The Factors of \((a, b, c, f, g, h)(x, y, z)^2 - \lambda(x^2 + y^2 + z^2)\).

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If \( f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy, \)
and \( S = x^2 + y^2 + z^2, \)
\( f - \lambda S \) is the product of two factors of the form \( ax + \beta y + \gamma z \) if \( \lambda \) is a root of a discriminating cubic
\[
\begin{vmatrix}
  a - \lambda & h & g \\
  h & b - \lambda & f \\
  g & f & c - \lambda
\end{vmatrix} = 0.
\]

A well-known proof of the reality of the roots of the cubic is as follows:—

Write
\[
\phi(\lambda) = (\lambda - a)((\lambda - b)(\lambda - c) - f^2) - ((\lambda - b)g^2 + (\lambda - c)h^2 + 2fg),
\]
and \( \psi(\lambda) = (\lambda - b)(\lambda - c) - f^2. \)

Suppose that \( a > b > c; \)
when \( \lambda = +\infty, b, c, -\infty, \)
\( \psi(\lambda) = +\infty, -f^2, -c^2, +\infty. \)

Hence, (see figure), the equation \( \psi(\lambda) = 0 \) has two real roots, \( a \) and \( \beta, \) such that
\( a > b > c > \beta. \)

When
\( \lambda = +\infty, - (\sqrt{a - bg} \pm \sqrt{a - ch}), (\sqrt{b - \beta} g \pm \sqrt{c - \beta} h), -\infty. \)

Hence the cubic, \( \phi(\lambda) = 0, \) has three real roots, \( \lambda_1, \lambda_2, \lambda_3, \) such that \( \lambda_1 > a > \lambda_2 > \beta > \lambda_3. \)
Now

\[ f - \lambda S = \frac{1}{b - \lambda} \left[ \{hx + (b - \lambda)y + fz\}^2 + \frac{1}{\psi(\lambda)} \{x\psi(\lambda) - x(hf - b - \lambda g)\}^2 \right]. \]

Therefore if \( \lambda = \lambda_1, \ b - \lambda < 0 \) and \( \psi(\lambda) > 0 \), and \( f - \lambda S \) is of the form \(- (u^2 + v^2)\), where \( u \) and \( v \) are linear functions of \( x, y, z \), with real coefficients. If \( \lambda = \lambda_2, \ b - \lambda < 0 \) and \( \psi(\lambda) < 0 \), and \( f - \lambda S \) is of the form \( \pm (u^2 - v^2) \). If \( \lambda = \lambda_3, \ b - \lambda > 0 \) and \( \psi(\lambda) > 0 \), and \( f - \lambda S \) is of the form \( u^2 + v^2 \).

The only value of \( \lambda \) for which \( f - \lambda S \) is the product of factors with real coefficients is therefore the mean value \( \lambda_m \).

The result can be applied to find the real circular sections of the conicoid \( f(x, y, z) = 1 \). Write the equation

\[ f(x, y, z) - \lambda(x^2 + y^2 + z^2) + \lambda(x^2 + y^2 + z^2) - 1 = 0, \]

and it appears that if \( f - \lambda S = 0 \) represents a pair of planes, the planes cut the conicoid in circles. The real circular sections are given by the mean root of the discriminating cubic.