POSITIVE SOLUTIONS OF NONLOCAL SINGULAR BOUNDARY VALUE PROBLEMS

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Abstract. The paper presents the existence result for positive solutions of the differential equation \( (g(x))'' = f(t, x, (g(x))') \) satisfying the nonlocal boundary conditions \( x(0) = x(T), \min\{x(t) : t \in J\} = 0 \). Here the positive function \( f \) satisfies local Carathéodory conditions on \([0, T] \times (0, \infty) \times (\mathbb{R} \setminus \{0\}) \) and \( f \) may be singular at the value 0 of both its phase variables. Existence results are proved by Leray-Schauder degree theory and Vitali’s convergence theorem.

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1. Introduction. Let \( T \) be a positive number, \( J = [0, T] \) and \( \mathbb{R}_0 = \mathbb{R} \setminus \{0\} \). We shall discuss the singular differential equation

\[
(g(x(t)))'' = f(t, x(t), (g(x(t)))'),
\]

(1.1)

where \( g \in C^0([0, \infty)) \) and the positive function \( f \) satisfies local Carathéodory conditions on \( J \times (0, \infty) \times \mathbb{R}_0 \) (\( f \in Car(J \times (0, \infty) \times \mathbb{R}_0) \)) and \( f \) may be singular at the value 0 of both its phase variables.

Furthermore we shall deal with the nonlocal boundary conditions

\[
x(0) = x(T), \quad \min\{x(t) : t \in J\} = 0.
\]

(1.2)

We say that \( x \in C^0(J) \) is a solution of the boundary value problem (BVP for short) (1.1), (1.2) if \( g(x) \in AC^1(J) \) (functions having absolutely continuous derivative on \( J \)), \( x \) satisfies the boundary conditions (1.2) and (1.1) holds a.e. on \( J \).

In this paper we are interested in finding conditions on the functions \( g \) and \( f \) in (1.1) that guarantee the existence of positive solutions to BVP (1.1), (1.2). The existence

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result is proved by regularization and sequential techniques. Any positive solution \( x \) and \( (g(x))^\prime \) for BVP (1.1), (1.2) ‘go through’ singularities of \( f \) somewhere inside of \( J \).

We show that our existence result for BVP (1.1), (1.2) can be applied to obtain solutions of BVP (1.3), (1.2), where

\[
(g(x(t))x'(t))^\prime = f(t, x(t), g(x(t))x'(t)). \tag{1.3}
\]

By a solution of BVP (1.3), (1.2) we understand a function \( x \in C^1(J) \) satisfying (1.2), \( g(x)x' \in AC(J) \) and (1.3) is true almost everywhere on \( J \).

We note that only a few papers in the literature are devoted to the study of BVPs for differential equations of the form (1.1) (see [1], [7] and references therein). In [1] the authors consider via the method of lower and upper functions the Dirichlet problem with the differential equation \((P(x))^\prime \prime + f_1(t, x) = 0\) where \( P(z) = \int_0^z r(x) dx \), \( r \) is a continuous function and \( f_1 \) satisfies local Carathéodory conditions. Existence results for solutions of \((P(x))^\prime \prime = q(t)f_2(t, x, x')\) with continuous \( f_2 \) satisfying the Dirichlet boundary conditions are given in [7]. Differential equations of the form \(g(x)x'f = f_3(t, x, x')\) and two-point boundary conditions were considered (in the regular case also) in [7]. The Dirichlet problem for differential equations of the form \((r(x)x')^\prime = \mu q(t)f_4(t, x)\) where \( f_4 \) is singular at the value 0 of its phase variable \( x \) was studied in [8]–[10]. In [2] the authors give conditions for the existence of positive solutions of a more general equation \((g(x)(x')^\alpha))^\prime = \mu q(t)f_2(t, x)(x')^\beta\) with \( \alpha \in (0, \infty) \) and \( \beta \in [0, 1) \) satisfying the Dirichlet boundary conditions. Existence results for a functional differential equation with a nonlinear functional left hand side and nonlocal boundary conditions are presented in [4]. In all the papers above, BVPs are considered only for local boundary conditions and, in the case that differential equations are singular at their phase variables solutions ‘start’ and/or ‘finish’, at singular points (with the exception of [4] and [9]).

In this paper the following assumptions will be used.

\((H_1)\) \( g \in C^0([0, \infty)) \) is increasing, \( g(0) = 0 \) and \( \lim_{u \to \infty} g(u) = \infty \).

\((H_2)\) \( g \in C^0([0, \infty)) \) is positive and \( \lim_{u \to \infty} G(u) = \infty \), where

\[
G(u) = \int_0^u g(s) ds, \quad u \in [0, \infty). \tag{1.4}
\]

\((H_3)\) \( f \in Car(J \times (0, \infty) \times \mathbb{R}_0) \) and there exists a positive constant \( a \leq 1/2 \) such that

\[ a \leq f(t, x, y) \quad \text{for a.e. } t \in J \text{ and each } (x, y) \in (0, \infty) \times \mathbb{R}_0. \]

\((H_4)\) For a.e. \( t \in J \) and each \( (x, y) \in (0, \infty) \times \mathbb{R}_0 \),

\[ f(t, x, y) \leq (h_1(x) + h_2(x))(\omega_1(|y|) + \omega_2(|y|)), \]

where \( h_1, \omega_1 \in C^0([0, \infty)) \) are non-negative and non-decreasing, \( h_2, \omega_2 \in C^0((0, \infty)) \) are positive and non-increasing.

\((H_5)\) \( \int_0^1 h_2(g^{-1}(s^2))\omega_2(s) ds < \infty \) and

\[
\lim_{u \to \infty} \int_0^u \frac{1}{K^{-1}(H_1(s))} ds > \frac{T}{2}. \]
where
\[ K(u) = \int_0^u \frac{s}{\omega_1(s+1) + \omega_2(s)} \, ds, \quad u \in [0, \infty), \quad (1.5) \]
\[ H_1(u) = \int_0^u [h_1(g^{-1}(s) + 1) + h_2(g^{-1}(s))] \, ds, \quad u \in [0, \infty). \quad (1.6) \]

\( (H_6) \int_0^1 h_2(G^{-1}(s^2))\omega_2(s) \, ds < \infty \) and
\[ \lim_{u \to \infty} \int_0^u \frac{1}{K^{-1}(H_2(s))} \, ds > \frac{T}{2}, \]
where
\[ H_2(u) = \int_0^u [h_1(G^{-1}(s) + 1) + h_2(G^{-1}(s))] \, ds, \quad u \in [0, \infty). \quad (1.7) \]

**Remark 1.1.** Let assumptions \((H_4)\) and \((H_5)\) be satisfied. We show that the integral \(\int_0^u (1/K^{-1}(H_1(s))) \, ds\) is convergent for all \(u > 0\). Since \(h_1(g^{-1}(u) + 1) + h_2(g^{-1}(u)) \geq h_2(g^{-1}(1))\) and \(\omega_1(u + 1) + \omega_2(u) \geq \omega_2(1)\) for \(u \in [0, 1]\), we have \(H_1(u) \geq h_2(g^{-1}(1))u\) and \(K(u) \leq u^2/(2\omega_2(1))\) for these \(u\). Hence \(K^{-1}(H_1(u)) \geq \sqrt{2h_2(g^{-1}(1))\omega_2(1)u}\) for \(u \in [0, \tau]\) with a \(\tau > 0\) and since \(K^{-1}(H_1)\) is positive and continuous on \([0, \infty)\), we see that \(\int_0^u (1/K^{-1}(H_1(s))) \, ds < \infty\) for all \(u > 0\). Analogously we can verify that \(\int_0^u (1/K^{-1}(H_2(s))) \, ds < \infty\) for \(u > 0\) if assumptions \((H_4)\) and \((H_6)\) are satisfied.

The paper is organized as follows. In Section 2 we prove that the solvability of BVP (1.1), (1.2) is equivalent to that of BVP (2.1), (1.2) (Lemma 2.1). Section 3 deals with a sequence of auxiliary regular BVPs to BVP (2.1), (1.2) where the nonlinearities \(f_n\) in the differential equations are regular functions on \(J \times \mathbb{R}^2\). We give *a priori* bounds for their solutions \(x_n\) (Lemma 3.3) and prove their existence (Lemma 3.4) using Leray-Schauder degree theory (see, for example, [5]). In addition, we show that the sequence \([f_n(t, g^{-1}(x_n(t))), x_n'(t)]\) is uniformly absolutely continuous on \(J\) (Lemma 3.5). In Section 4 we present our main results: the existence of a positive solution to BVP (1.1), (1.2) (Theorem 4.1) and to BVP (1.3), (1.2) (Corollary 4.2). In limiting processes we use the Vitali’s convergence theorem (see, for example, [3], [6]) since it is impossible to find a Lebesgue integrable majorant function for the sequence \([f_n(t, g^{-1}(x_n(t))), x_n'(t)]\) which is necessary for applying the Lebesgue dominated convergence theorem. We include also two examples (Examples 4.3 and 4.4) to illustrate our theory.

**2. Lemma.** Let assumptions \((H_1)\) and \((H_3)\) be satisfied. Together with the differential equation (1.1) we consider the differential equation
\[ x''(t) = f(t, g^{-1}(x(t)), x'(t)). \quad (2.1) \]

We say that \(x\) is a solution of equation (2.1) if \(x \in AC^1(J)\) and \(x\) satisfies (2.1) a.e. on \(J\).

In the next lemma we give relations between solutions of BVP (1.1), (1.2) and BVP (2.1), (1.2).
Lemma 2.1. Let assumptions \((H_1)\) and \((H_3)\) be satisfied. If \(x(t)\) is a solution of BVP (1.1), (1.2), then the function \(u(t) = g(x(t))\), \(t \in J\), is a solution of BVP (2.1), (1.2) and also conversely, if \(x(t)\) is a solution of BVP (2.1), (1.2), then the function \(u(t) = g^{-1}(x(t))\), \(t \in J\), is a solution of BVP (1.1), (1.2).

Proof. Let \(x\) be a solution of BVP (1.1), (1.2). Then \(x \in C^0(J), g(x) \in AC^1(J)\) and \(x\) satisfies (1.2). Set \(u(t) = g(x(t))\) for \(t \in J\). Then \(u(0) = u(T), \min\{u(t) : t \in J\} = 0\), \(u \in AC^1(J)\) and \(u''(t) = (g(x(t)))'' = f(t, x(t), (g(x(t)))') = f(t, g^{-1}(u(t)), u'(t))\) a.e. on \(J\). Hence \(u\) is a solution of BVP (2.1), (1.2).

Let \(x\) be a solution of BVP (2.1), (1.2). Then \(x\) satisfies (1.2) and \(x \in AC^1(J)\). Let \(u(t) = g^{-1}(x(t))\), \(t \in J\). Then (1.2) holds with \(u\) instead of \(x\), \(u \in C^0(J), g(u) = x \in AC^1(J)\) and \((g(u(t)))'' = x''(t) = f(t, g^{-1}(x(t)), x'(t)) = f(t, u(t), (g(u(t)))')\) a.e. on \(J\). Thus \(u\) is a solution of BVP (1.1), (1.2).

Remark 2.2. From Lemma 2.1 we see that solving BVP (1.1), (1.2) is equivalent to solving BVP (2.1), (1.2).

3. Auxiliary regular BVPs. For each \(n \in \mathbb{N}\), define \(f_n \in Car(J \times \mathbb{R}^2)\) by

\[
f_n(t, x, y) = \begin{cases} 
    f(t, x, y) & \text{for } t \in J, x \geq \frac{1}{n}, |y| \geq \frac{1}{n}, \\
    f(t, \frac{1}{n}, y) & \text{for } t \in J, x < \frac{1}{n}, |y| \geq \frac{1}{n}, \\
    g_n(t, x, y) & \text{for } t \in J, x \in \mathbb{R}, y \in (-\frac{1}{n}, \frac{1}{n}). 
\end{cases}
\]

Then \((H_3)\) and \((H_4)\) yield (for \(n \in \mathbb{N}\))

\[a \leq f_n(t, x, y) \quad \text{for a.e. } t \in J \text{ and each } (x, y) \in \mathbb{R}^2\] (3.1)

and

\[f_n(t, x, y) \leq (h_1(x + 1) + h_2(x))(\omega_1(|y| + 1) + \omega_2(|y|))\] (3.2)

for a.e. \(t \in J\) and each \((x, y) \in (0, \infty) \times \mathbb{R}_0\).

Also define \(\hat{g} \in C^0(\mathbb{R})\) by

\[\hat{g}(u) = \begin{cases} 
    g(u) & \text{for } u \in [0, \infty), \\
    -g(-u) + 2g(0) & \text{for } u \in (-\infty, 0). 
\end{cases}\]

If \(g\) satisfies assumption \((H_1)\), then \(\hat{g}\) is increasing on \(\mathbb{R}\), which is the domain of the inverse function \(\hat{g}^{-1}\) to \(\hat{g}\).

Consider the family of regular differential equations

\[x''(t) = \lambda f_n(t, \hat{g}^{-1}(x(t)), x'(t)) + (1 - \lambda)a \quad (E)_n\]

depending on the parameters \(\lambda \in [0, 1]\) and \(n \in \mathbb{N}\), where \(a\) appears in \((H_3)\).

Lemma 3.1. Let assumptions \((H_1)\) and \((H_3)\) be satisfied and let \(x\) be a solution of BVP \((E)_n\), (1.2). Then there exists a unique \(\xi \in (0, T)\) such that

\(a\) \(x(\xi) = 0 \) and \(x(t) > 0\) for \(t \in [0, \xi) \cup (\xi, T]\).
(b) \( x' \) is increasing on \( J \), \( x'(\xi) = 0 \) and \( |x'(t)| \geq a|\xi - t| \) for \( t \in J \).

(c) \( x(t) \geq \frac{a}{2}(t - \xi)^2 \) for \( t \in J \).

**Proof.** By (3.1),

\[
x''(t) \geq a \quad \text{for a.e. } t \in J.
\]

From (3.3) it follows that \( x' \) is increasing on \( J \) and then \( x(0) = x(T) \) implies that \( x' \) vanishes at a unique point \( \xi \in (0, T) \) and \( x \) is decreasing on \([0, \xi]\) and increasing on \([\xi, T]\). Hence the condition \( \min \{x(t) : t \in J\} = 0 \) yields \( x(\xi) = 0 \) and \( x > 0 \) on \([0, \xi) \cup (\xi, T] \). The validity of the inequalities in (b) and (c) follows immediately by integration of (3.3) and using \( x(\xi) = x'(\xi) = 0 \).

**REMARK 3.2.** Lemma 3.1 shows that any solution \( x \) of BVP \((E)_{\lambda}^\alpha \), (1.2) with \( \lambda \in [0, 1] \) and \( n \in \mathbb{N} \) satisfies the inequality \( x(t) > 0 \) for \( t \in [0, \xi) \cup (\xi, T] \) where \( \xi \in (0, T) \) is the unique zero of \( x \). Hence \( g^{-1}(x(t)) = g^{-1}(x(t)) \) for \( t \in J \).

**LEMMA 3.3.** Let assumptions \((H_1)\) and \((H_3) - (H_5)\) be satisfied. Let \( x \) be a solution of BVP \((E)_{\lambda}^\alpha \), (1.2). Then there exists a positive constant \( P \) independent of \( \lambda \in [0, 1] \) and \( n \in \mathbb{N} \) such that

\[
\|x\| = \sup_{t \in J} |x(t)| < P, \quad \|x'\| < P.
\]

**Proof.** By Lemma 3.1, there exists a unique \( \xi \in (0, T) \) such that \( x(\xi) = x'(\xi) = 0 \), \( x(t) > 0 \) on \([0, \xi) \cup (\xi, T] \) and \( x' \) is increasing on \( J \). Hence

\[
\|x\| = x(0) (= x(T)), \quad \|x'\| = \max\{|x'(0)|, x'(T)|\}.
\]

In addition (see (3.2) and Remark 3.2)

\[
x''(t) \leq [h_1(g^{-1}(x(t)) + 1) + h_2(g^{-1}(x(t)))][\omega_1(|x'(t)| + 1) + \omega_2(|x'(t)|)] \tag{3.6}
\]

for a.e. \( t \in J \). Integrating the inequality (for a.e. \( t \in [0, \xi] \))

\[
\frac{x'(t)x'(t)}{\omega_1(-x'(t)) + \omega_2(-x'(t))} \geq [h_1(g^{-1}(x(t)) + 1) + h_2(g^{-1}(x(t)))]|x'(t)|
\]

from \( t \in [0, \xi] \) to \( \xi \), we get

\[
\int_0^{-x'(t)} \frac{s}{\omega_1(s + 1) + \omega_2(s)} \, ds \leq \int_0^{x(t)} \left[ h_1(g^{-1}(s)) + 1 + h_2(g^{-1}(s)) \right] \, ds.
\]

Hence \( K(-x'(t)) \leq H_1(x(t)) \), where \( K \) and \( H_1 \) are defined by (1.5) and (1.6), respectively. Then

\[
-x'(t) \leq K^{-1}(H_1(x(t))) \quad \text{for } t \in [0, \xi], \tag{3.7}
\]

and integrating

\[
-x'(t) \leq K^{-1}(H_1(x(t))) \leq 1 \quad \text{(where } 0 \leq t < \xi\text{)}.
\]
over \([0, \xi]\), we have
\[
\int_0^{x(0)} \frac{1}{K^{-1}(H_1(s))} \, ds \leq \xi. \tag{3.8}
\]
Arguing as above on the inequality (for a.e. \(t \in [\xi, T]\))
\[
\frac{x''(t)x'(t)}{\omega_1(x'(t) + 1) + \omega_2(x'(t))} \leq [h_1(g^{-1}(x(t)) + 1) + h_2(g^{-1}(x(t)))]x'(t)
\]
now on the interval \([\xi, T]\), we get
\[
x'(t) \leq K^{-1}(H_1(x(t))) \quad \text{for } t \in [\xi, T] \tag{3.9}
\]
and
\[
\int_0^{x(T)} \frac{1}{K^{-1}(H_1(s))} \, ds \leq T - \xi. \tag{3.10}
\]
Then (3.5), (3.8) and (3.10) imply
\[
\int_0^{[x]} \frac{1}{K^{-1}(H_1(s))} \, ds \leq \frac{T}{2}. \tag{3.11}
\]
By (\(H_5\)), there is a positive constant \(V\) such that
\[
\int_0^{u} \frac{1}{K^{-1}(H_1(s))} \, ds > \frac{T}{2},
\]
for all \(u \geq V\). Hence (3.11) yields \(\|x\| < V\). Letting \(t = 0\) in (3.7), \(t = T\) in (3.9) and using the last inequality, we get \(-x'(0) < K^{-1}(H_1(V))\) and \(x'(T) < K^{-1}(H_1(V))\). Then (see (3.5)) \(\|x'\| < K^{-1}(H_1(V))\) and so (3.4) is true with \(P = \max\{V, K^{-1}(H_1(V))\}\). \(\square

**Lemma 3.4.** Let assumptions (\(H_1\)) and (\(H_5\)) – (\(H_2\)) be satisfied. Then BVP (E)\(^n\), (1.2) has a solution \(x\) for each \(n \in \mathbb{N}\) and (3.4) is true with a positive constant \(P\) given by Lemma 3.3.

**Proof.** Fix \(n \in \mathbb{N}\). Let
\[
\Omega = \left\{(x, A) : (x, A) \in C^1(J) \times \mathbb{R}, \|x\| < \max\left\{P, \frac{aT^2}{4}\right\}, \|x'\| < \max\left\{P, \frac{aT}{2}\right\}, |A| < \max\left\{P, \frac{aT^2}{8}\right\}\right\}
\]
and the operator \(S : \Omega \to C^1(J) \times \mathbb{R}\) be defined by the formula
\[
S(x, A) = \left(A + \int_0^T S(t, s)f_\nu(s, \hat{g}^{-1}(x(s)), x'(s)) \, ds, A + \min\{x(t) : t \in J\}\right), \tag{3.12}
\]
where
\[
S(t, s) = \begin{cases} \frac{s}{T} - 1 & \text{for } 0 \leq s \leq t \leq T, \\ t - 1 & \text{for } 0 \leq t < s \leq T. \end{cases} \tag{3.13}
\]
We see that $S \in C^0(J \times J)$ and
\[ S(t, s) < 0 \quad \text{for} \quad (t, s) \in (0, T) \times (0, T). \]
Assume that $(x_0, A_0) \in \overline{\Omega}$ is a fixed point of $S$; that is $S(x_0, A_0) = (x_0, A_0)$. Then
\[ x_0(t) = A_0 + \int_0^T S(t, s)f_n(s, \hat{g}^{-1}(x_0(s)), x'_0(s)) \, ds, \quad t \in J, \quad (3.14) \]
\[ \min\{x_0(t) : t \in J\} = 0. \quad (3.15) \]
From (3.14) we deduce that $x_0(0) = x_0(T) = A_0$, $x_0 \in AC^1(J)$ and $x''_0(t) = f_n(t, \hat{g}^{-1}(x_0(t)), x'_0(t))$ for a.e. $t \in J$. Hence $x_0$ is a solution of BVP $(E)_{A_0}$, (1.2). Therefore, to prove the existence of a solution of BVP $(E)_{A_0}$, (1.2) it is sufficient to verify that
\[ D(I - S, \Omega, 0) \neq 0, \quad (3.16) \]
where “$D$” stands for the Leray-Schauder degree and $I$ is the identity operator on $C^1(J) \times \mathbb{R}$. The validity of (3.16) will be proved by the homotopy property. We first define the operator $L : \overline{\Omega} \times [0, 1] \rightarrow C^1(J) \times \mathbb{R}$ by
\[ L(x, A, \lambda) = \left( A + \frac{a}{2} t(t - T), A + (1 - \lambda)x \left( \frac{T}{2} \right) + \lambda \min\{x(t) : t \in J\} \right). \quad (3.17) \]
Then $L$ is a continuous operator and also $L(\overline{\Omega} \times [0, 1])$ is relatively compact in $C^1(J) \times \mathbb{R}$. Set $V = I - L(\cdot, \cdot, 0)$. Then $V(x, A) = (x(t) - A - at(t - T)/2, -x(T/2))$ for $(x, A) \in \overline{\Omega}$. We claim that $V(-x, -A) \neq vV(x, A)$, for all $(x, A) \in \partial \Omega$ and $v \in [1, \infty)$, so that
\[ D(I - L(\cdot, \cdot, 0), \Omega, 0) \neq 0, \quad (3.18) \]
by Theorem 8.3 in [5]. If not, there exist $(x_*, A_*) \in \partial \Omega$ and $v_* \in [1, \infty)$ such that $V(-x_*, -A_*) = v_* V(x_*, A_*)$, we then have
\[ -x_*(t) + A_* - \frac{a}{2} t(t - T) = v_* \left( x_*(t) - A_* - \frac{a}{2} t(t - T) \right), \quad t \in J, \quad (3.19) \]
\[ x_* \left( \frac{T}{2} \right) = -v_* x_* \left( \frac{T}{2} \right). \quad (3.20) \]
From (3.20) we obtain that $x_*(T/2) = 0$ and then (3.19) with $t = T/2$ gives $A_* = \frac{v_* - 1}{v_* + 1} \frac{aT^2}{8}$. Hence $0 \leq A_* < aT^2/8$, and so (see (3.19))
\[ |x_*(t)| = \left| A_* + \frac{a(v_* - 1)}{2(v_* + 1)} t(t - T) \right| < \frac{aT^2}{4}, \quad |x'_*(t)| = \left| \frac{a(v_* - 1)}{2(v_* + 1)} (2t - T) \right| < \frac{aT}{2}. \]
We have proved that $(x_*, A_*) \notin \partial \Omega$ and so (3.18) is true. Assume now that $L(\hat{x}, \hat{A}, \hat{\lambda}) = (\hat{x}, \hat{A})$, for some $(\hat{x}, \hat{A}) \in \overline{\Omega}$ and $\hat{\lambda} \in [0, 1]$. Then
\[ \hat{x}(t) = \hat{A} + \frac{a}{2} t(t - T), \quad t \in J, \quad (3.21) \]
\[ (1 - \hat{\lambda})\hat{x} \left( \frac{T}{2} \right) + \hat{\lambda} \min\{\hat{x}(t) : t \in J\} = 0. \quad (3.22) \]
From (3.21) we conclude that \( \hat{x} \) is a solution of equation (E)\(_n\), \( \hat{x}(0) = \hat{x}(T) = \hat{A} \) and \( \min\{\hat{x}(t) : t \in J\} = \hat{x}(T)/2 \). Then (3.22) gives \( \min\{\hat{x}(t) : t \in J\} = 0 \), and so \( \hat{x} \) is a solution of BVP (E)\(_n\), (1.2). By Lemma 3.3, \( \|\hat{x}\| < P, \|\hat{x}'\| < P \) and then \( |\hat{A}| = |\hat{x}(0)| < P \). Hence \( (\hat{x}, \hat{A}) \notin \partial \Omega \). Thus (3.18) and the homotopy property yield

\[
D(\mathcal{I} - \mathcal{L}(\cdot, \cdot, 1), \Omega, 0) = D(\mathcal{I} - \mathcal{L}(\cdot, \cdot, 0), \Omega, 0) \neq 0. \tag{3.23}
\]

Finally, define \( \mathcal{K} : \overline{\Omega} \times [0, 1] \to C^1(J) \times \mathbb{R} \) by

\[
\mathcal{K}(x, A, \lambda) = \left( A + \int_0^T S(t, s)(\lambda f_n(s, x, \hat{x}'(s)), x') + (1 - \lambda)a \right) ds,
\]

\[
A + \min\{x(t) : t \in J\}.
\]

Then \( \mathcal{K}(\cdot, \cdot, 0) = \mathcal{L}(\cdot, \cdot, 1) \) and \( \mathcal{K}(\cdot, \cdot, 1) = \mathcal{S} \). If we verify that

(i) \( \mathcal{K} \) is a compact operator and

(ii) \( \mathcal{K}(x, A, \lambda) \neq (x, A) \) for \( (x, A) \in \partial \Omega \) and \( \lambda \in [0, 1] \),

then (3.23) guarantees the validity of (3.16). Since \( f_n \in \text{Car}(J \times \mathbb{R}^2) \), standard arguments show that \( \mathcal{K} \) is a compact operator. To verify (ii), assume that \( \mathcal{K}(x_n, A_n, \lambda_n) = (x_n, A_n) \), for some \( (x_n, A_n) \in \overline{\Omega} \) and \( \lambda_n \in [0, 1] \). Then \( x_n \) is a solution of BVP (E)\(_n\), (1.2) and \( x_n(0) = A_n \). According to Lemma 3.3, \( \|x_n\| < P, \|x'_n\| < P \) and then \( |A_n| = |x_n(0)| < P \). Therefore \( (x_n, A_n) \notin \partial \Omega \) and \( \mathcal{K} \) has property (ii).

LEMMA 3.5. Let assumptions (H\(_1\)) and (H\(_3\)) – (H\(_5\)) be satisfied and let \( x_n \) be a solution of BVP (E)\(_n\), (1.2). Then the sequence

\[
\{f_n(t, g^{-1}(x_n(t)), x'_n(t))\} \subset L^1(J)
\]

is uniformly absolutely continuous (UAC) on \( J \); that is for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\int_{\mathcal{M}} f_n(t, g^{-1}(x_n(t)), x'_n(t)) \, ds < \varepsilon \quad (n \in \mathbb{N}),
\]

whenever \( \mathcal{M} \subset J \) is measurable and \( \mu(\mathcal{M}) < \delta \), where \( \mu(\mathcal{M}) \) denotes the Lebesgue measure of \( \mathcal{M} \).

**Proof.** By Lemmas 3.1 and 3.3,

\[
x_n(t) \geq \frac{a}{2}(\xi_n - t)^2, \quad |x'_n(t)| \geq a|\xi_n - t| \quad \text{for } t \in J \text{ and } n \in \mathbb{N}, \tag{3.25}
\]

where \( \xi_n \in (0, T), x_n(\xi_n) = x'_n(\xi_n) = 0 \) and

\[
\|x_n\| < P, \quad \|x'_n\| < P \quad \text{for } n \in \mathbb{N}, \tag{3.26}
\]

where \( P \) is a positive constant. Then \( g^{-1}(x_n(t)) < g^{-1}(P) \) for \( t \in J, n \in \mathbb{N} \) and (see (3.1) and (3.2))

\[
a \leq f_n(t, g^{-1}(x_n(t)), x'_n(t)) \leq [h_1(g^{-1}(P) + 1) + h_2(g^{-1}(x_n(t)))]\alpha_1(P + 1) + \omega_2(|x'_n(t)|), \tag{3.27}
\]
for a.e. \( t \in J \) and for \( n \in \mathbb{N} \). Now, from (3.27) and the inequalities
\[
\begin{align*}
\left( h_2 \left( g^{-1}(x_n(t)) \right) \right) \omega_2 \left( |x'_n(t)| \right) & \geq h_2 \left( g^{-1}(P) \right) \omega_2 \left( |x'_n(t)| \right), \\
\left( h_2 \left( g^{-1}(x_n(t)) \right) \right) \omega_2 \left( |x'_n(t)| \right) & \geq h_2 \left( g^{-1}(x_n(t)) \right) \omega_2(P),
\end{align*}
\]
we see that the sequence (3.24) is UAC on \( J \) if \( \{ h_2 \left( g^{-1}(x_n(t)) \right) \omega_2 \left( |x'_n(t)| \right) \} \) is.

From the structure of the measurable set on \( J \) we deduce that the sequence \( \{ h_2 \left( g^{-1}(x_n(t)) \right) \omega_2 \left( |x'_n(t)| \right) \} \) is UAC on \( J \) if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any at most countable set \( \{(a_j, b_j)\}_{j \in \mathbb{N}} \) of mutually disjoint intervals \( (a_j, b_j) \subset J \),

\[
\sum_{j \in \mathbb{N}} \int_{a_j}^{b_j} h_2 \left( g^{-1}(x_n(t)) \right) \omega_2 \left( |x'_n(t)| \right) dt < \varepsilon \quad (n \in \mathbb{N}).
\]

Therefore, let \( \{(a_j, b_j)\}_{j \in \mathbb{N}} \) be an at most countable set of mutually disjoint intervals \( (a_j, b_j) \subset J \) and set
\[
J^1_n = \{ j : j \in \mathbb{J}, (a_j, b_j) \subset (0, \xi_n) \}, \quad J^2_n = \{ j : j \in \mathbb{J}, (a_j, b_j) \subset (\xi_n, T) \}.
\]

Then for \( i \in J^1_n \) and \( j \in J^2_n \) we have (see (3.25))
\[
\begin{align*}
\int_{a_i}^{b_i} h_2 \left( g^{-1}(x_n(t)) \right) \omega_2 \left( |x'_n(t)| \right) dt & \leq \int_{a_i}^{b_i} h_2 \left( g^{-1} \left( \frac{a}{2} (\xi_n - t)^2 \right) \right) \omega_2 \left( a(\xi_n - t) \right) dt \\
& = \frac{1}{a} \int_{a(\xi_n - b_i)}^{a(\xi_n - a_i)} h_2 \left( g^{-1} \left( \frac{s^2}{2a} \right) \right) \omega_2(s) ds,
\end{align*}
\]
\[
\begin{align*}
\int_{a_j}^{b_j} h_2 \left( g^{-1}(x_n(t)) \right) \omega_2 \left( |x'_n(t)| \right) dt & \leq \int_{a_j}^{b_j} h_2 \left( g^{-1} \left( \frac{a}{2} (\xi_n - t)^2 \right) \right) \omega_2 \left( a(t - \xi_n) \right) dt \\
& = \frac{1}{a} \int_{a(t - \xi_n)}^{a(b_j - \xi_n)} h_2 \left( g^{-1} \left( \frac{s^2}{2a} \right) \right) \omega_2(s) ds.
\end{align*}
\]

If \( a_{j_n} < \xi_n < b_{j_n} \) for some \( j_n \in \mathbb{J} \), then
\[
\begin{align*}
\int_{a_{j_n}}^{b_{j_n}} h_2 \left( g^{-1}(x_n(t)) \right) \omega_2 \left( |x'_n(t)| \right) dt & \leq \int_{a_{j_n}}^{\xi_n} h_2 \left( g^{-1} \left( \frac{a}{2} (\xi_n - t)^2 \right) \right) \omega_2 \left( a(\xi_n - t) \right) dt \\
& + \int_{\xi_n}^{b_{j_n}} h_2 \left( g^{-1} \left( \frac{a}{2} (\xi_n - t)^2 \right) \right) \omega_2 \left( a(t - \xi_n) \right) dt \\
& = \frac{1}{a} \int_{0}^{a(\xi_n - a_{j_n})} h_2 \left( g^{-1} \left( \frac{s^2}{2a} \right) \right) \omega_2(s) ds \\
& + \int_{0}^{a(b_{j_n} - \xi_n)} h_2 \left( g^{-1} \left( \frac{s^2}{2a} \right) \right) \omega_2(s) ds.
\end{align*}
\]
Set

\[ M_n^1 = \mathcal{E}_n^1 \cup \bigcup_{i \in J_n^1} (a(\xi_n - b_i), a(\xi_n - a_i)), \quad M_n^2 = \mathcal{E}_n^2 \cup \bigcup_{j \in J_n^2} (a(b_j - \xi_n), a(b_j - \xi_n)), \]

where

\[ \mathcal{E}_n^1 = \begin{cases} \emptyset & \text{if } \mathcal{J} = J_n^1 \cup J_n^2, \\ (0, a(\xi_n - a_{j_n})) & \text{if } \{j_n\} = J_n \setminus (J_n^1 \cup J_n^2), \end{cases} \]

\[ \mathcal{E}_n^2 = \begin{cases} \emptyset & \text{if } \mathcal{J} = J_n^1 \cup J_n^2, \\ (0, a(b_{j_n} - \xi_n)) & \text{if } \{j_n\} = J_n \setminus (J_n^1 \cup J_n^2). \end{cases} \]

Then

\[
\sum_{j \in J} \int_{a_j}^{b_j} h_2(g^{-1}(x_n(t))) \omega_2(|x_n'(t)|) \, dt \\
\leq \int_{M_n^1} h_2 \left( g^{-1} \left( \frac{s^2}{2a} \right) \right) \omega_2(s) \, ds + \int_{M_n^2} h_2 \left( g^{-1} \left( \frac{s^2}{2a} \right) \right) \omega_2(s) \, ds.
\]

By (H3), \( h_2(g^{-1}(s^2/(2a)))\omega_2(s) \in L_1([0, aT]) \) and, since \( \mu(M_n^k) \leq a \sum_{j \in J}(b_j - a_j) \) for \( n \in \mathbb{N} \) and \( k = 1, 2 \), we see that \( h_2(g^{-1}(x_n(t))\omega_2(|x_n'(t)|)) \) is UAC on \( J \) which finishes the proof. \( \square \)

4. Existence results and examples.

**Theorem 4.1.** Let assumptions (H1) and (H3) – (H5) be satisfied. Then BVP (1.1), (1.2) has a solution.

**Proof.** By Lemma 2.1 (see also Remark 2.2), the solvability of BVP (1.1), (1.2) is equivalent to that of BVP (2.1), (1.2). Theorem 4.1 will be proved if BVP (2.1), (1.2) has a solution.

By Lemma 3.4, BVP (E)1, (1.2) has a solution \( x_n \) for each \( n \in \mathbb{N} \). Also Lemmas 3.1 and 3.3 guarantee the validity of inequalities (3.25) and (3.26), where \( P \) is a positive constant and \( \xi_n \in (0, T), x_n(\xi_n) = x_n'(\xi_n) = 0 \). In addition (see Lemma 3.5), \( \{f_n(t, g^{-1}(x_n(t)), x_n'(t))\} \) is UAC on \( J \) and therefore \( \{x_n'(t)\} \) is equicontinuous on \( J \). Going if necessary to a subsequence, we can assume, by the Arzelà-Ascoli theorem and the compactness principle, that \( \{x_n\} \) is convergent in \( C^1(J) \) and \( \{\xi_n\} \) in \( \mathbb{R} \). Let \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} \xi_n = \xi \). Then \( x \) satisfies the boundary conditions (1.2) and (see (3.25)) \( x(t) \geq (a/2)(\xi - t)^2, |x'(t)| \geq a|\xi - t| \) for \( t \in J \). Thus, \( x(t) > 0 \) and \( |x'(t)| > 0 \) for \( t \in J \setminus \{\xi\} \), and \( f(t, g^{-1}(x(t)), x'(t)) \) is defined almost everywhere. Also

\[
\lim_{n \to \infty} f_n(t, g^{-1}(x_n(t)), x_n'(t)) = f(t, g^{-1}(x(t)), x'(t)) \quad \text{for a.e. } t \in J.
\]

Now, by the Vitali’s convergence theorem, \( f(t, g^{-1}(x(t)), x'(t)) \in L_1(J) \) and

\[
\lim_{n \to \infty} \int_0^t f_n(s, g^{-1}(x_n(s)), x_n'(s)) \, ds = \int_0^t f(s, g^{-1}(x(s)), x'(s)) \, ds \quad (t \in J).
\]
Letting \( n \to \infty \) in the equalities
\[
\chi_n'(t) = \chi_n'(0) + \int_0^t f_n(s, g^{-1}(\chi_n(s)), \chi_n'(s)) \, ds \quad (t \in J, \ n \in \mathbb{N}),
\]
we get
\[
\chi'(t) = \chi'(0) + \int_0^t f(s, g^{-1}(\chi(s)), \chi'(s)) \, ds \quad (t \in J).
\]
Hence \( \chi \in AC^1(J) \) and \( \chi \) is a solution of BVP (2.1), (1.2).

**Corollary 4.2.** Let assumptions \((H_2) - (H_4)\) and \((H_6)\) be satisfied. Then BVP (1.3), (1.2) has a solution.

**Proof.** Using the function \( G \) defined in (1.4) we can write equation (1.3) in the form
\[
(G(x(t))') = f(t, x(t), (G(x(t)))'),
\]
which is equation (1.1) with \( G \) instead of \( g \). Since assumption \((H_6)\) is obtained from assumption \((H_5)\) with \( G \) instead of \( g \), we see that BVP (4.1), (1.2) has a solution \( \chi \), by Theorem 4.1, such that \( \chi \in C^6(J) \) and \( G(x) \in AC^1(J) \). Set \( y(t) = G(x(t)) \) for \( t \in J \). Then \( y \in AC^1(J) \) and from \( x(t) = G^{-1}(y(t)) \) we see that \( x \in C^1(J) \) by \((H_2)\), and so \( g(x)\chi' = (G(x))' \in AC(J) \). Consequently, \( \chi \) is a solution of BVP (1.3), (1.2). \( \square \)

**Example 4.3.** Consider the differential equation
\[
(x^\rho)' = c_0 \left[ 1 + c_1 x^\alpha + \frac{c_2}{x^\beta} \right] \left[ 1 + c_3 |(x^\rho)'|^\gamma + \frac{c_4}{|x^\rho|^\delta} \right],
\]
where \( \rho \in (0, \infty), \ c_0, c_4 \in (0, \infty), \ c_1, c_3 \in [0, \infty), \ \alpha, \beta, \gamma \in (0, \infty), \ \delta \in (0, 1) \) and \( 2\beta < p(1 - \delta) \). Equation (4.2) is the special case of (1.1) with \( g(u) = u^\rho \) satisfying \((H_1)\), and
\[
f(t, x, y) = c_0 \left[ 1 + c_1 x^\alpha + \frac{c_2}{x^\beta} \right] \left[ 1 + c_3 |y|^\gamma + \frac{c_4}{|y|^\delta} \right].
\]
We see that \((H_3)\) is true with \( a = \min\{1/2, c_0\} \) and \((H_4)\) with
\[
h_1(u) = c_0(1 + c_1 u^\alpha), \quad h_2(u) = \frac{c_0 c_2}{u^\beta}, \quad \omega_1(u) = 1 + c_3 u^\gamma, \quad \omega_2(u) = \frac{c_4}{u^\delta}.
\]
We now verify \((H_5)\). Notice that
\[
\int_0^1 h_2(g^{-1}(s^2)) \omega_2(s) \, ds = c_0 c_2 c_4 \int_0^1 s^{-(\delta + \frac{3\gamma}{2})} \, ds = \frac{pc_0 c_2 c_4}{(1 - \delta)p - 2\beta} < \infty
\]
and by a calculation we can show that there exist positive constants \( A, B \) and \( u_0 \in (0, \infty) \) such that for \( u \geq u_0 \) we have
\[
H_1(u) = \int_0^u \left[ h_1(g^{-1}(s) + 1) + h_2(g^{-1}(s)) \right] \, ds < \begin{cases} \frac{Au^{\frac{1}{1-\rho}}}{1-\rho} u^{-\frac{\alpha}{1-\rho}} & \text{if } c_1 > 0, \\ \frac{Au}{1-\rho} & \text{if } c_1 = 0, \end{cases}
\]
\[ K(u) = \int_0^u \frac{s}{\omega_1(s+1) + \omega_2(s)} \, ds > \begin{cases} Bu^{2-\gamma} & \text{if } c_3 > 0, \\ Bu^2 & \text{if } c_3 = 0. \end{cases} \]

Hence there exists \( u_1 \geq u_0 \) such that for \( u \geq u_1 \) we have
\[
K^{-1}(H_1(u)) < \begin{cases} \sqrt{\frac{A}{B} u^{\frac{p+\alpha}{p}}} & \text{if } c_1 > 0, c_3 = 0, \\ \sqrt{\frac{A}{B} u} & \text{if } c_1 = 0, c_3 = 0, \\ 2^{-\gamma} \sqrt{\frac{A}{B} u} & \text{if } c_1 > 0, c_3 > 0, \\ 2^{-\gamma} \sqrt{\frac{A}{B} u^{\frac{1}{\gamma}}} & \text{if } c_1 = 0, c_3 > 0. \end{cases}
\]

Finally, from the last inequalities we deduce that if one of the cases
(a) \( \alpha < p \) if \( c_1 > 0, c_3 = 0 \),
(b) \( c_1 = c_3 = 0 \),
(c) \( \alpha < p(1-\gamma) \) if \( c_1 > 0, c_3 > 0 \),
(d) \( \gamma \in (0, 1) \) if \( c_1 = 0 \) and \( c_3 > 0 \)
occurs, we have
\[
\lim_{u \to \infty} \int_0^u \frac{1}{K^{-1}(H_1(s))} \, ds = \infty.
\]

Applying Theorem 4.1, BVP (4.2), (1.2) has a solution if one of the cases (a)–(d) is satisfied.

**EXAMPLE 4.4.** Consider the differential equation
\[
\left( \frac{x'(t)}{\max\{1, x(t)\}^p} \right)' = c_0(x(t))^\alpha + \frac{c_1}{(x(t))^{\beta}} + \frac{c_2}{|x'(t)|^\gamma}, \tag{4.4}
\]
where \( p \in (0, 1), \alpha, \beta, \gamma, c_i \) are positive constants \( (i = 0, 1, 2) \) and
\[
2\beta + \gamma < 1, \quad \alpha < 1 - p. \tag{4.5}
\]

Equation (4.4) is the special case of (1.3) with \( g(u) = 1/(\max\{1, u\})^p \) satisfying \((H_2)\) since
\[
G(u) = \int_0^u g(s) \, ds = \begin{cases} u & \text{for } u \in [0, 1], \\ u^{1-p} - \frac{1}{1-p} & \text{for } u \in (1, \infty), \end{cases}
\]
and
\[
f(t, x, y) = c_0 x^\alpha + \frac{c_1}{x^\beta} + \frac{c_2}{(\max\{1, x\})^\psi |y|^\gamma}.
\]

We can see that \((H_3)\) is satisfied with \( a = \min\{1/2, c_0, c_1\} \) and \((H_4)\) with
\[
h_1(u) = cu^\alpha, \quad h_2(u) = c \left( \frac{1}{u^\beta} + \frac{1}{(\max\{1, u\})^{\psi}} \right), \quad \omega_1(u) = 1, \quad \omega_2(u) = \frac{1}{u^\nu}.
\]
where \( c = \max\{c_0, c_1, c_2\} \). We shall show that (4.5) guarantees the validity of \((H_6)\). Since
\[
G^{-1}(u) = \begin{cases} 
  u & \text{for } u \in [0, 1], \\
  \frac{1}{\sqrt[2]{1-p}}u + p & \text{for } u \in (1, \infty),
\end{cases}
\]
we have
\[
\int_0^1 h_2(G^{-1}(s^2))\omega_2(s) \, ds = c\int_0^1 \left( \frac{1}{s^{\beta+\gamma}} + \frac{1}{s^{\gamma}} \right) \, ds < \infty.
\]
Further for \( u \geq 1 \),
\[
H_2(u) = \int_0^u \left[ h_1(G^{-1}(s) + 1) + h_2(G^{-1}(s)) \right] \, ds = c\int_0^1 \left[ (s + 1)^\alpha + \frac{1}{s^\beta} + 1 \right] \, ds
+ c\int_1^u \left[ \frac{1}{\sqrt[2]{1-p}s + p + 1} \right] \, ds
+ \frac{1}{(1-\sqrt[2]{1-p}s + p)^p} \right) \, ds
\]
and, for \( u \geq 0 \), we have
\[
K(u) = \int_0^u \frac{s}{\omega_1(s+1) + \omega_2(s)} \, ds = \int_0^u \frac{s^{1+\gamma}}{1+s^\gamma} \, ds.
\]
Thus there exist a positive constant \( A \) and \( u_1 \in (1, \infty) \) such that
\[
H_2(u) < Au^{1+\frac{\alpha}{p}}, \quad K(u) > Au^2 \quad \text{for } u \geq u_1.
\] (4.6)
Now from (4.6) we deduce that
\[
K^{-1}(H_2(u)) < \sqrt{u^{1+\frac{\alpha}{p}}} \quad (u \geq u_2),
\] (4.7)
where \( u_2 \) \((\geq u_1)\) is a sufficiently large number. Since \( \alpha < 1 - p \) by (4.5), we see that
\[
\lim_{u \to \infty} \int_0^u \frac{1}{K^{-1}(H_2(s))} \, ds = \infty.
\]
We have verified that \((H_6)\) is true. Applying Theorem 4.1, BVP (4.4), (1.2) has a solution.

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