On families of finite sets no two of which intersect in a singleton

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Let $X$ be a finite set of cardinality $n$, and let $F$ be a family of $k$-subsets of $X$. In this paper we prove the following conjecture of P. Erdős and V.T. Sós.

If $n > n_0(k)$, $k \geq 4$, $|F| > \binom{n-2}{k-2}$ then we can find two members $F$ and $G$ in $F$ such that $|F \cap G| = 1$.

1. Introduction and some lemmas

Let $X$ be a finite set of cardinality $n$ and let $F$ be a family of $k$-subsets of $X$. Let us define

$$F_x = \{F-x \mid x \in F \in F\}.$$ 

We say that a family of sets is intersecting if any two members of it have non-empty intersection. Let $L$ be a set of non-negative integers. We say that $F$ is an $(n, L, k)$-system if, for any two different members $F, G$ of $F$, $|G \cap F| \in L$.

The Erdős-Ko-Rado Theorem (Erdős, Ko, and Rado [4]) states that if $F$ is an $(n, \{t, t+1, \ldots, k-1\}, k)$-system and $n > n_0(k)$; then $|F| \leq \binom{n-t}{k-t}$ with equality holding if and only if for some $t$-element subset $Y$ of $X$, $F = \{F \subseteq X \mid |F| = k, Y \subseteq F\}$, where $t$ is a positive integer.

Erdős and Sós made the following conjecture (see Erdős [2]):

If $F$ is an $(n, \{0, 2, 3, \ldots, k-1\}, k)$-system, $k \geq 4$, $n \geq n_0(k)$,
then \( |F| \leq \binom{n-2}{k-2} \).

The aim of this paper is to prove this conjecture. For the case \( k = 4 \) it was proved by Katona [5].

Obviously the condition is equivalent to that for every \( x \in X \), \( F_x \) is an intersecting family.

If \( G \) is an intersecting family of \((k-1)\)-subsets of \( X \) then let us define

\[
G^* = \left\{ E \subseteq X \mid E \neq \emptyset, \exists G_1, G_2, \ldots, G_k \mid E \right\}
\]

such that \( G_i \cap G_j = E \), \( 1 \leq i < j \leq k \mid E \right\},

\( B(G) = \{ B \in G^* \mid \exists E \in G^* \text{ such that } E \subseteq B \} \).

From the definition it is evident that \( G \subseteq G^* \) and consequently for every \( G \in G \), there exists \( B \in B(G) \) such that \( B \subseteq G \). Therefore we call \( B \) the \( \Delta \)-base of \( G \).

A family \( C = \{C_1, \ldots, C_s\} \) is called a \( \Delta \)-system of cardinality \( s \) if for some set \( K \subseteq C_1 \) we have \( C_i \cap C_j = K \) for \( 1 \leq i < j \leq s \). \( K \) is called the kernel of the \( \Delta \)-system. Erdős and Rado [3] proved that there exists a function \( f(k, s) \) such that any family consisting of \( f(k, s) \) different \( k \)-sets contains a \( \Delta \)-system of cardinality \( s \).

**Lemma 1.** Let \( F \) be an \((n, \{0, 2, 3, \ldots, k-1\}, k)\)-system, \( k \geq 4 \), \( x \in X \), \( 1 \leq i \leq k-1 \). Then we cannot find sets \( B_1, \ldots, B_{k^i} \in B(F_x) \), forming a \( \Delta \)-system of cardinality \( k^i \) and satisfying further \( |B_j| = i + 1 \) for \( 1 \leq j \leq k^i \).

**Proof.** Let us suppose that for \( B_1, \ldots, B_{k^i} \) the lemma fails; let \( K \) be the kernel of the corresponding \( \Delta \)-system.

By the definition of the \( \Delta \)-base there exist sets \( E_{j}^r \in F_x \) for \( 1 \leq r \leq k^i \), \( 1 \leq j \leq k^{i+1} \), such that for \( 1 \leq j < j^* \leq k^{i+1} \),
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\[ E_j^P \cap E_j^{P*} = B_j. \] As the sets \( E_j^1 - B \) are pairwise disjoint and

\[ \left| \bigcup_{r=2}^{k^i} B_r \right| < k \cdot k^i = k^{i+1}, \]

we can find a \( j, 1 \leq j \leq k^{i+1} \) such that \( E_j^1 - B_j \) is disjoint to the union of the \( B^* \)'s. Let us define \( D_1 = E_j^1 \). Let us suppose that \( D_r \) is defined already for \( r = 1, \ldots, s-1 \). Let us set

\[ C_s = \left\{ \bigcup_{r=1}^{s-1} D_r \right\} \cup \left\{ \bigcup_{r=s+1}^{k^{i+1}} B_r \right\}. \]

Then \( |C_s| < k^{i+1} \).

Hence among the pairwise disjoint sets \( E_j^1 - B_s, \ldots, E_j^{k^{i+1}} - B_s \) we can find one, say \( E_j^s - B_s \), which is disjoint from \( C_s \); let us define \( D_s = E_j^s \). Let us continue this procedure until \( s = k^i \). From the definition of the \( D_s \)'s it follows that they belong to \( F_x \), and that they form a \( \Delta \)-system of cardinality \( k^i \geq k^{\lfloor k \rfloor} \) with kernel \( K \), yielding \( K \in F_x^k \). But this is a contradiction as \( K \notin F_x \). //

In view of Lemma 1, \( B(F_x) \) contains at most \( f(i+1, k^i) \) \((i+1)\)-element sets for \( i = 1, 2, \ldots, k-1 \). Now the next lemma is obvious.

**LEMMA 2.** Let \( F \) be an \((n, \{0, 2, 3, \ldots, k-1\}, k)\)-system consisting of subsets of \( X \), \( x \in X \), \( k \geq 4 \). Suppose that \( F_x^k \) does not contain any 1-element set and let \( B_1, \ldots, B_v \) be the 2-element sets in it. Then

\[ |F_x - \{E \in F_x \mid \exists j, 1 \leq j \leq v, B_j \subset E\}| < k \cdot f(k-1, k^{k-2}) \cdot \binom{n-h}{k-4}. \]

We need one more lemma.

**LEMMA 3.** Let \( F \) be an \((n, \{0, 2, 3, \ldots, k-1\}, k)\)-system
consisting of subsets of \( X \). Let \( y, x \) be two not necessarily different elements of \( X \). If \( B \in \mathcal{B}(F_x), \ C \in \mathcal{B}(F_y) \), then
\[
|\{B \cup x\} \cap \{C \cup y\}| \neq 1.
\]

Proof. If \( B \in F_x, \ C \in F_y \), then the statement follows from the definition of \((n, \{0, 2, 3, \ldots, k-1\}, k)\)-systems. So we may assume that, for example, \( B \notin F_x \). By the definition of \( \mathcal{B}(F_x) \) there exist \( F_1, \ldots, F_k \in F \) forming a \( \Delta \)-system with kernel \( B \cup x \). As the sets \( F_i - (B \cup x) \) are pairwise disjoint and in the case \((B \cup x) \cap (C \cup y) = 1, \]
\[
|(C \cup y) - (B \cup x)| \leq k - 1,
\]
we can find an index \( j, \ 1 \leq j \leq k \), such that \( |F_j \cap (C \cup y)| = 1 \). If \( (C \cup y) \in F \) then this is a contradiction to the definition of \((n, \{0, 2, 3, \ldots, k-1\}, k)\)-systems. If \( C \notin F_y \) then by the definition of \( \mathcal{B}(F_y) \) there exist \( G_1, G_2, \ldots, G_k \in F \) which form a \( \Delta \)-system with kernel \( C \cup y \). As the sets \( G_i - (C \cup y) \) are pairwise disjoint and \( |F_j - (C \cup y)| = k - 1 \), we can find an index \( i, \ 1 \leq i \leq k \), such that \( (F_j - (C \cup y)) \cap G_i = \emptyset \); that is, \( |F_j \cap G_i| = 1 \), a contradiction which proves the lemma.

2. The proof of the result

Let us first prove a slightly weaker result which, however, implies the conjecture of Erdös-Sós.

THEOREM 1. Let \( F \) be an \((n, \{0, 2, 3, \ldots, k-1\}, k)\)-system consisting of subsets of \( X \), \( n > n_0(k) \). Then one of the following cases occurs:

(i) \( |F| < \binom{n-2}{k-2} \);

(ii) there exist \( x \neq y \in X \) such that
\[
F = \{F \subset X \mid |F| = k, \ x \in F, \ y \in F\} ;
\]

(iii) there exists \( x \in X \) such that \( |F_x| < \binom{n-3}{k-3} \);

(iv) there exist \( x \neq y \in X \) such that
Proof. Let us argue indirectly. By Lemma 2 we may assume that for every \( x \in X \), \( B(F_x) \) contains a set of cardinality at most 2. If it contains a 1-element set, say \( \{ y \} \), then the intersection property implies \( B(F_x) = \{ y \} \) and conversely \( B(F_y) = \{ x \} \). Indeed, if for some \( F \in F \), \( |F \cap \{ x, y \}| = 1 \) holds then let us consider a \( \Delta \)-system of cardinality \( k \) and with kernel \( \{ x, y \} \), consisting of members of \( F \) - such a system exists by the definition of \( B(F_x) \). Now as \( |F-\{ x, y \}| = k - 1 \), there is a member of the \( \Delta \)-system, say \( G \), which is disjoint from it; that is, \( |F \cap G| = 1 \), a contradiction.

Now let us suppose that \( B(F_x) \) consists of sets of cardinality at least 2, and let \( B_1, \ldots, B_v \) be the 2-element sets belonging to it. By Lemma 3 the \( B_i \)'s form an intersecting family of 2-sets, and by Lemma 1 this family does not contain a \( \Delta \)-system of cardinality \( k \). As \( k \geq 4 \), it follows \( v < k \). Now Lemma 2 and \( |F_x| \geq \binom{n-3}{k-3} \) imply, for \( n > n_0(k) \), that there exists an \( i \), \( 1 \leq i \leq v \), such that
\[
|\{ F \in F_x \mid B_i \subseteq G \}| > 1/k \binom{n-3}{k-3} .
\]

(1)

By symmetry reasons we may assume that (1) holds for \( i = 1 \). Let us suppose first that for some \( G \in F_x \), \( G \not\supseteq B_1 \) holds.

If \( G \cap B_1 = \emptyset \) then let us choose \( k \) sets \( G_1, \ldots, G_k \) belonging to \( F_x \) and forming a \( \Delta \)-system with kernel \( B_1 \). Then \( |G-B_1| = k-1 \) implies that \( G \) is disjoint from at least one of the \( G_i \)'s, say from \( G_u \). As \( F_x \) is an intersecting family of sets, we see that \( G \cap B_1 = \emptyset \) is impossible. Hence if \( G \not\supseteq B_1 \) then \( |G \cap B_1| = 1 \). We prove now that this is impossible, too.

Let us define \( H = G \cup B_1 \cup \cdots \cup B_v \), and
\[
E_1 = \{ F-B_1 \mid B_1 \subseteq F \in F_x, \ (F-B_1) \cap H = \emptyset \} .
\]
From (1), and \( n > n_0(k) \) it follows that, for example,

\[
|E_1| > \frac{1}{2k} \binom{n-3}{k-3}.
\]

Hence there exists an element \( z \) of \( X - (B_1 \cup x) \) which satisfies

\[
|\{E \in E_1 \mid z \in E\}| > \frac{1}{2k} \binom{n-4}{k-4}.
\]

(2)

In the case \( k = 4 \) we just choose \( z = E - (B_1 \cup x) \) for some \( E \in E_1 \). By a result of Erdős [7] if a family \( V \) of \( s \)-subsets of \( X \), \( s \geq 1 \), does not contain \( f(s) \) pairwise disjoint members, then

\[
|V| = O\left(\binom{n}{s-1}\right).
\]

We apply this theorem for \( V = \{E - z \mid E \in E_1, z \in E\} \), \( s = k - 4 \), \( f(s) = s + 4 \), to prove that there exist \( k \) members of \( E_1 \), say \( C_1, \ldots, C_k \), such that \( C_i \cap C_j = \{z\} \), \( 1 \leq i < j \leq k \). In the case \( k = 4 \) we can choose \( C_1 = \ldots = C_k = \{z\} \).

Let \( B \) be a member of \( B(F) \) for which \( |\{F \in F \mid B \subset F\}| \) is maximal. Then, as we proved it already for \( x \), it follows that

\[
|\{F \in F \mid B \subset F\}| > \frac{1}{k} \binom{n-3}{k-3}.
\]

(3)

From Lemma 3 we know that \( (B \cup z) \cap (B_1 \cup x) = \emptyset \) is impossible. We prove now that these two sets cannot be disjoint either. Otherwise from a \( \Lambda \)-system \( F_1, \ldots, F_k \) consisting of members of \( F \), and having kernel \( B \cup z \), we could choose a set, say \( F_i \), satisfying \( F_i \cap (B_1 \cup x) = \emptyset \).

But then there is an index \( j \), \( 1 \leq j \leq k \), such that \( C_j \cap F_i = \{z\} \).

Now setting \( G_j = (C_j \cup B_1 \cup x \cup z) \in F \), \( |G_j \cap F_i| = 1 \) is a contradiction, proving \( (B \cup z) \cap (B_1 \cup x) \neq \emptyset \).

As \( |B| \leq 2 \), it follows now that \( |B| = 2 \) and \( B \subset (B_1 \cup x) \).

If \( B = B_1 \) then from a \( \Lambda \)-system \( F_1, \ldots, F_k \) consisting of members of \( F \) and having kernel \( B \cup z \) we can choose a set, say \( F_i \), which is...
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disjoint from \( G - B \). But then we have \(|F_1 \cap (G \cup x)| = 1\), a contradiction.

Hence \( x \in B \). This in turn implies \( ((B \cup z) - x) \notin B(F_x) \), a contradiction since \( x \notin (B_1 \cup ... \cup B_v) \). This final contradiction proves that \( G \perp B_1 \) is impossible; that is, for every \( G \in F_x \), \( G \subseteq B_1 \) holds, and in particular \( v = 1 \). Hence \( B(F_x) = \{B_1\} \).

Let \( B_1 = \{y_1, y_2\} \). We assert that \( B(F_{y_1}) = \{\{y_2\}\} \).

Otherwise it follows from the definition of \( B(F_{y_1}) \) that \( \{x, y_2\} \) is a member of it. Then repeating the argument applied to \( x \) for \( y_1 \) we obtain that \( B(F_{y_1}) = \{\{x, y_2\}\} \); that is, every member of \( F \) which contains \( y_1 \) contains \( x \) and \( y_2 \) as well. Consequently we have

\[
|\{F \in F \mid F \cap \{x, y_1\} \neq \emptyset\}| \leq |\{F \subseteq X \mid |F| = k, \{x, y_1, y_2\} \subseteq F\}|
\]

\[
= \binom{n-3}{k-3} < \binom{n-3}{k-3} + \binom{n-4}{k-3},
\]

contradicting the indirect assumptions.

So far we have proved that for every \( x \in X \) either there exists a \( y \in X \) such that \( B(F_x) = \{\{y\}\} \), \( B(F_y) = \{\{x\}\} \), or there exist \( y, z \in X \) such that \( B(F_x) = \{\{y, z\}\} \), \( B(F_y) = \{\{z\}\} \), \( B(F_z) = \{\{y\}\} \).

Now let \( \{x_{i1}, y_{i1}\}, ..., \{x_{i\omega}, y_{i\omega}\} \) be the collection of all the different unordered pairs satisfying \( x_{i}, y_{i} \in X \), \( B(F_{x_{i}}) = \{\{x_{i}\}\} \), \( B(F_{x_{i}}) = \{\{y_{i}\}\} \), \( 1 \leq i \leq \omega \). By Lemma 3 all the elements \( x_{i}, y_{i} \) are different; that is, they form \( \omega \) pairwise disjoint 2-subsets of \( X \).

As we proved it is possible to divide the remaining elements of \( X \) into \( \omega \) classes \( Z_1, ..., Z_{\omega} \) such that for \( 1 \leq i \leq \omega \), \( x_i \in Z_i \), we have \( B(F_{z_i}) = \{\{x_i, y_i\}\} \). So we proved that \( F \) is contained in the following family of subsets of \( X \):
\[ F^* = \{ F \subset X \mid |F| = k, F \cap (Z_i \cup \{x_i, y_i\}) \neq \emptyset \} \]

implies

\[ \{x_i, y_i\} \subset F, 1 \leq i \leq w \].

If \( w = 1 \) then either (i) or (ii) holds. So we may assume \( w \geq 2 \).

All we have to prove now is that in this case \( |F^*| < \binom{n-2}{k-2} \).

We prove this by induction on \( w \) and for every \( n > k \).

By symmetry reasons we may assume that \( |Z_1| \leq |Z_2| \). We count the number of members of \( F^* \) according to the cardinality of their intersection with \( Z_1 \cup \{x_1, y_1\} \). Let us define \( n_1 = |Z_1 \cup \{x_1, y_1\}| \).

Then \( |Z_1| \leq |Z_2| \) implies \( n_1 \leq n - n_1 \). Using the induction hypothesis or the estimate for the case \( w = 1 \), we obtain

\[
|F^*| \leq \binom{n_1-2}{k-2} + \sum_{i=2}^{k-2} \binom{n_1-2}{k-2-i} \binom{n-n_1-2}{i-2} + \binom{n-n_1-2}{i-2}. \tag{4}
\]

As

\[
\binom{n_1-2}{k-2} = \left(\frac{(n_1-2)/(k-2)}{k-3}\right) \binom{n_1-3}{k-3} < \binom{n-n_1-2}{k-3},
\]

and

\[
\binom{n_1-2}{k-2-i} \leq \binom{n_1}{k-i}, \quad i = 2, \ldots, k-2,
\]

it follows from (4),

\[
|F^*| < \binom{n-n_1-2}{k-3} + \sum_{i=2}^{k-2} \binom{n_1}{k-i} \binom{n-n_1-2}{i-2} + \binom{n-n_1-2}{k-2} = \sum_{i=2}^{k-2} \binom{n_1}{k-i} \binom{n-n_1-2}{i-2} = \binom{n-2}{k-2}. \quad //
\]

**Theorem 2.** Let \( F \) be an \((n, \{0, 2, 3, \ldots, k-1\}, \kappa)\)-system, \( k \geq 4 \). Suppose that \( n > n_0(k) + 2 \left\lfloor \frac{n_0(k)}{k} \right\rfloor \), where \( n_0(k) \) is the bound from Theorem 1. Then either there exist two different elements \( x, y \) such that \( F = \{ F \subset X \mid |F| = k, \{x, y\} \subset F \} \) or \( |F| < \binom{n-2}{k-2} \).
Proof. Let us argue indirectly and let \( F \) be a counter-example. Let \( |F| = \binom{n-2}{k-2} + d \), where \( d \) is a non-negative integer. We may apply Theorem 1 to \( F \). Hence either there exists \( x \in X \) such that
\[
|F_x| < \binom{n-3}{k-3}
\]
or there exist two different elements \( x, y \) in \( X \) such that
\[
|\{F \in F \mid F \cap \{x, y\} \neq \emptyset\}| < \binom{n-3}{k-3} + \binom{n-4}{k-3}.
\]

In the first case let us define \( X_1 = X - x \) and in the second \( X_1 = X - \{x, y\} \). In both cases we define \( F_1 = \{F \subset X_1 \mid F \in F\} \). Then \( F_1 \) is an \( (|X_1|, \{0, 2, 3, \ldots, k-1\}, k) \)-system of cardinality at least
\[
\left\lfloor \frac{|X_1| - 2}{k-2} \right\rfloor + d + 1.
\]

Now we apply Theorem 1 to the family \( F_1 \), and we construct a set \( X_2 \) and a family of subsets, \( F_2 \), of \( X_2 \) such that \( F_2 \) is an \( (|X_2|, \{0, 2, 3, \ldots, k-1\}, k) \)-system of cardinality at least
\[
\left\lfloor \frac{|X_2| - 2}{k-2} \right\rfloor + d + 2,
\]
and so on, and so on until we get a set \( X_r \) and a family of \( k \)-subsets of \( X_r \), \( F_r \) such that \( |X_r| \leq n_0(k) \).

Now the method of construction implies that
\[
|F_r| \geq \left\lfloor \frac{|X_r| - 2}{k-2} \right\rfloor + d + \binom{n_0(k)}{k} > \binom{n_0(k)}{k},
\]
a contradiction since the number of \( k \)-subsets of \( X_r \) is
\[
\binom{|X_r|}{k} \leq \binom{n_0(k)}{k}, \quad //
\]

REMARK. One might conjecture that for an arbitrary integer \( s \) and
\( k > k_0(s), n > n_0(k) \), any family of more than \( \binom{n-s-1}{k-s-1} \) \( k \)-subsets of an \( n \)-set contains two members intersecting in a set of cardinality \( s \). The author can prove it only for \( c_k \binom{n-s-1}{k-s-1} \), where \( c_k \) is a large constant.
depending only on $k$.

References


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