Basics on Categories

In this monograph some basic concepts of the theory of categories are used frequently. For the convenience of the reader, we recall them here. At some places later in the text, however, the requirements are higher than those we provide in this chapter. In each case we give a precise reference to the literature where the reader can find the full information needed.

2.1 Additive and Abelian Categories

Definition 2.1. A category $\mathcal{C}$ consists of a class of objects $\text{Ob}(\mathcal{C})$ and for every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$ a set of morphisms $\text{Mor}_\mathcal{C}(X,Y)$ together with a composition law,

$$\circ : \text{Mor}_\mathcal{C}(X,Y) \times \text{Mor}_\mathcal{C}(Y,Z) \to \text{Mor}_\mathcal{C}(X,Z),$$

$$(f, g) \mapsto g \circ f,$$

for every triple of objects $X,Y,Z$. These data are subject to the following conditions:

1. The composition law is associative.
2. For every object $X$ there is a morphism $\text{id}_X \in \text{Mor}_\mathcal{C}(X,X)$ which is a left and right identity for the composition law, i.e. $f \circ \text{id}_X = f$ for all $Y$ and all $f : X \to Y$ and $\text{id}_X \circ g = g$ for all $Z$ and all $g : Z \to X$.

A category is $\mathbb{Z}$-linear (or $\mathbb{Q}$-linear) if the morphism sets are given the structure of abelian groups (or $\mathbb{Q}$-vector spaces) and the composition of morphisms is bilinear. We usually write $\text{Hom}_\mathcal{C}$ instead of $\text{Mor}_\mathcal{C}$ in this case. It is additive if it is $\mathbb{Z}$-linear and, in addition, has finite direct sums. The direct sum of two objects $X$ and $Y$ is denoted $X \oplus Y$. As the particular case of the
empty direct sum, this also requires the existence of a 0-object characterised uniquely by the property that
\[ \text{Hom}(0, X) = \text{Hom}(X, 0) = 0 \]
for all objects \( X \).

An additive category \( \mathcal{A} \) is called **abelian** if the following two properties are satisfied:

1. Every morphism \( f: X \rightarrow Y \) in \( \mathcal{A} \) has a kernel and cokernel.
2. For every morphism \( f: X \rightarrow Y \) in \( \mathcal{A} \) the natural map \( X/\ker(f) \rightarrow \text{im}(f) \) is an isomorphism.

In this abstract setting, the image of a morphism is defined as the kernel of the cokernel.

**Example 2.2.** For every ring \( R \), the category of \( R \)-modules is abelian. The category of finitely generated modules is additive. It is abelian if \( R \) is noetherian.

Many examples of additive and abelian categories are going to be used throughout the book. In order of appearance:

**Examples 2.3.**
1. Let \( k \) be a field. Then the category of connected commutative algebraic groups schemes over \( k \) is additive, but not abelian. We refer the reader to Chapter 4 for details.
2. Let \( K, L \subset \mathbb{C} \) be subfields. Then the category \( (K, L)\text{-Vect} \) introduced in Section 7.2 is \( \mathbb{Q} \)-linear and abelian.
3. Let \( k \) be a field. Then the category of filtered \( k \)-vector spaces is additive but not abelian. Every morphism has a kernel and a cokernel, but the isomorphism between \( X/\ker(f) \) and \( \text{im}(f) \) fails in general.
4. Let \( k \) be an algebraically closed field of characteristic 0. The category \( 1\text{-Mot}_k \) of iso-1-motives is abelian. A thorough review is given in Chapter 8.

Given a \( \mathbb{Z} \)-linear category \( \mathcal{A} \), we obtain a \( \mathbb{Q} \)-linear category \( \mathcal{A} \otimes \mathbb{Q} \) with the same objects as \( \mathcal{A} \) and morphism
\[ \text{Hom}_{\mathcal{A} \otimes \mathbb{Q}}(X, Y) = \text{Hom}_{\mathcal{A}}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}. \]
We refer to it as the **isogeny category of** \( \mathcal{A} \). If \( \mathcal{A} \) is additive or abelian, then so is \( \mathcal{A} \otimes \mathbb{Q} \).

**Examples 2.4.**
1. Let \( k \) be a field. If \( \mathcal{A} \) is the category of abelian varieties over \( k \), then \( \mathcal{A} \otimes \mathbb{Q} \) is what is referred to as the **category of abelian varieties up to isogeny** in the literature. It is abelian. The same remark also applies to the category of connected commutative group schemes.
2. By Definition 8.1, 1-Mot_k is defined as the isogeny category of the category of 1-motives over k.

For any category C, we define the additive hull \( \mathbb{Z}[C] \), where the objects are formal direct sums \( \bigoplus_{i=1}^{n} X_i \) for \( n \geq 0 \) and \( X_1, \ldots, X_n \in C \). We interpret the empty direct sum as an object 0. Morphisms are defined by the formula

\[
\text{Hom}_{\mathbb{Z}[C]} \left( \bigoplus_{i=1}^{n} X_i, \bigoplus_{j=1}^{m} Y_j \right) = \bigoplus_{i,j} \mathbb{Z}[\text{Hom}_C(X_i, Y_j)].
\]

Here for a set \( S \), we denote by \( \mathbb{Z}[S] \) the free abelian group with basis \( S \).

### 2.2 Subcategories

Given a category \( C \), a subcategory of \( C \) is a category \( C' \) such that every object and every morphism of \( C' \) is an object and morphism in \( C \), respectively. The composition of morphisms in \( C' \) is defined as their composition in \( C \) and the identity morphisms in \( C' \) agree with the identity morphisms in \( C \). A subcategory is called full if

\[
\text{Mor}_{C'}(X, Y) = \text{Mor}_C(X, Y)
\]

for all objects \( X, Y \) of \( C' \).

**Remark 2.5.** If \( C \) is additive or abelian, then a subcategory \( C' \) is not necessarily additive or abelian itself. If \( C \) and \( C' \) are both abelian, this does not imply that the kernels and cokernels are the same when computed in \( C \) or \( C' \). We are not going to consider such pathological situations, which only appear if the subcategory is not full.

**Example 2.6.** The category of abelian varieties (up to isogeny) is a full subcategory of the category of connected commutative algebraic groups (up to isogeny).

**Example 2.7.** The category of \( \mathbb{Q} \)-vector spaces is a full subcategory of the category of abelian groups.

Let \( \mathcal{A} \) be an abelian category. A subquotient of an object \( X \) in \( \mathcal{A} \) is a quotient of a subobject of \( X \), or equivalently, a subobject of a quotient of \( X \).

**Definition 2.8.** Let \( \mathcal{A} \) be an abelian category and \( X \in \mathcal{A} \) an object. We define \( \langle X \rangle \) as the smallest full subcategory closed under subquotients containing \( X \).

More explicitly, this means that \( \langle X \rangle \) contains \( X \) and all the quotients and subobjects of every object \( Y \) of \( \langle X \rangle \).
Lemma 2.9. The category $\langle X \rangle$ is abelian. We have

$$\mathcal{A} = \bigcup_{X \in \mathcal{C}} \langle X \rangle.$$  

Proof. Let $f: Y \to Z$ be a morphism in $\langle X \rangle$. Then $\ker(f) \subset Y$ exists in $\mathcal{A}$. As a subobject of an object in $\langle X \rangle$, it is itself an object of $\langle X \rangle$. The universal property of a kernel holds because it holds in $\mathcal{A}$. The same argument gives the existence of cokernels. The natural map $Y/\ker(f) \to \im(f)$ has an inverse in $\mathcal{A}$ because the category is abelian. This inverse is in $\langle X \rangle$ because the subcategory is full.

Obviously all objects of $\mathcal{A}$ are contained in the union of all $\langle X \rangle$. We have to check that the same is true for morphisms. Let $f: X \to Y$ be a morphism in $\mathcal{A}$. Both $X$ and $Y$ are subobjects of $X \oplus Y$, hence they are both objects of $\langle X \oplus Y \rangle$. As $\langle X \oplus Y \rangle \subset \mathcal{A}$ is a full subcategory, the morphism $f$ is a morphism in $\langle X \oplus Y \rangle$. $\square$

2.3 Functors

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A covariant functor $F: \mathcal{C} \to \mathcal{C}'$ is an assignment $F: \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{C}')$ together with a map

$$\text{Mor}_\mathcal{C}(X, Y) \to \text{Mor}_{\mathcal{C}'}(F(X), F(Y))$$

for every pair of objects $X, Y$ of $\mathcal{C}$. It is subject to the following conditions.

1. Compatibility with composition: $F(g) \circ F(f) = F(g \circ f)$ for all objects $X, Y, Z$ in $\mathcal{C}$ and morphisms $f: X \to Y$, $g: Y \to Z$.

2. Compatibility with identities: $F(\text{id}_X) = \text{id}_{F(X)}$ for all objects $X$ of $\mathcal{C}$.

In the case of a contravariant functor, we are given maps

$$\text{Mor}_\mathcal{C}(X, Y) \to \text{Mor}_{\mathcal{C}'}(F(Y), F(X))$$

and the compatibility condition reads $F(f) \circ F(g) = F(g \circ f)$ instead.

A functor $F: \mathcal{C} \to \mathcal{C}'$ is called faithful, full or fully faithful if for all objects $X, Y \in \mathcal{C}$ the natural map

$$\text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_{\mathcal{C}'}(F(X), F(Y))$$

is injective, surjective or bijective, respectively.

Example 2.11. If $F$ is the inclusion of a subcategory of a category, then it is faithful. If the subcategory is full, the inclusion is fully faithful.
A functor $F: C \to C'$ between $\mathbb{Z}$-linear or $\mathbb{Q}$-linear categories is called \textit{additive} or $\mathbb{Q}$-linear if for all $X, Y \in C$ the map
\[ \text{Hom}_C(X, Y) \to \text{Hom}_{C'}(F(X), F(Y)) \]
is $\mathbb{Z}$-linear or $\mathbb{Q}$-linear, respectively. Such a functor automatically respects direct sums, provided that they exist (e.g. because the categories are additive).

An additive functor $F: \mathcal{A} \to \mathcal{A}'$ between abelian categories is called \textit{exact} if it sends short exact sequences to short exact sequences.

\textbf{Lemma 2.12.} Let $F: \mathcal{A} \to \mathcal{A}'$ be an exact functor between abelian categories. Then $F$ is faithful if and only if for all $X$ in $\mathcal{A}$ the assumption $F(X) = 0$ implies $X \cong 0$.

\textit{Proof} Assume that $F$ is faithful and that $X$ is an object of $\mathcal{A}$ such that $F(X) = 0$. Then this implies $F(0) = F(\text{id}_X)$. By faithfulness this gives $0 = \text{id}_X$ and hence $X \cong 0$.

Conversely assume the condition on objects. Let $f$ be in the kernel of the map $\text{Hom}_\mathcal{A}(X, Y) \to \text{Hom}_\mathcal{A}'(F(X), F(Y))$. The functor $F$ maps the exact sequence
\[ 0 \to \ker(f) \to X \xrightarrow{f} Y \to \text{coker}(f) \to 0 \]
to the exact sequence
\[ 0 \to F(\ker(f)) \to F(X) \xrightarrow{F(f) = 0} F(Y) \to F(\text{coker}(f)) \to 0. \]
This gives $F(\ker(f)) \cong F(X)$ and $F(Y) \cong F(\text{coker}(f))$. Now consider the short exact sequence
\[ 0 \to \ker(f) \to X \to X/\ker(f) \to 0 \]
and its image
\[ 0 \to F(\ker(f)) \to F(X) \to F(X/\ker(f)) \to 0. \]
We had established that the first map is an isomorphism, so $F(X/\ker(f)) \cong 0$. By assumption this implies that $X/\ker(f) \cong 0$ or $\ker(f) \cong X$. The same type of argument also shows that $Y \cong \text{coker}(f)$. Taken together this means that $f = 0$. \hfill $\square$

Faithful functors allow us to test for inclusions.

\textbf{Lemma 2.13.} Let $F: \mathcal{A} \to \mathcal{A}'$ be a faithful exact functor between abelian categories, $X \in \mathcal{A}$ an object and $X_1, X_2 \subseteq X$ subobjects. If $F(X_2) \subseteq F(X_1)$, then $X_2 \subseteq X_1$.
Proof  Let $X_3 = X_1 \cap X_2$ (or, more abstractly, let $X_3$ be the pull-back of $X_1 \to X$ via $X_2 \to X$). We need to show that the natural inclusion $X_3 \to X_2$ is an isomorphism, whence $X_2 \subset X_1$. By the exactness of $F$, we have $F(X_3) \cong F(X_1) \cap F(X_2)$. By assumption this is $F(X_2)$. We apply $F$ to the exact sequence

$$0 \to X_3 \to X_2 \to C \to 0.$$ 

As $F(X_3) = F(X_2)$, we get $F(C) = 0$. By the faithfulness of $F$, this implies $C \cong 0$. □

As a consequence of our results on transcendence and the Period Conjecture, we are also going to establish results on fullness of certain functors; see Proposition 8.17, Theorem 9.14 and Theorem 13.5. The following criterion will be useful.

**Lemma 2.14.** Let $F: A \to A'$ be a faithful exact functor between abelian categories. Assume that the image of $F$ is closed under subquotients, i.e. if

$$0 \to Y' \to F(X) \to Y'' \to 0$$

is an exact sequence in $A'$, then there is a short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $A$ mapping to the given exact sequence in $A'$. Then $F$ is full.

Proof  Let $f: F(Y_1) \to F(Y_2)$ be a morphism in $A'$ and $\Gamma \subset F(Y_1) \times F(Y_2)$ its graph. We find the graph as the image of $F(Y_1)$ under the map

$$F(Y_1) \xrightarrow{\Delta} F(Y_1) \times F(Y_1) \xrightarrow{(\text{id},f)} F(Y_1) \oplus F(Y_2).$$

It is a subobject. By assumption, there is $G \subset Y_1 \times Y_2$ in $A$ such that $F(G) = \Gamma$. The projection $p: G \to Y_1 \times Y_2 \to Y_1$ is an isomorphism because this is true for the image $\Gamma \to F(Y_1)$ and $F$ is faithful. Let $i$ be its inverse. The composition

$$Y_1 \xrightarrow{i} G \subset Y_1 \times Y_2 \to Y_2$$

is the preimage of $f$ we were looking for. □