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PERTURBATIONS FROM INDEFINITE SYMMETRIC ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study the multiplicity of solutions for the following problem:

$$\begin{cases} -\Delta u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = g(x,u) + \theta h(x,u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\alpha \geq 2$, Ω is a smooth bounded domain in $\mathbb{R}^N$, θ is a parameter and $g,\ h \in C(\bar{\Omega} \times \mathbb{R})$. Under the assumptions that $g(x,u)$ is odd and locally superlinear at infinity in u, we prove that for any $j \in \mathbb{N}$ there exists $\varepsilon_j > 0$ such that if $|\theta| \leq \varepsilon_j$, the above problem possesses at least j distinct solutions. Our results generalize some known results in the literature and are new even in the symmetric situation.

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1. Introduction and main results. Consider the following quasilinear Schrödinger equations:

$$\begin{cases} -\Delta u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = g(x,u) + \theta h(x,u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $\alpha \geq 2$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, θ is a parameter, and $g,\ h \in C(\bar{\Omega} \times \mathbb{R})$.

The quasilinear elliptic equation (1), referred as Modified Schrödinger equation due to the quasilinear and nonconvex term $\Delta(|u|^{\alpha})|u|^{\alpha-2}u$, is derived from several models of mathematical physics (see [**7, 9, 17**]). Compared to the semilinear elliptic equation, the quasilinear case becomes much more difficult because of the effects of the quasilinear term. The main difficulty is that there is no suitable space on which the energy functional is well defined and belongs to the class of C^1. In recent years, several approaches have been developed to overcome this difficulty, such as the Nehari method (see [**13, 20**]), constrained minimization (see [**11**]), dual approach (see [**11, 25, 27**]), perturbation method (see [**14, 15, 26**]). Recently, Liu and Zhao [**16**] obtained the existence of infinitely many solutions of the quasilinear problem under broken symmetry situations. This kind of problem is referred to as perturbation from symmetry problem, and the main feature is that the symmetry of the corresponding energy functional is broken by non-odd perturbed terms. It is worth pointing that the multiple critical values can be maintained by restricting the growth range of the perturbed terms with suitable bounds, and the perturbation from symmetry problem for elliptic equations and systems has been extensively studied (see [**2–4, 8, 18, 21–23, 28, 29**] and the references therein).

In this paper, we consider the perturbation from symmetry problem in another direction. Roughly speaking, if $g(x, t)$ is odd and locally superlinear at infinity in t for a.e. $x \in \Omega$, $h \in C(\bar{\Omega} \times \mathbb{R})$ with no growth and symmetric conditions, we study the multiplicity of solutions for problem (1). As far as we know, a few papers have discussed this problem. There are several difficulties to study this problem. First, when $\theta \neq 0$, the perturbation term h may break the symmetry of the energy functional for problem (1), the classical multiple critical point theorems cannot be used directly. On the other hand, apart from continuity, we do not impose any condition on h, so there is no hope of obtaining multiple solutions of problem (1) by the methods in [**2–4, 8, 18, 21–23, 28, 29**]. Li and Liu [**10**] studied a similar perturbation problem for semilinear elliptic equation, their proof is based on the approach developed by Degiovanni and Lancelotti [**6**]. Since $g(x, t)$ is assumed to be locally superlinear at infinity in t, the method in [**10**] cannot be applied directly. Our approach is different from the method used in [**10**]. Next, we explain our method briefly. First, we introduce an orthogonal sequence on a Banach space E due to the indefinite property of g, and then a sequence of families of subsets on E can be constructed. When we control the parameter θ small enough, the effect of the perturbation term h is so small that the critical values of the energy functional for problem (1) can be reconstructed by minimax procedure over the families of subsets on E introduced above. In detail, we obtain the following results.

THEOREM 1.1. *Assume that g satisfy the following conditions:*

(g_1) *there exist constants* $2\alpha < p < 2^*\alpha$ *and* $C_0 > 0$ *such that*

$$|g(x, t)| \leq C_0(1 + |t|^{p-1}), \quad (x, t) \in \Omega \times \mathbb{R},$$

where $2^* := 2N/(N-2)$ *if* $N \geq 3$ *and* $2^* := \infty$ *if* $N = 1, 2$;

(g_2) *there exist constants* $\mu > 2\alpha$, $1 < \alpha_1 < 2\alpha$ *and* $C_1 > 0$ *such that*

$$|\mu G(x, t) - tg(x, t)| \leq C_1\big(|t|^{\alpha_1} + 1\big), \quad (x, t) \in \Omega \times \mathbb{R},$$

where $G(x, t) := \int_0^t g(x, s)ds$;

(g_3) *there exists a nonempty open subset* $\Lambda \subset \Omega$ *such that*

$$\lim_{|t|\to\infty} \frac{G(x, t)}{|t|^{2\alpha}} = \infty, \quad a.e.\ x \in \Lambda,$$

and there exists $r_0 \geq 0$ *such that*

$$G(x, t) \geq 0, \quad (x, t) \in \Lambda \times \mathbb{R} \ \text{ and } \ |t| \geq r_0;$$

(g_4) $g(x, t) = -g(x, -t)$ *for* $(x, t) \in \Omega \times \mathbb{R}$.

Then, for any $j \in \mathbb{N}$, *there exists* $\theta_j > 0$ *such that if* $|\theta| \leq \theta_j$, *then problem (1) possesses at least j distinct solutions.*

THEOREM 1.2. *Assume that* (g_1)–(g_4) *are satisfied. Then, there exists an unbounded sequence of solutions for problem (1) with* $\theta = 0$.

REMARK 1.1. When $\theta = 0$, problem (1.1) is in the symmetric situation, and our results are also new. In fact, condition (g_3) implies that $g(x, t)$ is only of locally superlinear growth in t as $|t| \to \infty$. There are some functions satisfying condition (g_3), for example, $g(x, t) = a(x)|t|^{p-2}t$, where $a(x) \in C(\bar{\Omega}, \mathbb{R})$ changes sign in Ω and $2\alpha < p < 2^*\alpha$. But this function does not satisfy the globally superlinear growth conditions presented in the reference.

The paper is organized as follows. In Section 2, we establish the variational framework associated with problem (1), and we also give some preliminary lemmas which are useful in the sequel. The proofs of our main results are given in Section 3.

2. Variational setting and preliminaries. First, we introduce some function spaces. For $1 \leq s < +\infty$, let

$$\|u\|_s := \left(\int_\Omega |u|^s dx \right)^{1/s}, \quad u \in L^s(\Omega).$$

Let $E := H_0^1(\Omega)$ be the usual Sobolev space with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}.$$

It is well known that E is continuously embedded into $L^\nu(\Omega)$ for $1 \leq \nu \leq 2^*$, i.e., there exists $\tau_\nu > 0$ such that $\|u\|_\nu \leq \tau_\nu \|u\|$, $u \in E$. Moreover, E is compactly embedded into $L^\nu(\Omega)$ only for $1 \leq \nu < 2^*$.

By direct computation, problem (1) is the Euler–Lagrange equation associated with the energy functional $J_\theta : \mathbb{R} \times E \to \mathbb{R}$ given by

$$J_\theta(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2\alpha} \int_\Omega |\nabla(|u|^\alpha)|^2 dx - \int_\Omega G(x, u) dx - \theta \int_\Omega H(x, u) dx, \tag{2}$$

where $H(x, t) := \int_0^t h(x, s) ds$. It is obvious that J_θ may not be well defined in $\mathbb{R} \times E$. To overcome this difficulty, we adapt a dual approach as in [**5, 11**]. More precisely, the main idea of dual approach is that the quasilinear equation can be reduced to a semilinear equation by the use of a suitable function f, and then the classical Sobolev space framework can be used as the working space. In the spirit of the transformation introduced in [**1**], we make the change of variables by $v = f^{-1}(u)$, where the function f can be defined by

$$f'(t) = \left(1 + \alpha |f(t)|^{2(\alpha-1)}\right)^{-\frac{1}{2}}, \quad t \in [0, +\infty) \text{ and } f(-t) = -f(t), \quad t \in (-\infty, 0].$$

Next, we collect some properties of the function f, which will be used frequently in the sequel of the paper. Detailed proofs can be found in [**1**].

LEMMA 2.1. *The function f and its derivative have the following properties:*
(f_1) *f is uniquely defined C^∞ function and invertible;*
(f_2) $0 < f'(t) \leq 1$ *and* $|f(t)| \leq |t|$, $\forall\, t \in \mathbb{R}$;
(f_3) $\lim_{t \to 0} \frac{|f(t)|}{|t|} = 1$ *and* $\lim_{t \to \infty} \frac{|f(t)|^\alpha}{|t|} = \sqrt{\alpha}$;

(f_4) *there exists a positive constant C such that* $|f(t)|^{\alpha-1}f'(t) \le C, \ \forall\, t \in \mathbb{R}$;
(f_5) $f''(t)f(t) = (\alpha-1)(f'(t))^2\big((f'(t))^2-1\big), \ \forall\, t \in \mathbb{R}$.

Therefore, by a change of variables and (2), we obtain the following functional:

$$I_\theta(v) := J_\theta(f(v)) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega G(x, f(v))dx \\ - \theta \int_\Omega H(x, f(v))dx, \quad (\theta, v) \in \mathbb{R} \times E.$$

Under suitable hypotheses on g and h, for fixed $\theta_0 \in \mathbb{R}$, $I_{\theta_0} \in C^1(E, \mathbb{R})$ and

$$\langle I'_{\theta_0}(v), w\rangle = \int_\Omega \nabla v \nabla w dx - \int_\Omega g(x, f(v))f'(v)w dx - \theta_0 \int_\Omega h(x, f(v))f'(v)w dx$$

for any v, $w \in E$. Moreover, the critical points of I_{θ_0} are the weak solutions of the following problem:

$$\begin{cases} -\Delta v = \big(1 + \alpha|f(v)|^{2(\alpha-1)}\big)^{-\frac{1}{2}}\big(g(x, f(v)) + \theta_0 h(x, f(v))\big), & x \in \Omega, \\ v = 0, \quad x \in \partial\Omega. \end{cases}$$

Arguing similarly as in the proof of Lemma 2.6 and Remark 2.7 in [**1**], if $v_0 \in E$ is a critical point of the functional I_{θ_0}, then $u_0 = f(v_0) \in E$ is a weak solution of problem (1.1) with $\theta = \theta_0$.

Since we only know $h \in C(\bar{\Omega} \times \mathbb{R})$, we cannot apply the variational methods to I_θ directly. To overcome this difficulty, we use several cut-off functions to introduce some truncated functionals, then we seek multiple critical points of these truncated functionals, and finally we can prove that the critical points of these truncated functionals are also critical points of I_θ that yield multiple solutions for problem (1).

For any $k \in \mathbb{N}$, we introduce cut-off functions $\zeta_k \in C(\mathbb{R}, \mathbb{R})$ satisfying

$$\begin{cases} \zeta_k(t) = 1, & |t| \le k, \\ 0 \le \zeta_k(t) \le 1, & k < |t| < k+1, \\ \zeta_k(t) = 0, & |t| \ge k+1. \end{cases} \tag{3}$$

By the use of these cut-off functions, define

$$h_k(x, t) := \zeta_k(t)h(x, t), \quad (x, t) \in \Omega \times \mathbb{R}, \tag{4}$$

and $H_k(x, t) := \int_0^t h_k(x, s)ds$. First, we introduce the functionals

$$I_{\theta_k}(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G(x, f(v))dx - \theta \int_\Omega H_k(x, f(v))dx. \tag{5}$$

By (g_1), (3) and (4), for any $(\theta, k) \in \mathbb{R} \times \mathbb{N}$, I_{θ_k} is well defined on E. Moreover, for any $(\theta, k) \in \mathbb{R} \times \mathbb{N}$, I_{θ_k} is of class $C^1(E, \mathbb{R})$ with its derivative given by

$$\langle I'_{\theta_k}(v), w\rangle = \int_\Omega \nabla v \nabla w dx - \int_\Omega g(x, f(v))f'(v)w dx - \theta \int_\Omega h_k(x, f(v))f'(v)w dx \tag{6}$$

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for any $v,\ w \in E$. Next, we define a functional $I_0 : E \to \mathbb{R}$ given by

$$I_0(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G(x, f(v))dx, \quad v \in E. \tag{7}$$

Under assumption (g_1), I_0 is of class $C^1(E, \mathbb{R})$ and its derivative is

$$\langle I_0'(v), w\rangle = \int_\Omega \nabla v \nabla w dx - \int_\Omega g(x, f(v))f'(v)w dx, \quad \forall\ v,\ w \in E.$$

LEMMA 2.2. *Suppose that* (g_1) *and* (g_2) *are satisfied. Then,*
(H_1) *for every* $(\theta, k) \in \mathbb{R} \times \mathbb{N}$, I_{θ_k} *satisfies the Palais–Smale condition;*
(H_2) *for any* $(\theta, k) \in \mathbb{R} \times \mathbb{N}$, *there exists a positive constant* C_k *depending on* k *such that*

$$|I_{\theta_k}(v) - I_0(v)| \le C_k|\theta|, \quad \forall\ v \in E. \tag{8}$$

Proof. For any $(\theta, k) \in \mathbb{R} \times \mathbb{N}$, we show that I_{θ_k} satisfies the Palais–Smale condition. Assume that $\{v_n\}_{n\in\mathbb{N}} \subset E$ is a (PS) sequence, i.e.,

$$|I_{\theta_k}(v_n)| \le M \ \text{ and } \ I'_{\theta_k}(v_n) \to 0, \quad n \to \infty, \tag{9}$$

where M is a positive constant. Next, we need to prove that $\{v_n\}$ has a convergent subsequence. First, we show that $\{v_n\}$ is bounded. By (f_3) in Lemma 2.1, there exist positive constants M_0 and C_2 such that

$$|f(t)| \le C_2|t|^{1/\alpha}, \quad |t| \ge M_0. \tag{10}$$

For any $v \in E$, it follows from (f_2) and (10) that

$$\begin{aligned}\int_\Omega |f(v)|^{\alpha_1} dx &= \int_{\Omega_0} |f(v)|^{\alpha_1} dx + \int_{\Omega\setminus\Omega_0} |f(v)|^{\alpha_1} dx \\ &\le C_2 \int_{\Omega_0} |v|^{\alpha_1/\alpha} dx + \int_{\Omega\setminus\Omega_0} |v|^{\alpha_1} dx \\ &\le C_2 \int_\Omega |v|^{\alpha_1/\alpha} dx + M_0^{\alpha_1}\text{meas}(\Omega),\end{aligned} \tag{11}$$

where $\Omega_0 := \{x \in \Omega : |v(x)| \ge M_0\}$. By Hölder's inequality, (g_2) and (11), there exists a positive constant C_3 such that

$$\int_\Omega \big|\mu G(x, f(v)) - g(x, f(v))f(v)\big| dx \le C_3\big(\|v\|^{\alpha_1/\alpha} + 1\big), \quad \forall\ v \in E. \tag{12}$$

Set $\psi = \frac{f(v)}{f'(v)}$, $\forall\ v \in E$. Then, by (f_5) in Lemma 2.1 and direct computation, there exists a positive constant C_4 independent of v such that

$$\|\psi\| \le C_4\|v\|, \quad \forall\ v \in E. \tag{13}$$

In view of (3) and (4), there exists a constant $C_k > 0$ depending on k such that

$$\Big|\int_\Omega H_k(x, f(v))dx\Big| \le C_k, \quad \Big|\int_\Omega h_k(x, f(v))f(v)dx\Big| \le C_k, \quad v \in E. \tag{14}$$

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By (5), (6) and (14),

$$I_{\theta_k}(v_n) - \frac{1}{\mu}\Big\langle I'_{\theta_k}(v_n), \frac{f(v_n)}{f'(v_n)}\Big\rangle \geq \frac{\mu-2\alpha}{2\mu}\|v_n\|^2 + \frac{\alpha-1}{\mu}\int_\Omega (f'(v_n))^2|\nabla v_n|^2dx - C_3\big(\|v_n\|^{\alpha_1/\alpha}+1\big) - 2C_k|\theta|. \quad (15)$$

In combination with (9), (13) and (15), $\{v_n\}$ is bounded in E, that is, there exists a constant $A>0$ such that $\|v_n\| \leq A$, $n \in \mathbb{N}$. Since E is a reflexive space, passing to a subsequence, also denoted by $\{v_n\}$, it can be assumed that $v_n \rightharpoonup v_0$, $n \to \infty$. By the fact that E is compactly embedded into $L^\nu(\Omega)$ for any $\nu \in [1, 2^*)$, up to a subsequence, also denoted by $\{v_n\}$,

$$v_n \to v_0 \ \text{ in } L^\nu(\Omega), \quad (16)$$

as $n \to \infty$ for any $\nu \in [1, 2^*)$.

For any $v, w \in E$, by Hölder's inequality, (10), (f_2) and (f_4) in Lemma 2.1,

$$\begin{aligned}\int_\Omega |f(v)|^{p-1}f'(v)|w|dx &= \int_{\Omega_0} |f(v)|^{p-1}f'(v)|w|dx + \int_{\Omega\setminus\Omega_0} |f(v)|^{p-1}f'(v)|w|dx \\ &\leq CC_2\int_{\Omega_0} |v|^{\frac{p-\alpha}{\alpha}}|w|dx + \int_{\Omega\setminus\Omega_0}|v|^{p-1}|w|dx \\ &\leq CC_2\|v\|_{\frac{p}{\alpha}}^{\frac{p-\alpha}{\alpha}}\|w\|_{\frac{p}{\alpha}} + M_0^{p-1}\|w\|_1, \end{aligned} \quad (17)$$

where $\Omega_0 := \{x \in \Omega : |v(x)| \geq M_0\}$. In view of (g_1), (16), (17) and (f_2) in Lemma 2.1,

$$\begin{aligned}\left|\int_\Omega g\big(x, f(v_n)\big)f'(v_n)(v_n - v_0)dx\right| &\leq C_0\int_\Omega (1+|f(v_n)|^{p-1})f'(v_n)|v_n - v_0|dx \\ &\leq C_0\big(CC_2\|v_n\|_{\frac{p}{\alpha}}^{\frac{p-\alpha}{\alpha}}\|v_n - v_0\|_{\frac{p}{\alpha}} \\ &\quad + (M_0^{p-1}+1)\|v_n - v_0\|_1\big) \\ &\leq o_n(1). \end{aligned} \quad (18)$$

Similarly, we also have

$$\left|\int_\Omega g\big(x, f(v_0)\big)f'(v_0)(v_n - v_0)dx\right| \leq o_n(1). \quad (19)$$

In combination with (18) and (19),

$$\left|\int_\Omega \big[g\big(x, f(v_n)\big)f'(v_n) - g\big(x, f(v_0)\big)f'(v_0)\big](v_n - v_0)dx\right| \to 0, \quad n \to \infty. \quad (20)$$

By (4), (16) and (f_2) in Lemma 2.1,

$$\left|\int_\Omega h_k\big(x, f(v_n)\big)f'(v_n)(v_n - v_0)dx\right| \to 0, \quad n \to \infty. \quad (21)$$

On the other hand, by the fact that $v_n \rightharpoonup v_0$ and (9),

$$\big|\langle I'_{\theta_k}(v_n) - I'_0(v_0), v_n - v_0\rangle\big| := \epsilon_n \to 0, \ \ n \to \infty. \quad (22)$$

In view of (5), (6) and (7),

$$\|v_n - v_0\|^2 \leq \left| \int_\Omega \left[g\big(x, f(v_n)\big)f'(v_n) - g\big(x, f(v_0)\big)f'(v_0)\right](v_n - v_0)dx \right| + |\theta| \left| \int_\Omega h_k\big(x, f(v_n)\big)f'(v_n)(v_n - v_0)dx \right| + \epsilon_n,$$

which implies that $v_n \to v_0$ in E by (20), (21) and (22). Hence, I_{θ_k} satisfies Palais–Smale condition.

To prove (H_2), in view of (5), (7) and (14), (8) holds. □

LEMMA 2.3. *There exists a normalized orthogonal sequence* $\{\phi_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$ *such that supp* $\phi_n \subset \Lambda$, $n \in \mathbb{N}$, *where* Λ *is the nonempty open set given in* (g_3).

Proof. Since Λ is a nonempty open set, there exist a point $x_0 \in \Lambda$ and $\delta_0 > 0$ such that $B(x_0, \delta_0) \subset \Lambda$, where $B(x_0, \rho)$ denotes the open ball of radius ρ centred at x_0, and $\bar{B}$ denotes the closure in $\mathbb{R}^N$. Choose a strictly increasing sequence $\{r_n\}_{n=1}^{\infty}$ such that $0 < r_1 < r_2 < \cdots < r_n < \cdots \to \delta_0/4$. Define $O_n = B(x_0, r_{n+1}) \backslash \bar{B}(x_0, r_n)$, $n \in \mathbb{N}$. Let $x_n \in O_n$ and choose $d_n > 0$ such that

$$\bar{B}(x_n, d_n) \subset O_n, \quad n \in \mathbb{N}. \tag{23}$$

Define

$$\phi(x) = \begin{cases} e^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{24}$$

In view of (24), define ϕ_n as follows:

$$\phi_n(x) = \phi\big((x - x_n)/d_n\big), \quad n \in \mathbb{N}. \tag{25}$$

In combination with (24) and (25), $\phi_n \in C_0^{\infty}(\Omega)$, $n \in \mathbb{N}$. Replace ϕ_n by $\|\phi_n\|^{-1}\phi_n$, also denoted by ϕ_n, and then $\|\phi_n\| = 1$. By (23) and (25), supp $\phi_n \subset O_n \subset \Lambda$, then the supports of ϕ_n are disjoint to each other, so $(\phi_i, \phi_j) = \delta_{ij}$, where δ_{ij} stands for Kronecker's symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. So, $\{\phi_n\}_{n=1}^{\infty}$ forms a normalized orthogonal sequence in E. □

Let $D_n = \text{span}\{\phi_1, \ldots, \phi_n\}$, $n \in \mathbb{N}$. It is obvious that D_n is a finite dimensional subspace in E. Next, we prove that there exists a strictly increasing sequence of numbers R_n such that

$$I_0(v) \leq 0, \quad v \in D_n \backslash B_{R_n}, \tag{26}$$

where B_{R_n} denotes the open ball of radius R_n centred at 0 in E, and $\bar{B}_{R_n}$ denotes the closure of B_{R_n} in E.

LEMMA 2.4. *Under assumption* (g_3)*, for any finite dimensional subspace* $D_n \subset E$*,*

$$I_0(v) \to -\infty, \quad \|v\| \to \infty, \quad v \in D_n. \tag{27}$$

Proof. We prove (27) by contradiction. If not, there exists a sequence $\{v_m\} \subset D_n$ with $\|v_m\| \to \infty$, there exists $M_1 > 0$ such that $I_0(v_m) \geq -M_1$ for all $m \in \mathbb{N}$. Set $w_m =$

$v_m/\|v_m\|$, then $\|w_m\| = 1$. Passing to subsequence, we may assume $w_m \rightharpoonup w$ in E. Since D_n is a finite dimensional space, $w_m \to w \in D_n$ and $\|w\| = 1$. Set $\Pi = \{x \in \Omega : w(x) \neq 0\}$. Since $\|w\| = 1$, meas$(\Pi) > 0$. Moreover, by Lemma 2.3, $\Pi \subset \Lambda$ and

$$\lim_{m\to\infty} |v_m(x)| = \infty, \quad \text{a.e. } x \in \Pi. \tag{28}$$

For $0 \leq a < b$, let $\Omega_m(a, b) = \left\{x \in \Lambda : a \leq \left|f\left(v_m(x)\right)\right| < b\right\}$. By (f_3) in Lemma 2.1 and (28), $\Pi \subset \Omega_m(r_0, \infty)$ for large $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} |f\left(v_m(x)\right)| = \infty, \quad \text{a.e. } x \in \Pi. \tag{29}$$

In view of Lemma 2.3, (g_1), (29) and Fatou's Lemma,

$$\begin{aligned}
0 &\leq \limsup_{m\to\infty} \frac{I_0(v_m)}{\|v_m\|^2} = \limsup_{m\to\infty} \left[\frac{1}{2} - \int_\Omega \frac{G(x, f(v_m))}{\|v_m\|^2} dx\right] \\
&= \limsup_{m\to\infty} \left[\frac{1}{2} - \int_\Lambda \frac{G(x, f(v_m))}{\|v_m\|^2} dx\right] \\
&= \limsup_{m\to\infty} \left[\frac{1}{2} - \int_{\Omega_m(0,r_0)} \frac{G(x, f(v_m))}{\|v_m\|^2} dx - \int_{\Omega_m(r_0,\infty)} \frac{G(x, f(v_m))}{|v_m|^2} |w_m|^2 dx\right] \\
&\leq \limsup_{m\to\infty} \left[\frac{1}{2} + C_0\left(r_0 + r_0^p\right)\text{meas}(\Lambda)\|v_m\|^{-2} - \int_{\Omega_m(r_0,\infty)} \frac{G(x, f(v_m))}{|v_m|^2} |w_m|^2 dx\right] \\
&\leq \frac{1}{2} - \liminf_{m\to\infty} \int_{\Omega_m(r_0,\infty)} \frac{G(x, f(v_m))}{f^{2\alpha}(v_m)} \cdot \frac{f^{2\alpha}(v_m)}{|v_m|^2} |w_m|^2 dx \\
&= \frac{1}{2} - \liminf_{m\to\infty} \int_\Lambda \frac{G(x, f(v_m))}{f^{2\alpha}(v_m)} \cdot \frac{f^{2\alpha}(v_m)}{|v_m|^2} |w_m|^2 \left[\chi_{\Omega_m(r_0,\infty)}(x)\right] dx \\
&\leq \frac{1}{2} - \int_\Lambda \liminf_{m\to\infty} \frac{G(x, f(v_m))}{f^{2\alpha}(v_m)} \cdot \frac{f^{2\alpha}(v_m)}{|v_m|^2} |w_m|^2 \left[\chi_{\Omega_m(r_0,\infty)}(x)\right] dx \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Thus, (27) holds. □

3. Proofs of the main results. First, we introduce some continuous maps in E to construct a sequence of minimax values of I_0. Set

$$\Gamma_n = \{h \in C(F_n, E) |\ h \text{ is odd and } h = \text{id on } \partial B_{R_n} \cap D_n\}, \tag{30}$$

where $F_n := \bar{B}_{R_n} \cap D_n$. By (30), we define a sequence of minimax values:

$$b_n = \inf_{h\in\Gamma_n} \max_{v\in F_n} I_0(h(v)). \tag{31}$$

Since E is a speratable Hilbert space, there exists a total orthonormal basis $\{e_j\}$ of E. Define $X_j = \mathbb{R}e_j, j \in \mathbb{N}$ and

$$Y_n = \oplus_{j=1}^n X_j, \quad Z_n = \overline{\oplus_{j=n+1}^\infty X_j}, \quad n \in \mathbb{N}. \tag{32}$$

By (32), it is clear that $E = Y_n \oplus Z_n$ and $Z_n = Y_n^\perp$, $n \in \mathbb{N}$.

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In order to get the lower bound of the minimax values b_n, we give an intersection property which has been essentially proved by Rabinowitz in Proposition 9.23 of [**19**].

LEMMA 3.1. *Let $\rho > 0$. For any $n \in \mathbb{N}$, $\rho < R_n$ and $h \in \Gamma_n$, $h(F_n) \cap \partial B_\rho \cap Z_{n-1} \neq \emptyset$.*

LEMMA 3.2. *Assume that (g_1) holds. Then,*

$$b_n \to \infty, \quad n \to \infty. \tag{33}$$

Proof. By Lemma 3.1, for any $h \in \Gamma_n$ and $\rho < R_n$ there exists $v_n \in h(F_n) \cap \partial B_\rho \cap Z_{n-1}$, such that

$$\max_{v \in F_n} I_0(h(v)) \geq I_0(v_n) \geq \inf_{v \in \partial B_\rho \cap Z_{n-1}} I_0(v). \tag{34}$$

In view of (g_1) and (f_3) in Lemma 2.1, there exists a positive constant C_5 such that

$$\int_\Omega \left|G(x, f(v))\right| dx \leq C_5\left(\|v\|_{\frac{p}{\alpha}}^{\frac{p}{\alpha}} + 1\right), \quad v \in E. \tag{35}$$

By a similar proof in Lemma 3.8 in [**24**],

$$\beta_n := \sup_{v \in Z_n,\ \|v\|=1} \|v\|_{\frac{p}{\alpha}} \to 0, \quad n \to \infty. \tag{36}$$

In combination with (7), (35) and (36), for $v \in Z_{n-1}$,

$$I_0(v) \geq \frac{\|v\|^2}{2} - C_5\big(\beta_{n-1}^{\frac{p}{\alpha}} \|v\|^{\frac{p}{\alpha}} + 1\big). \tag{37}$$

By (37), if $v \in \partial B_\rho \cap Z_{n-1}$,

$$I_0(v) \geq \rho^2\left(\frac{1}{2} - C_5 \beta_{n-1}^{\frac{p}{\alpha}} \rho^{\frac{p-2\alpha}{\alpha}}\right) - C_5. \tag{38}$$

In view of (38), choose $\rho_n := (4C_5\beta_{n-1}^{\frac{p}{\alpha}})^{\frac{\alpha}{2\alpha-p}}$, when $v \in \partial B_{\rho_n} \cap Z_{n-1}$,

$$I_0(v) \geq \frac{1}{4}\rho_n^2 - C_5. \tag{39}$$

By (31), (34), (36) and (39), (33) holds. □

Next, we introduce some continuous maps in E. Set

$$\begin{aligned} \Lambda_n := \big\{ & H \in C(U_n, E) |\ H|_{F_n} \in \Gamma_n \text{ and } H = \text{id for} \\ & v \in Q_n := (\partial B_{R_{n+1}} \cap D_{n+1}) \cup \big((B_{R_{n+1}} \backslash \bar{B}_{R_n}) \cap D_n\big)\big\}, \end{aligned} \tag{40}$$

where

$$U_n := \Big\{ v = t\phi_{n+1} + \omega |\ t \in [0, R_{n+1}],\ \omega \in \bar{B}_{R_{n+1}} \cap D_n,\ \|v\| \leq R_{n+1} \Big\}. \tag{41}$$

In view of Lemma 3.2, it is impossible that $b_{n+1} \leq b_n$ for all large n. Next, we can construct critical values of I_{θ_k} as follows.

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LEMMA 3.3. *Let n be a positive integer satisfying $b_{n+1} > b_n > 0$. For any $\delta \in (0, b_{n+1} - b_n)$, define*

$$\Lambda_n(\delta) = \left\{H \in \Lambda_n \mid I_0(H(v)) \leq b_n + \delta \text{ for } v \in F_n\right\}. \tag{42}$$

For any $k \in \mathbb{N}$ and $|\theta| < 2C_k^{-1}(b_{n+1} - b_n - \delta)$, where C_k is given in Lemma 2.2, define

$$c_n(\theta) = \inf_{H \in \Lambda_n(\delta)} \max_{v \in U_n} I_{\theta_k}(H(v)). \tag{43}$$

Then, $c_n(\theta)$ is a critical value of I_{θ_k}.

Proof. By (H_2) in Lemma 2.2, we have

$$I_0(v) - C_k|\theta| \leq I_{\theta_k}(v) \leq I_0(v) + C_k|\theta|, \quad \forall\, v \in E. \tag{44}$$

For any $H \in \Lambda_n(\delta)$, since $F_{n+1} = U_n \cup (-U_n)$, H can be continuously extended to F_{n+1} as an odd function $\bar{H}$. Moreover, $\bar{H} \in \Gamma_{n+1}$. Since I_0 is even, by the construction of $\bar{H}$,

$$\max_{v \in U_n} I_0(H(v)) = \max_{v \in F_{n+1}} I_0(\bar{H}(v)). \tag{45}$$

In combination with (31), (44) and (45),

$$\begin{aligned} \max_{v \in U_n} I_{\theta_k}(H(v)) &\geq \max_{v \in U_n} I_0(H(v)) - C_k|\theta| \\ &= \max_{v \in F_{n+1}} I_0(\bar{H}(v)) - C_k|\theta| \\ &\geq b_{n+1} - C_k|\theta|. \end{aligned} \tag{46}$$

Since H is an arbitrary map in $\Lambda_n(\delta)$, by (43) and (46),

$$c_n(\theta) \geq b_{n+1} - C_k|\theta| > b_n + \delta + C_k|\theta|. \tag{47}$$

If we choose $H_n \in \Lambda_n(\delta)$, then H_n can be continuously extended to F_{n+1} as an odd function $\bar{H}_n$. Moreover, $\bar{H}_n \in \Gamma_{n+1}$. Define

$$c_n = \max_{v \in U_n} I_0(H_n(v)). \tag{48}$$

It is obvious that $c_n < +\infty$ and c_n is independent of θ and k. By (31) and (48),

$$c_n = \max_{v \in U_n} I_0(H_n(v)) = \max_{v \in F_{n+1}} I_0(\bar{H}_n(v)) \geq b_{n+1}. \tag{49}$$

Moreover, by (43), (44) and (48),

$$c_n(\theta) \leq c_n + C_k|\theta|. \tag{50}$$

Next, we show that $c_n(\theta)$ is a critical value of I_{θ_k}. If $c_n(\theta)$ is a regular value of I_{θ_k}, define

$$\bar{\varepsilon} = (c_n(\theta) - b_n - \delta - C_k|\theta|)/2. \tag{51}$$

In view of (47), $\bar{\varepsilon} > 0$. By (H_1) in Lemma 2.2 and the Deformation Theorem (see **[19]**), there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0, 1] \times E, E)$ such that

$$\eta(1, v) = v, \quad \text{if } \; I_{\theta_k}(v) \notin [c_n(\theta) - \bar{\varepsilon}, c_n(\theta) + \bar{\varepsilon}], \tag{52}$$

and if $I_{\theta_k}(v) \le c_n(\theta) + \varepsilon$, then

$$I_{\theta_k}(\eta(1, v)) \le c_n(\theta) - \varepsilon. \tag{53}$$

By (43), there exists $H_0 \in \Lambda_n(\delta)$ such that

$$\max_{v \in U_n} I_{\theta_k}(H_0(v)) < c_n(\theta) + \varepsilon. \tag{54}$$

Define

$$\bar{H}_0(\cdot) = \eta(1, H_0(\cdot)). \tag{55}$$

Next, we prove $\bar{H}_0 \in \Lambda_n(\delta)$. It is obvious that $\bar{H}_0 \in C(U_n, E)$. In view of $H_0 \in \Lambda_n(\delta)$, (42), (44) and (51),

$$I_{\theta_k}(H_0(v)) \le I_0(H_0(v)) + C_k|\theta| \le b_n + \delta + C_k|\theta| < c_n(\theta) - \bar{\varepsilon}, \quad v \in F_n. \tag{56}$$

In combination with (52), (55) and (56), $\bar{H}_0(v) = \eta(1, H_0(v)) = H_0(v)$, $v \in F_n$, which yields that

$$\bar{H}_0|_{F_n} \in \Gamma_n \ \text{ and } \ I_0(\bar{H}_0(v)) = I_0(H_0(v)) \le b_n + \delta, \quad v \in F_n. \tag{57}$$

In view of $H_0 \in \Lambda_n(\delta)$ and the definitions of R_n and R_{n+1}

$$H_0(v) = v \ \text{ and } \ I_0(H_0(v)) \le 0, \quad v \in Q_n. \tag{58}$$

By (44), (51) and (58), we have

$$I_{\theta_k}(H_0(v)) \le I_0(H_0(v)) + C_k|\theta| \le C_k|\theta| < c_n(\theta) - \bar{\varepsilon}, \quad v \in Q_n. \tag{59}$$

In combination with (52), (55) and (59),

$$\bar{H}_0(v) = \eta(1, H_0(v)) = H_0(v) = v, \quad v \in Q_n. \tag{60}$$

In view of (57) and (60), $\bar{H}_0 \in \Lambda_n(\delta)$. Moreover, by (53) and (54),

$$\max_{v \in U_n} I_{\theta_k}\big(\bar{H}_0(v)\big) = \max_{v \in U_n} I_{\theta_k}\big(\eta(1, H_0(v))\big) \le c_n(\theta) - \varepsilon,$$

which is a contradiction to (43). □

Proof of Theorem 1.1. For any $j \in \mathbb{N}$, choose strictly increasing integers p_i $(1 \le i \le j+1)$ such that

$$b_{p_i+1} > b_{p_i} > 0 \ \text{ and } \ b_{p_{(i+1)}} > c_{p_i}, \quad 1 \le i \le j. \tag{61}$$

By Lemma 3.3, for every $k \in \mathbb{N}$, there exists $\varepsilon_k' > 0$ such that if $|\theta| \le \varepsilon_k'$, then $c_{p_i}(\theta)$ $(1 \le i \le j)$ defined by (43) are critical values of I_{θ_k}. Moreover, in view of (47) and (50),

$$b_{p_i} - C_k|\theta| \le c_{p_i}(\theta) \le c_{p_i} + C_k|\theta|, \quad 1 \le i \le j. \tag{62}$$

By (61) and (62), for every $k \in \mathbb{N}$, choose $\varepsilon_k'' > 0$ such that if $|\theta| \le \varepsilon_k''$,

$$c_{p_i} + C_k|\theta| < b_{p_{(i+1)}} - C_k|\theta|, \quad c_{p_i}(\theta) \le b_{p_{(j+1)}} \tag{63}$$

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for $1 \le i \le j$. In view of (3) and (4), for every $k \in \mathbb{N}$, there exists $\varepsilon_k''' > 0$ such that if $|\theta| \le \varepsilon_k'''$,

$$|\theta||h_k(x,t)t| < 1, \quad |\theta||H_k(x,t)| < 1, \quad (x,t) \in \Omega \times \mathbb{R}. \tag{64}$$

For every $k \in \mathbb{N}$, define $\varepsilon_k = \min\{\varepsilon_k', \varepsilon_k'', \varepsilon_k'''\}$. By (62) and (63), for every $k \in \mathbb{N}$, $|\theta| \le \varepsilon_k$, I_{θ_k} has at least j distinct critical values $c_{p_1}(\theta), c_{p_2}(\theta), \ldots, c_{p_j}(\theta)$ such that

$$c_{p_1}(\theta) < c_{p_2}(\theta) < \cdots < c_{p_j}(\theta) \le b_{p_{(j+1)}}. \tag{65}$$

By (65), for every $k \in \mathbb{N}$, $|\theta| \le \varepsilon_k$, I_{θ_k} has at least j distinct critical points $v_i(\theta)$, $1 \le i \le j$. By (5) and (6), there are j distinct critical points $v_i(\theta)$ $(1 \le i \le j)$ of I_{θ_k} such that

$$c_{p_i}(\theta) = \frac{1}{2}\|v_i(\theta)\|^2 - \int_\Omega G(x, f(v_i(\theta)))dx - \theta \int_\Omega H_k(x, f(v_i(\theta)))dx, \tag{66}$$

and

$$\begin{aligned}\alpha\|v_i(\theta)\|^2 &= (\alpha-1)\int_\Omega (f'(v_i(\theta)))^2|\nabla v_i(\theta)|^2dx - \int_\Omega g(x, f(v_i(\theta)))f(v_i(\theta))dx \\ &\quad - \theta\int_\Omega h_k(x, f(v_i(\theta)))f(v_i(\theta))dx. \end{aligned}\tag{67}$$

By (12) and (63)–(67),

$$b_{p_{(j+1)}} \ge c_{p_i}(\theta) \ge \frac{\mu - 2\alpha}{2\mu}\|v_i(\theta)\|^2 - C_3\big(\|v_i(\theta)\|^{\alpha_1/\alpha} + 1\big) - 2\mathrm{meas}(\Omega) \tag{68}$$

for $1 \le i \le j$. In view of (g_1), (g_2) and (68), there exists a positive constant C_j only depending on j such that $\|v_i(\theta)\| \le C_j$, $1 \le i \le j$. By classical elliptic theory, there exists a positive constant C_j' only depending on j such that for every $k \in \mathbb{N}$, $|\theta| \le \varepsilon_k$, $\|v_i(\theta)\|_{C(\bar\Omega)} \le C_j'$, $1 \le i \le j$. So, we can choose $k > C_j'$, for any θ with $|\theta| \le \varepsilon_k$, I_{θ_k} has at least j distinct critical points $v_1(\theta), v_2(\theta), \ldots, v_j(\theta)$ and $\|v_i(\theta)\|_{C(\bar\Omega)} \le C_j'$, $1 \le i \le j$. Moreover, by (f_2) in Lemma 2.1, for any θ with $|\theta| \le \varepsilon_k$,

$$\|f\big(v_i(\theta)\big)\|_{C(\bar\Omega)} \le C_j', \quad 1 \le i \le j. \tag{69}$$

Since $k > C_j'$, by (3), (4), (6) and (69), for any θ with $|\theta| \le \varepsilon_k$, $v_1(\theta), v_2(\theta), \ldots, v_j(\theta)$ are also j distinct critical points of I_θ. So, for any θ with $|\theta| \le \varepsilon_k$, problem (1.1) has at least j distinct solutions. □

Proof of Theorem 1.2. If $\theta = 0$, by Deformation Theorem and Lemma 3.2, we can prove that $\{b_n\}$ is a sequence of critical values of I_0 which converge to $+\infty$. Hence, the corresponding critical points are solutions of problem (1.1) with $\theta = 0$. □

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