A RESULT CONCERNING ADDITIVE MAPS
ON THE SET OF QUATERNIONS AND AN APPLICATION

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We determine all additive $F, G : H \to R$ and multiplicative $M : H \to R$ satisfying the functional equation $F(x) + M(x)G(x^{-1}) = 0$. As an application we generalise Kurepa's solution of one of Halperin's problem concerning quadratic functionals.

The functional equation (FE) $F(x) + M(x)G(x^{-1}) = 0$, for additive $F, G,$ and multiplicative $M$, and its special cases have been studied by many authors. Kurepa [5] came across the functional equation (FE) with $M(x) = x^2$ on the reals $R$. Through this equation he obtained the general form of functionals $Q$ on $R$-vector spaces satisfying the parallelogram law and the homogeneity $Q(\lambda x) = \lambda^2 Q(x)$, thus answering a question raised by I. Halperin in 1963 in Paris. With this result over $R$ he further solved [6] the problem on vector spaces over the field of complex numbers $C$ or the skew field of quaternions $H$ under the homogeneity $Q(\lambda x) = \lambda^2 Q(x)$. P. Vrbová [10] managed to solve (FE) with $F = G$ and $M(x) = |x|^2$ on $C$ and gave a shorter proof of the result of S. Kurepa on complex spaces. These and further results of many other authors were unified and generalised in the work of Ng [9]. He determined the general solution of (FE) on a commutative field $k$ of characteristic $\neq 2$ in order to generalise the results of various authors concerning the Halperin problem on quadratic functionals. The other motivation for the study of (FE) is its application in information theory [8]. The problem of characterising multiplicative-type recursive measures of information in $n$ dimensions [3] leads to (FE) with $G = F$ and $M, F : \mathbb{R}^n \to \mathbb{R}$ [1]. Working in this area Ebanks [2] extended the result of Ng [9] by solving the (FE) for additive $F, G : k^n \to k$ and multiplicative $M : k^n \to k$, where $k$ is a commutative field of characteristic $\neq 2$. Some functional equations closely related to (FE) have been treated by Vukman on Banach algebras [11].

One can obtain further generalisations of results of Ng and Ebanks by studying all triples $F, G, M : R \to k$ of additive $F, G$ and multiplicative $M$ satisfying (FE) on the subset of all invertible elements $R^* \subset R$, where $k$ is a commutative field with

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characteristic 0 and $R$ is a commutative ring with identity and regularity property. The regularity property is defined similarly as in [7]: A ring $R$ with identity has regularity property if for any $x \in R$ there exists a positive integer $n_x$ such that for any $m \in \mathbb{N}$, $m \geq n_x$, the element $(m + x)$ is invertible. For example, all Banach algebras with identity have this property. The regularity property implies that there are a lot of invertible elements in $R$. This seems to be a reason that the general form of $F, G,$ and $M | R^*$ in this more general setting can be described in almost the same way as in the special case treated by Ng [9]. We shall omit the details since the basic ideas are exactly the same as those of Ng.

In our note we shall determine all additive $F, G : \mathbb{H} \rightarrow \mathbb{R}$ and multiplicative $M : \mathbb{H} \rightarrow \mathbb{R}$ satisfying (FE). This result shows us that the solution of (FE) in a noncommutative case can be essentially different from that in the commutative case.

We shall apply this result in the theory of quadratic functionals. Let $X$ be a quaternionic vector space. Let us recall that a mapping $Q : X \rightarrow \mathbb{R}$ is called a quadratic functional if it satisfies the parallelogram law

\[(i)\] $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$

and the homogeneity

\[(ii)\] $Q(\lambda x) = |\lambda|^2 Q(x)$.

A mapping $B : X \times \mathbb{H} \rightarrow \mathbb{H}$ is called a sesquilinear functional if

\[(iii)\] $B(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 B(x_1, y) + \lambda_2 B(x_2, y),$  
\[(iv)\] $B(x, \mu_1 y_1 + \mu_2 y_2) = B(x, y_1)\bar{\mu_1} + B(x, y_2)\bar{\mu_2},$

where $\bar{\mu}$ denotes the conjugate of $\mu$. One of the questions posed by I. Halperin can be formulated as follows: Does a quadratic functional $Q$ possess the property that

$$B(x, y) = m(x, y) + im(x, iy) + jm(x, jy) + km(x, ky)$$

where $m(x, y) = (1/4)(Q(x + y) - Q(x - y))$

is a sesquilinear functional. It has been proved by Kurepa [6] that the answer to this problem is in the affirmative. We shall generalise this result by allowing a very general notion of homogeneity on $Q$. More precisely, we shall replace (ii) by a weaker assumption that $Q$ is functionally homogeneous, that is, for some scalar function $M : \mathbb{H} \rightarrow \mathbb{R}$ the relation $Q(\lambda x) = M(\lambda)Q(x)$ holds for all $\lambda \in \mathbb{H}$ and all $x \in X$.

Let us recall that a derivation $D$ on the reals is an additive mapping satisfying $D(ts) = sD(t) + tD(s)$. For an arbitrary quaternion $\lambda = t_1 + t_2 i + t_3 j + t_4 k$, the notations $\bar{\lambda}$ and $|\lambda|$ are used for usual conjugation and norm on $\mathbb{H}$. We denote the set of all nonzero quaternions by $\mathbb{H}^*$. 


THEOREM 1. Let additive $F, G : \mathbb{H} \rightarrow \mathbb{R}$ and multiplicative $M : \mathbb{H} \rightarrow \mathbb{R}$ be nonzero maps satisfying the equation $F(\lambda) + M(\lambda)G(\lambda^{-1}) = 0$ on $\mathbb{H}^*$. Then they are of the form:

\begin{align*}
F(t_1 + t_2 i + t_3 j + t_4 k) &= b_1 t_1 + b_2 t_2 + b_3 t_3 + b_4 t_4, \\
G(t_1 + t_2 i + t_3 j + t_4 k) &= -b_1 t_1 + b_2 t_2 + b_3 t_3 + b_4 t_4,
\end{align*}

and

$M(\lambda) = |\lambda|^2$,

where $b_1, b_2, b_3, b_4$ are real constants satisfying $(b_1, b_2, b_3, b_4) \neq (0, 0, 0, 0)$. The converse is also true.

PROOF: Comparing the equations

$F(\lambda) + M(\lambda)G(\lambda^{-1}) = 0$ and $F(-\lambda) + M(-\lambda)G(-\lambda^{-1}) = 0$

we obtain $G(\lambda^{-1})(M(\lambda) - M(-\lambda)) = 0$. As $G$ is nonzero we have necessarily $M(\mu) = M(-\mu)$ for at least one $\mu \in \mathbb{H}^*$. Using the multiplicativity of $M$ we get

$M(\lambda) = M(\mu)M(\mu^{-1}\lambda) = M(-\mu)M(\mu^{-1}\lambda) = M(-\lambda)$

for any $\lambda \in \mathbb{H}$. For every $\lambda \in \mathbb{H}$ we can find a quaternion $\mu$ such that $\lambda = \mu^2$. Thus, we have $M(\lambda) = M(\mu^2) \geq 0$ for any $\lambda \in \mathbb{H}$.

Since $F$ and $G$ are additive they can be written as

\begin{align*}
F(t_1 + t_2 i + t_3 j + t_4 k) &= \sum_{i=1}^{4} f_i(t_i), \\
G(t_1 + t_2 i + t_3 j + t_4 k) &= \sum_{i=1}^{4} g_i(t_i),
\end{align*}

where $f_i, g_i : \mathbb{R} \rightarrow \mathbb{R}$ are additive functions. Substituting $t$, $t_i$, $t_j$, and $t_k$ in $(FE)$ we get

$f_i(t) + s_i M(t) g_i(t^{-1}) = 0, \quad t \in \mathbb{R}^*$,

where $s_1 = 1$, $s_2 = -M(i)$, $s_3 = -M(j)$, and $s_4 = -M(k)$. The restriction $M|_{\mathbb{R}}$ is not identically zero since otherwise all $f_i$ would be zero functions which would contradict the fact that $F$ is nonzero. The multiplicativity of $M$ implies now that $M(t)$ is a nonzero real number for every nonzero real $t$. Moreover, we have $M(1) = M(-1) = M(i) = M(j) = M(k) = 1$. It follows that for every $i$, $i = 1, 2, 3, 4$, additive functions $f_i$ and $s_i g_i$ are either both nonzero or both identically equal to zero. We may now
apply [9, Corollary 4.2] in order to derive from \( f_i(t) + s_i M(t) g_i(t^{-1}) = 0 \) that \( f_i, g_i \), and the restriction of \( M \) to the field of reals are either of the form

\[
f_i(t) = D_i(t) + b_i t, \quad g_i(t) = u_i(D_i(t) - b_i t), \quad M(t) = t^2,
\]

where \( D_i, i = 1, 2, 3, 4 \), are derivations on the field of real numbers, \( b_i, i = 1, 2, 3, 4 \), are real constants, and \( u_1 = 1 \), while \( u_2 = u_3 = u_4 = -1 \); or

\[
f_i(t) = \text{Im}(a_i \phi(t)) + \text{Re}(b_i \phi(t)), \quad g_i(t) = u_i(\text{Im}(a_i \phi(t)) - \text{Re}(b_i \phi(t))), \quad M(t) = |\phi(t)|^2,
\]

where \( \phi : \mathbb{R} \rightarrow \mathbb{C} \) is a nontrivial embedding, \( a_i, b_i, i = 1, 2, 3, 4 \), are real constants, \( u_1 = 1 \), \( u_2 = u_3 = u_4 = -1 \), and \( \text{Im} \phi, \text{Re} \phi \) denote respectively the imaginary part of \( \phi \), the real part of \( \phi \).

Let us first consider the case that \( M(t) = t^2 \) holds for all real numbers \( t \). Let \( \lambda \) be a strictly imaginary quaternion, that is \( \lambda = -\overline{\lambda} \). For such \( \lambda \) we have

\[
|\lambda|^4 = M(|\lambda|^2) = M(\lambda \overline{\lambda}) = M(\lambda)M(-\lambda) = (M(\lambda))^2,
\]

and consequently, \( M(\lambda) = |\lambda|^2 \). For an arbitrary \( \lambda \in \mathbb{H} \) one can always find strictly imaginary quaternions \( \lambda_1 \) and \( \lambda_2 \) such that \( \lambda = \lambda_1 \lambda_2 \). It follows that we have \( M(\lambda) = |\lambda|^2 \) for all \( \lambda \in \mathbb{H} \). We can easily derive from (FE) that for an arbitrary quaternion \( \lambda = t_1 + t_2 i + t_3 j + t_4 k \) with \( |\lambda| = 1 \) the relation

\[
D_1(t_1) + D_2(t_2) + D_3(t_3) + D_4(t_4) = 0
\]

is valid. It follows also that \( D_1(t_1) + D_2(-t_2) + D_3(-t_3) + D_4(-t_4) = 0 \), and consequently, \( D_1(t) = 0 \) for all \( t, 0 \leq t \leq 1 \). From additivity of \( D_1 \) we get that \( D_1 \) is identically equal to zero. Clearly, the same must be true for \( D_i, i = 2, 3, 4 \). One can now easily verify that

\[
F(t_1 + t_2 i + t_3 j + t_4 k) = \sum_{i=1}^{4} b_i t_i,
\]

\[
G(t_1 + t_2 i + t_3 j + t_4 k) = -b_1 t_1 + \sum_{i=2}^{4} b_i t_i,
\]

and

\[
M(\lambda) = |\lambda|^2
\]

satisfy the (FE).

In the case that we have \( M(t) = |\phi(t)|^2 \) for all real numbers we get, in the same way as above, that \( M(\lambda) = |\phi(|\lambda|)|^2 \) holds for all quaternions \( \lambda \). The same method as in the first case gives us

\[
\text{Im}(a_1 \phi(t_1) + \ldots + a_4 \phi(t_4)) = 0,
\]
Additive maps on the set of quaternions

and \( a_1 \text{Im} \phi = \ldots = a_4 \text{Im} \phi = 0 \). Since \( \text{Im} \phi \) cannot be identically zero, we have \( a_1 = \ldots = a_4 = 0 \). Substituting quaternions \( \lambda = 1 + t_3 i + t_4 j + t_4 k \) and \( \bar{\lambda} \) in (FE), and comparing the results so obtained, we get

\[
b_1 \left( 1 - \left| \phi(\lambda)^2 \right| \text{Re} \phi \left| \lambda^{-2} \right| \right) = 0.
\]

As \( \phi \) is a nontrivial ring morphism, this implies \( b_1 = 0 \). Similarly, one can verify that \( b_2 = b_3 = b_4 = 0 \). This completes the proof. \( \square \)

The following theorem is an extension of Kurepa’s solution of Halperin’s problem concerning quadratic forms on quaternionic vector space [6, Theorem 2].

**Theorem 2.** Let \( X \) be a vector space over the skew field of quaternions \( \mathbb{H} \), \( M \) a real function on \( \mathbb{H} \), and \( Q \) a nonzero real functional on \( X \) satisfying the parallelogram law and the homogeneity \( Q(\lambda x) = M(\lambda)Q(x) \). Then \( M(\lambda) = |\lambda|^2 \), the functional \( B : X \times X \rightarrow \mathbb{H} \) given by

\[
B(x,y) = m(x,y) + im(x,iy) + jm(x,jy) + km(x,ky),
\]

where

\[
m(x,y) = (1/4)(Q(x + y) - Q(x - y))
\]

is a sequilinear functional, and

\[
B(x,x) = Q(x)
\]

holds for all \( x \in X \).

**Proof:** It is well known that the functional \( m(x,y) \) is biadditive [5, Lemma 1]. Obviously, we have \( m(\lambda x, \lambda y) = M(\lambda)m(x,y) \). Assume that \( m(x,y) = 0 \) for all \( x,y \in X \). Then we have \( Q(x + y) = Q(x - y) \). Putting \( x = y = z/2 \) in this equation we get \( Q(z) = 0 \) for all \( z \in X \) which is a contradiction with our assumption that \( Q \) is nonzero. Fix now \( x_0, y_0 \in X \) such that \( m(x_0, y_0) \neq 0 \). Comparing relations

\[
m(\lambda \mu x_0, \lambda \mu y_0) = M(\lambda \mu)m(x_0, y_0)
\]

and

\[
m(\lambda \mu x_0, \lambda \mu y_0) = M(\lambda)m(\mu x_0, \mu y_0) = M(\lambda)M(\mu)m(x_0, y_0)
\]

we see that \( M \) is multiplicative. For any nonzero \( \lambda \in \mathbb{H} \) we have

\[
m(\lambda x_0, y_0) - M(\lambda)m(x_0, \lambda^{-1} y_0) = 0.
\]

Consequently, mappings \( M : \mathbb{H} \rightarrow \mathbb{R} \) and \( F,G : \mathbb{H} \rightarrow \mathbb{R} \) given by

\[
F(\lambda) = m(\lambda x_0, y_0), \quad G(\lambda) = -m(x_0, \lambda y_0),
\]

satisfy the (FE). Obviously, mappings \( F \) and \( G \) are additive and nonzero. According to Theorem 1 the mapping \( M \) is of the form \( M(\lambda) = |\lambda|^2 \). Using [6, Theorem 2] one can complete the proof. \( \square \)
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