{\langle 0,a\rangle} \in [v_{\langle 0,a\rangle}]$ and $[v_{\langle 0,a\rangle}] \cap U_{\langle 0,a\rangle} = \emptyset$. Let $e'_{\langle 0,a\rangle} \colon e_{\langle 0,a-1\rangle}[C_0] \to \Sigma_{j_{\langle 0,a\rangle}}$ be a coherent map such that $e'_{\langle 0,a\rangle}(e_{\langle 0,a-1\rangle}(v_a)) = v_{\langle 0,a\rangle}$. Put $e_{\langle 0,a\rangle} := e'_{\langle 0,a\rangle} \circ e_{\langle 0,a-1\rangle}$.

After subround $\langle 0, l_0 - 1\rangle$, the last subround of round 0, we set

$$j_0 := j_{\langle 0,l_0-1\rangle}, \quad i_1 := j_0 + 1, \quad e_0 := e_{\langle 0,l_0-1\rangle}, \quad C_1 := \bigcup_{v \in C_0} \{\, \sigma \in \Sigma_{i_1} : e_0(v) \prec \sigma \,\}.$$

Fix a natural number $n > 0$ and assume that the elements of the sequence

$$\langle i_0, C_0, j_0, e_0; \dots ; i_{n-1}, C_{n-1}, j_{n-1}, e_{n-1}; i_n, C_n\rangle$$

satisfy all relevant instances of (1)–(4).

Round n. Alice declares that the nth group will have length $l_n := |C_n|$. Let $\{v_j : j < l_n\} \subseteq \Sigma_{i_n}$ be an enumeration[3] of C_n. Suppose that Bob plays a closed nowhere dense subset $K_n \subseteq K$.

Subround $\langle n, 0\rangle$: Alice picks $y_{\langle n,0\rangle} \in [v_0] \setminus (X \cup K_n)$ and she plays the family $\mathcal{U}_{\langle n,0\rangle} := \mathcal{U}_{y_{\langle n,0\rangle}}$. Suppose that Bob replies by choosing $U_{\langle n,0\rangle} \in \mathcal{U}_{\langle n,0\rangle}$ with $K_n \subseteq U_{\langle n,0\rangle}$. Take $j_{\langle n,0\rangle} > i_n$ and $v_{\langle n,0\rangle} \in \Sigma_{j_{\langle n,0\rangle}}$ with $v_{\langle n,0\rangle} \succ v_0$ such that $y_{\langle n,0\rangle} \in [v_{\langle n,0\rangle}]$ and $[v_{\langle n,0\rangle}] \cap U_{\langle n,0\rangle} = \emptyset$. Let $e_{\langle n,0\rangle} \colon C_n \to \Sigma_{j_{\langle n,0\rangle}}$ be a coherent map such that $e_{\langle n,0\rangle}(v_0) = v_{\langle n,0\rangle}$.

Fix a natural number a with $0 < a < l_n$ and assume that the players have already defined the following sequences:

- $\langle y_{\langle n,b\rangle} : 0 \le b < a \rangle$ of elements $y_{\langle n,b\rangle} \in K \setminus (X \cup K_n)$;
- $\langle \mathcal{U}_{\langle n,b\rangle} : 0 \le b < a \rangle$ of covers of X by clopen subsets of K such that $\mathcal{U}_{\langle n,b\rangle} = \mathcal{U}_{y_{\langle n,b\rangle}}$;
- $\langle U_{\langle n,b\rangle} : 0 \le b < a \rangle$ of sets $U_{\langle n,b\rangle} \in \mathcal{U}_{\langle n,b\rangle}$ with $K_n \subseteq U_{\langle n,b\rangle}$;
- $\langle j_{\langle n,b\rangle} : 0 \le b < a \rangle$ which is an increasing sequence with $j_{\langle n,0\rangle} > i_n$;
- $\langle v_{\langle n,b\rangle} : 0 \le b < a \rangle$ of maps $v_{\langle n,b\rangle} \in \Sigma_{j_{\langle n,b\rangle}}$ such that $y_{\langle n,b\rangle} \in [v_{\langle n,b\rangle}]$, $v_b \prec v_{\langle n,b\rangle}$ and $[v_{\langle n,b\rangle}] \cap U_{\langle n,b\rangle} = \emptyset$;
- $\langle e_{\langle n,b\rangle} : 0 \le b < a \rangle$ of coherent maps $e_{\langle n,b\rangle} \colon C_n \to \Sigma_{j_{\langle n,b\rangle}}$ with $e_{\langle n,b\rangle}(v_b) = v_{\langle n,b\rangle}$ and for every $0 \le b_0 < b_1 < a$ and $j < l_n$ we have $e_{\langle n,b_0\rangle}(v_j) \prec e_{\langle n,b_1\rangle}(v_j)$.

Subround $\langle n, a\rangle$: Alice picks $y_{\langle n,a\rangle} \in [e_{\langle n,a-1\rangle}(v_a)] \setminus (X \cup K_n)$ and she plays the family $\mathcal{U}_{\langle n,a\rangle} := \mathcal{U}_{y_{\langle n,a\rangle}}$. Suppose that Bob replies by choosing $U_{\langle n,a\rangle} \in \mathcal{U}_{\langle n,a\rangle}$ such that $K_n \subseteq U_{\langle n,a\rangle}$. Take $j_{\langle n,a\rangle} > j_{\langle n,a-1\rangle}$ and $v_{\langle n,a\rangle} \in \Sigma_{j_{\langle n,a\rangle}}$ with $v_{\langle n,a\rangle} \succ e_{\langle n,a-1\rangle}(v_a)$

[3] Formally, we should have written $\{\, v^n_j : j < l_n \,\}$ instead of $\{\, v_j : j < l_n \,\}$, but we omit extra indices in order to shorten our notation.

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such that $y_{\langle n,a\rangle} \in [v_{\langle n,a\rangle}]$ and $[v_{\langle n,a\rangle}] \cap U_{\langle n,a\rangle} = \emptyset$. Let $e'_{\langle n,a\rangle} : e_{\langle n,a-1\rangle}[C_n] \to \Sigma_{j_{\langle n,a\rangle}}$ be a coherent map such that $e'_{\langle n,a\rangle}(e_{\langle n,a-1\rangle}(v_a)) = v_{\langle n,a\rangle}$. Put $e_{\langle n,a\rangle} := e'_{\langle n,a\rangle} \circ e_{\langle n,a-1\rangle}$.

After subround $\langle n, l_n - 1\rangle$, the last subround of round n, we set

$$j_n := j_{\langle n,l_n-1\rangle}, \quad i_{n+1} := j_n + 1, \quad e_n := e_{\langle n,l_n-1\rangle},$$
$$C_{n+1} := \bigcup_{v \in C_n} \{\sigma \in \Sigma_{i_{n+1}} : e_n(v) \prec \sigma\}.$$

This completes the definition of the strategy § in the weak grouped Menger game played on X. It remains to notice that any play in this game in which Alice uses § gives a sequence of objects we require in our lemma, such that conditions (1)–(4) are satisfied, and if Alice loses (and such a play exists because § cannot be winning), then also (5) is satisfied. ⊣

§6. The weak grouped Menger game and Sacks forcing. Again, by V we mean a ground model of ZFC and G_{ω_2} is an $\mathbb{S}_{\omega_2}$-generic filter over V.

Theorem 6.1. *Assume that V satisfies CH. In $V[G_{\omega_2}]$, suppose that $X \subseteq 2^\omega$ and for any perfect set $K \subseteq 2^\omega$ such that $K \setminus X$ is dense in K* Alice *has no winning strategy in the weak grouped Menger game played on $X \cap K$ in K. Then both X and $2^\omega \setminus X$ are unions of ω_1 compact sets.*

For the proof of Theorem 6.1, we need the following auxiliary result. Similarly to Lemma 4.2, it can be proved in the same way as [5, Lemma 5.10], a rather standard argument is left to the reader.

Lemma 6.2. *In $V[G_{\omega_2}]$, let $X \subseteq 2^\omega$. Then there exists a limit ordinal $\alpha < \omega_2$ of cofinality ω_1 such that*

(1) *$X \cap V[G_\alpha] \in V[G_\alpha]$, and if $K, K' \subseteq 2^\omega$ are closed crowded sets and coded in $V[G_\alpha]$, and $K' \subseteq K \setminus (X \cap V[G_\alpha])$, then $K' \subseteq K \setminus X$;*
(2) *There is a function in $V[G_\alpha]$ which assigns to every perfect set $K \subseteq 2^\omega$, coded in $V[G_\alpha]$, such that $K \setminus X$ is not dense in K, a nonempty clopen subset O of 2^ω with[4] $\emptyset \neq K \cap O \subseteq K \cap X$.*

Moreover, if for each perfect set $K \subseteq 2^\omega$ such that $K \setminus X$ is dense in K, Alice *has no winning strategy in the weak grouped Menger game played on $X \cap K$ in K, then we can in addition assume that*

(3) *in $V[G_\alpha]$, for each perfect set $K \subseteq 2^\omega$ such that $K \setminus (X \cap V[G_\alpha])$ is dense in K,* Alice *has no winning strategy in the weak grouped Menger game played on $X \cap K \cap V[G_\alpha]$ in K.*

Proof of Theorem 6.1. Let α be such as in Lemma 6.2. Working in $V[G_{\omega_2}]$, we claim that

$$X = \bigcup \{L \subseteq 2^\omega \ : \ L \subseteq X, L \text{ is compact, and } L \text{ is coded in } V[G_\alpha]\} \text{ and} \quad (6.2.1)$$

$$2^\omega \setminus X = \bigcup \{L \subseteq 2^\omega \ : \ L \subseteq 2^\omega \setminus X, L \text{ is compact, and } L \text{ is coded in } V[G_\alpha]\}. \quad (6.2.2)$$

[4]This function assigns to each code in $V[G_\alpha]$ for a perfect set a code in $V[G_\alpha]$ for a basic open set with the given properties.

Since in $V[G_\alpha]$ the remainder $\mathbb{S}_{\alpha,\omega_2}$ is order-isomorphic to $\mathbb{S}_{\omega_2}$, there is no loss of generality in assuming that $\alpha = 0$, i.e., that $V = V[G_\alpha]$.

First we prove (6.2.1). Let us pick $z \in X$ and let γ be the minimal ordinal with $z \in V[G_\gamma]$. From now on, whenever we do not specify that we work in some other model, we work in V. Let $r \in G_{\omega_2}$ and $\tau \in V^{\mathbb{S}_\gamma}$ be such that $\tau^{G_\gamma} = z$ and $r \Vdash \tau \in X$. Let $p_0 \geq r \restriction \gamma$ be such that $p_0 \in G_\gamma$ and

$$p_0 \Vdash_\gamma \tau \in (2^\omega \cap V[\dot{G}_\gamma]) \setminus \bigcup_{\beta<\gamma} V[\dot{G}_\beta].$$

We shall find $q \geq p_0$, $q \in \mathbb{S}_\gamma$ and a compact set $L \subseteq X$ coded in V such that $q \Vdash_{\omega_2} \tau \in L$. This would accomplish the proof: The genericity of G_γ implies that there is a condition q as above which lies in $G_\gamma \subseteq G_{\omega_2}$. Then $z = \tau^{G_\gamma} = \tau^{G_{\omega_2}} \in L$.

Take p and $F_n, k_n, l_n, y_{\sigma_n}$ from Lemma 3.7, applied to p_0 and τ. Let Σ_n be the set of all maps $\sigma\colon F_n \to 2^{k_n}$ consistent with p, where $n \in \omega$. By Lemma 3.9(1), for each map $\sigma \in \Sigma_n$, there are maps $\sigma' \neq \sigma'' \in \Sigma_{n+1}$ extending σ, which implies $[\sigma'] \cap [\sigma''] = \emptyset$. For $S := \operatorname{supp}(p)$, define a perfect set K, exactly in the same way as before Lemma 2.1, i.e.,

$$K := \bigcap_{n\in\omega} \bigcup \{\, [\sigma] : \sigma \in \Sigma_n \,\}.$$

Then the family of all sets $[\sigma]$, where $\sigma \in \bigcup_{n\in\omega} \Sigma_n$, is a basis for K. Note also that Lemma 3.9(3) ensures that S and the sequence $\langle\, k_n, F_n, \Sigma_n : n \in \omega \,\rangle$ satisfy (e_f) stated at the beginning of Section 2.

Let $h : K \to 2^\omega$ be defined in the same way as in the proof of Theorem 4.1. Thus $h\colon K \to h[K]$ is a homeomorphism, and hence $h[K]$ is perfect. Fix a natural number n. By Lemma 3.7(2) and Observation 3.2(3), the set $\{\, p|\sigma_n : \sigma_n \in \Sigma_n \,\}$ is a maximal antichain above p. Applying Lemma 3.7(4), we have that

$$p \Vdash \tau \restriction l_n \in \{\, y_{\sigma_n} : \sigma_n \in \Sigma_n \,\}.$$

It follows from the above and from the definition of the function h that $p \Vdash \tau \in h[K]$.

Assume that in $V[G_{\omega_2}]$, the set $h[K] \setminus X$ is dense in $h[K]$. We shall show that this is impossible. By the assumption, Alice has no winning strategy in $\mathrm{wgM}(h[K], X \cap h[K])$. We proceed in V. By Lemma 6.2(3) we have that Alice has no winning strategy in $\mathrm{wgM}(h[K], h[K] \cap X \cap V)$. Since h is a homeomorphism and $h^{-1}[X \cap V] = h^{-1}[X] \cap V = h^{-1}[X] \cap V \cap K$ (because h is defined in V and its domain is K), Alice has no winning strategy in $\mathrm{wgM}(K, h^{-1}[X] \cap V)$. Applying Lemma 5.10 to $S = \operatorname{supp}(p)$ and $h^{-1}[X] \cap V \subseteq K$, we can get a sequence $\langle\, \langle i_n, j_n, C_n\rangle : n \in \omega \,\rangle$ satisfying the conclusion of Lemma 5.10. Lemma 5.10(5) yields

$$K' := \bigcap_{n\in\omega} \bigcup \{\, [v] : v \in C_n \,\} \subseteq K \setminus (h^{-1}[X] \cap V),$$

and therefore $h[K'] \subseteq h[K] \setminus (X \cap V)$, both of these inclusions holding in V. Applying Lemma 6.2(1) we conclude that $h[K'] \subseteq h[K] \setminus X$ holds in $V[G_{\omega_2}]$.

Let $q \geq p$ be a condition given by Lemma 3.10. Note that $q \Vdash \tau \restriction l_{i_n} \in \{\, y_v : v \in C_n \,\}$ because $\{p|v : v \in C_n\}$ is predense above q, for all $n \in \omega$, which implies $q \Vdash \tau \in h[K']$. It follows from the above that $q \Vdash \tau \notin X$, which is impossible since $q^\frown r \restriction [\gamma, \omega_2) \Vdash \tau \in X$.

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Now assume that in $V[G_{\omega_2}]$, we have $\mathrm{Int}_{h[K]}(h[K] \cap X) \neq \emptyset$. Since $h[K]$ is coded in V, there is a clopen set O (namely the one assigned to $h[K]$ by the function given by Lemma 6.2(2)) such that $\emptyset \neq h[K] \cap O \subseteq X$. It follows from the definition of h that there exist $n \in \omega$ and $\sigma \in \Sigma_n$ with $h[[\sigma] \cap K] \subseteq O$, and hence $h[[\sigma] \cap K] \subseteq X$. Let $q := p|\sigma$. Since $q \Vdash \tau \in h[[\sigma] \cap K]$, the set $L := h[[\sigma] \cap K]$ is a perfect set coded in V which is a subset of X and $q \Vdash \tau \in L$. This completes the proof of Equation 6.2.1.

Next, we prove (6.2.2). The argument is very similar to that of (6.2.1), but we anyway give it for the sake of completeness. As above, let $r \in G_{\omega_2}$, $p_0 \geq r \restriction \gamma$, $p_0 \in G_\gamma$ and $\tau \in V^{\mathbb{S}_\gamma}$ be such that $r \Vdash \tau \in 2^\omega \setminus X$ and

$$p_0 \Vdash_\gamma \tau \in (2^\omega \cap V[\dot{G}_\gamma]) \setminus \bigcup_{\beta<\gamma} V[\dot{G}_\beta].$$

We shall find $q \geq p_0$, $q \in \mathbb{S}_\gamma$ and a compact set $L \subseteq 2^\omega \setminus X$ coded in V such that $q \Vdash_{\omega_2} \tau \in L$. As in the case of (6.2.1), this would accomplish the proof.

Again, take p and $F_n, k_n, l_n, y_{\sigma_n}$ from Lemma 3.7, applied to p_0 and τ. Let Σ_n, $[\sigma]$ for $\sigma \in \Sigma_n$, K, and $h : K \to 2^\omega$ be defined in the same way as in the proof of (6.2.1). We have $p \Vdash \tau \in h[K]$.

Assume that in $V[G_{\omega_2}]$, we have $\mathrm{Int}_{h[K]}(h[K] \cap X) \neq \emptyset$. We shall show that this case is impossible. Since $h[K]$ is coded in V, there is a clopen set O such that $\emptyset \neq h[K] \cap O \subseteq X$. It follows from the above that there exist $n \in \omega$ and $\sigma \in \Sigma_n$ with $h[[\sigma] \cap K] \subseteq O$, and hence $h[[\sigma] \cap K] \subseteq X$. Let $q := p|\sigma$. Since $q \Vdash \tau \in h[[\sigma] \cap K]$, the set $L := h[[\sigma] \cap K]$ is a perfect set coded in V which is a subset of X and $q \Vdash \tau \in L$. Consequently, $q \Vdash \tau \in X$, a contradiction to $r \Vdash \tau \in 2^\omega \setminus X$.

Now assume that in $V[G_{\omega_2}]$, the set $h[K] \setminus X$ is dense in $h[K]$. In V, we have that $h[K] \setminus (X \cap V)$ is dense in $h[K]$: If there were a clopen $K' \subseteq 2^\omega$ with

$$\emptyset \neq K' \cap h[K] \subseteq h[K] \setminus (X \cap V),$$

then we would get $K' \cap h[K] \subseteq h[K] \setminus X$ holding in $V[G_{\omega_2}]$ by Lemma 6.2(1).

By the assumption, Alice has no winning strategy in the game $\mathrm{wgM}(h[K], X \cap h[K])$. By Lemma 6.2(3), in V, Alice has no winning strategy in $\mathrm{wgM}(h[K], X \cap h[K] \cap V)$. Since h is a homeomorphism, Alice has no winning strategy in $\mathrm{wgM}(K, h^{-1}[X] \cap K \cap V)$. Now let $\langle \langle i_n, j_n, C_n \rangle : n \in \omega \rangle$, K' and q be the same as in the proof of (6.2.1). Again, since $K' \subseteq K \setminus (h^{-1}[X] \cap V)$ holds in V, we have $K' \subseteq K \setminus h^{-1}[X]$ in $V[G_{\omega_2}]$ by Lemma 6.2(1), or equivalently $h[K'] \subseteq h[K] \setminus X$. Repeating our previous arguments we get

$$q \Vdash \tau \in h[K'] \subseteq 2^\omega \setminus X.$$

It follows from the above that the set $L := h[K']$ is a perfect subset of $2^\omega \setminus X$ and $q \Vdash \tau \in L$, which completes the proof of (6.2.2). $\dashv$

Combining Theorem 6.1 with Proposition 5.3, Remark 5.8, and Proposition 5.9, we get the following result.

Corollary 6.3. *Assume that V satisfies GCH. In $V[G_{\omega_2}]$, the following assertions hold.*

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(1) *Each consonant (Hurewicz) subset of 2^ω as well as its complement are unions of $\mathfrak{d} = \omega_1$ many compact subspaces. In particular, there are $\mathfrak{c} = \omega_2$ consonant (Hurewicz) subspaces of 2^ω.*
(2) *Each perfectly meager subset of 2^ω has cardinality at most $\mathfrak{d} = \omega_1$, and its complement is a union of ω_1 compact sets.*

The first half of Corollary 6.3(2), namely that all perfectly meager subsets of 2^ω have cardinality at most ω_1 in the Sacks model, was established by Miller [15, Section 5].

Remark 6.4. In $V[G_{\omega_2}]$, considered in the above corollary, there is a Luzin subset of 2^ω, i.e., an uncountable set whose intersection with any meager set is at most countable, which is totally imperfect and Menger [12], but it is not (perfectly) meager. There is also a perfectly meager set that is not Menger: Since $\mathfrak{d} = \omega_1$ in $V[G_{\omega_2}]$, there is a dominating set $X = \{ x_\alpha : \alpha < \omega_1 \}$ in $[\omega]^\omega$, where $x_\beta \leq^* x_\alpha$ for all ordinal numbers $\beta < \alpha < \omega_1$. This set is not Menger. The set $X \cup \mathrm{Fin}$ satisfies the Hurewicz covering property [1, Theorem 10], and any totally imperfect set with this property is perfectly meager [14, Theorem 5.5]. Thus, the set X is perfectly meager, too. Alternatively, we could use here the main result of [17], which implies directly that X is perfectly meager since $\leq^*$ is a Borel subset of $\omega^\omega \times \omega^\omega$.

Theorems 4.1 and 6.1 motivate the following problem.

Problem 6.5. *In the Sacks model:*

(1) *Is every Menger space $X \subseteq 2^\omega$ a union of ω_1-many of its compact subspaces?*
(2) *Is the complement $2^\omega \setminus X$ of a Menger set $X \subseteq 2^\omega$ a union of ω_1-many of its compact subspaces?*
(3) *Are there only ω_2-many Menger subsets of 2^ω?*
(4) *Is the complement $2^\omega \setminus X$ of a totally imperfect Menger set $X \subseteq 2^\omega$ a union of ω_1-many of its compact subspaces?*

Regarding the last item of Problem 5.6, we do not know the answer even to the following question.

Problem 6.6. *In the Sacks model, is the complement $2^\omega \setminus X$ of any set $X \subseteq 2^\omega$ with $|X| = \omega_1$ a union of ω_1-many of its compact subspaces? In particular, is $2^\omega \setminus (2^\omega \cap V)$ a union of ω_1-many of its compact subspaces?*

§7. Menger sets and Hechler forcing. The results from the previous section lead to the question, whether $\mathfrak{d} < \mathfrak{c}$ implies that any totally imperfect Menger subset of 2^ω has cardinality at most $\mathfrak{d}$. We address this problem, showing that this is not the case. We also provide a consistent result that the size of the family of all Hurewicz subsets of 2^ω can be equal $2^\mathfrak{c}$ even if $\mathfrak{d} < \mathfrak{c}$. Let $\mathbb{P}$ be a definable ccc forcing notion of size $\mathfrak{c}$ which adds dominating reals over a ground model (e.g., the poset defined in [2, p. 95], nowadays commonly named Hechler forcing) and $\mathbb{P}_{\omega_1}$ be an iterated forcing of length ω_1 with finite support, where each iterand is equal to $\mathbb{P}$. In this section, by G_{ω_1} we mean a $\mathbb{P}_{\omega_1}$-generic filter over a ground model V of ZFC.

Proposition 7.1. *In $V[G_{\omega_1}]$, each subset of $2^\omega \cap V$ is Hurewicz.*

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PROOF. Let $X \subseteq 2^\omega \cap V$ and $\phi\colon X \to [\omega]^\omega$ be a continuous function. The function ϕ can be extended to a continuous function Φ defined on a G_δ-set A, containing X. Then there is an ordinal number $\alpha < \omega_1$ such that the function Φ and the set A are coded in $V[G_\alpha]$. It follows that $\Phi[X] \subseteq V[G_\alpha]$. Since there exists a function g_α, added in step α, which dominates any real from $V[G_\alpha]$, the set $\Phi[X]$ is bounded in $V[G_{\omega_1}]$. By the Hurewicz–Recław characterization of the Hurewicz property [18, Proposition 1], the set X is Hurewicz[5] in $V[G_{\omega_1}]$. $\dashv$

THEOREM 7.2. *Assume that V satisfies $\neg CH$. In $V[G_{\omega_1}]$, we have $\mathfrak{d} < \mathfrak{c}$ and the following assertions hold.*

(1) *There is a totally imperfect Hurewicz and Rothberger (and thus Menger) subset of 2^ω with cardinality $\mathfrak{c}$.*
(2) *There are $2^\mathfrak{c}$-many Hurewicz and Rothberger (and thus consonant) subsets of 2^ω.*

PROOF. Since $\mathbb{P}_{\omega_1}$ adds ω_1 dominating reals to the ground model, we have $\mathfrak{d} = \omega_1 < \mathfrak{c}$ in $V[G_{\omega_1}]$. Since the forcing $\mathbb{P}$ is ccc, the forcing $\mathbb{P}_{\omega_1}$ is ccc, too. Since $\mathbb{P}_{\omega_1}$ adds reals to the ground model, any subset of $2^\omega \cap V$ is totally imperfect in $V[G_{\omega_1}]$. By Proposition 7.1, any subset of $2^\omega \cap V$ is Hurewicz in $V[G_{\omega_1}]$.

Since Cohen reals are added by any tail of the considered iteration, any subset of $2^\omega \cap V$ is also Rothberger in $V[G_{\omega_1}]$, by virtue of an argument similar to that in the proof of Proposition 7.1, more details could be found in the proof of [22, Theorem 11]. $\dashv$

§8. Comments and open problems. Let $\mathrm{P}(\omega)$ be the power set of ω, the set of natural numbers. We identify each element of $\mathrm{P}(\omega)$ with its characteristic function, an element of 2^ω. In that way we introduce a topology on $\mathrm{P}(\omega)$. Let $[\omega]^\omega$ be the family of all infinite sets in $\mathrm{P}(\omega)$. Each set in $[\omega]^\omega$ we identify with an increasing enumeration of its elements, a function in the Baire space ω^ω. We have $[\omega]^\omega \subseteq \omega^\omega$ and topologies in $[\omega]^\omega$ induced from $\mathrm{P}(\omega)$ and ω^ω are the same. Let Fin be the family of all finite sets in $\mathrm{P}(\omega)$. A totally imperfect Menger set constructed by Bartoszyński and Tsaban, mentioned in Theorem 1.1 is a set the form $X \cup \mathrm{Fin} \subseteq \mathrm{P}(\omega)$ of size $\mathfrak{d}$, such that for any function $d \in [\omega]^\omega$, we have $|\{\, x \in X : x \leq^* d \,\}| < \mathfrak{d}$. In fact, any set with these properties is totally imperfect and Menger [1, Remark 18]. It has also a stronger covering property $\mathsf{S}_1(\Gamma, \mathrm{O})$, described in details in Section 8.2.

PROPOSITION 8.1. *There are at least $2^\mathfrak{d}$-many totally imperfect Menger subsets of 2^ω.*

PROOF. Let $X \subseteq [\omega]^\omega$ be a set of size $\mathfrak{d}$ such that all coordinates of elements in X are even numbers and for any function $d \in [\omega]^\omega$, we have $|\{\, x \in X : x \leq^* d \,\}| < \mathfrak{d}$. Let $\{\, x_\alpha : \alpha < \mathfrak{d} \,\}$ be a bijective enumeration of elements in X. Fix a function $y\colon \mathfrak{d} \to 2^\omega$. For each ordinal number $\alpha < \mathfrak{d}$, let $x_\alpha + y_\alpha$ be the function such that $(x_\alpha + y_\alpha)(n) := x_\alpha(n) + y_\alpha(n)$ for all n. Then $x_\alpha + y_\alpha \in [\omega]^\omega$ and $x_\alpha \leq^* x_\alpha + y_\alpha$ for all ordinal numbers $\alpha < \mathfrak{d}$. Thus, for any function $d \in [\omega]^\omega$, we have $\{\, \alpha : x_\alpha + y_\alpha \leq^* d \,\} \subseteq \{\, \alpha : x_\alpha \leq^* d \,\}$, and the latter set has size smaller than $\mathfrak{d}$. Then

[5] As noted by the referee, a similar argument actually gives that $\phi[X]$ is bounded for any Borel function $f : X \to \omega^\omega$.

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the set $\{ x_\alpha + y_\alpha : \alpha < \mathfrak{d} \} \cup \mathrm{Fin}$ is totally imperfect and Menger. It follows that for different functions $y : \mathfrak{d} \to 2^\omega$, we get different totally imperfect Menger sets. $\dashv$

8.1. Menger versus property $\mathsf{S}_1(\Gamma, \mathrm{O})$. A cover of a space is a γ-cover if it is infinite and any point of the space belongs to all but finitely many sets in the cover. A space satisfies property $\mathsf{S}_1(\Gamma, \mathrm{O})$ if for any sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open γ-covers of the space there are sets $U_0 \in \mathcal{U}_0$, $U_1 \in \mathcal{U}_1, \ldots$ such that the family $\{ U_n : n \in \omega \}$ covers the space. This property implies the Menger property. By the result of Just, Miller, Scheepers, and Szeptycki [14, Theorem 2.3] (see also the work of Sakai [20, Lemma 2.1]), any subset of 2^ω satisfying $\mathsf{S}_1(\Gamma, \mathrm{O})$ is totally imperfect. The following questions are one of the major open problems in the combinatorial covering properties theory.

Problem 8.2.
(1) *Is there a ZFC example of a totally imperfect Menger subset of* 2^ω *which does not satisfy* $\mathsf{S}_1(\Gamma, \mathrm{O})$?
(2) *Is there a subset of* 2^ω *whose continuous images into* 2^ω *are totally imperfect and Menger, which does not satisfy* $\mathsf{S}_1(\Gamma, \mathrm{O})$?

In the first item of the problem above we ask about ZFC examples because under CH there exists even a Hurewicz totally imperfect subspace of 2^ω which can be mapped continuously onto 2^ω, see [25]. The second item of Problem 8.2 is motivated by the fact that the property $\mathsf{S}_1(\Gamma, \mathrm{O})$ is preserved by continuous functions. By the results from Section 2, we can put this problem in a more specific context.

Problem 8.3. *Let* G_{ω_2} *ba an* $\mathbb{S}_{\omega_2}$*-generic filter over a ground model* V. *In* $V[G_{\omega_2}]$, *does any Menger set of cardinality* ω_1 *satisfy* $\mathsf{S}_1(\Gamma, \mathrm{O})$?

8.2. Other problems. The following problem is motivated by Remark 6.4.

Problem 8.4. *Is any perfectly meager subset of* 2^ω *contained in a Menger totally imperfect set*?

We do not know whether the conclusion of Corollary 5.5 holds in ZFC.

Problem 8.5. *Is it consistent that* $\mathcal{GM}$ *coincides with the family of all Menger subspaces of* 2^ω? *In other words, is it consistent that for every Menger* $X \subseteq 2^\omega$ Alice *has no winning strategy in the grouped Menger game on* X?

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INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRASSE 8-10/104, 1040 WIEN
AUSTRIA

E-mail: valentin.haberl.math@gmail.com
URL: https://www.tuwien.at/mg/valentin-haberl/

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INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS AND
NATURAL SCIENCE, COLLEGE OF SCIENCES, CARDINAL STEFAN WYSZYŃSKI
UNIVERSITY IN WARSAW
WÓYCICKIEGO 1/3, 01–938 WARSAW
POLAND
E-mail: p.szewczak@wp.pl
URL: http://piotrszewczak.pl

INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRASSE 8-10/104, 1040 WIEN
AUSTRIA
E-mail: lzdomsky@gmail.com
URL: https://dmg.tuwien.ac.at/zdomskyy/

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