REMARKS ON ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. A classical result of Laguerre says that if \( P \) is a polynomial of degree \( n \) such that \( P(z) \neq 0 \) for \( |z| < 1 \) then \( (\xi - z)P'(z) + nP(z) \neq 0 \) for \( |\xi| < 1 \) and \( |z| < 1 \). Rahman and Schmeisser have obtained an extension of that result to entire functions of exponential type: if \( f \) is an entire function of exponential type \( \tau \), bounded on \( \mathbb{R} \), such that \( h_f(\pi/2) = 0 \) then \( (\xi - 1)f'(z) + if(z) \neq 0 \) for \( \Im(z) > 0 \) and \( |\xi| < 1 \), whenever \( f(z) \neq 0 \) if \( \Im(z) > 0 \). We obtain a new proof of that result. We also obtain a generalization, to entire functions of exponential type, of a result of Szegő according to which the inequality \( |P(Rz) - P(z)| \leq R^\tau - 1, \quad |z| \leq 1, \quad R \geq 1 \), holds for all polynomials \( P \), of degree \( \leq n \), such that \( |P(z)| \leq 1 \) for \( |z| \leq 1 \).

1. Statement of the results. Let \( B_\tau \) denote the class of entire functions of exponential type \( \tau > 0 \) bounded on the real axis. The Phragmén–Lindelöf indicator function of \( f \in B_\tau \) is defined as

\[
h_f(\theta) := \lim_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r}, \quad 0 \leq \theta \leq 2\pi.
\]

Rahman and Schmeisser [8] have proved the following result:

**THEOREM 1.** Let \( f \in B_\tau \) such that \( h_f(\pi/2) = 0 \) and \( f(z) \neq 0 \) in \( \Im(z) > 0 \). Then, for \( |\xi| < 1 \) and \( \Im(z) > 0 \),

\[
(\xi - 1)f'(z) + if(z) \neq 0.
\]

This theorem represents an interesting generalization of a classical result of Laguerre (see [2] or [7, vol. II, chap. 2]) according to which

\[
(\xi - z)P'(z) + nP(z) \neq 0, \quad |\xi| < 1, \quad |z| < 1,
\]

for all polynomials \( P(z) := \sum_{\ell=0}^n a_\ell z^\ell \) such that \( P(z) \neq 0 \) in \( |z| < 1 \). In [8] the theorem is proved by using a property of \( B \)-operators. Here, we prove (1) with a method of approximation due to Lewitan [6] in a form given by Hörmander [5]; it will be done by adding only one hypothesis on the roots of the function \( f \). Using that method, we will also prove the
Theorem 2. Let \( f \in B \), such that \( h_{1}(\pi/2) \leq 0 \). If \(-\infty < X < \infty, -\infty < \eta < \infty\) and \( Y \leq -|\eta|\), then:

\[
\left| e^{-\eta Y} (f(X + iY - i\eta) - f(X - i\eta)) + e^{\eta Y} (f(X + iY + i\eta) - f(X + i\eta)) \right| \leq 2 (e^{-\gamma Y} - 1) \max_{-\infty < t < \infty} |f(t)|.
\]

If \( \eta = 0 \) in Theorem 2 we obtain the inequality

\[
|f(X + iY) - f(X)| \leq (e^{-\gamma Y} - 1) \max_{-\infty < t < \infty} |f(t)|, -\infty < X < \infty, Y \leq 0.
\]

In that inequality the hypothesis \( h_{1}(\pi/2) \leq 0 \) is not necessary. In fact, it may be deduced from the classical inequality of Bernstein [1], \( |f'(X)| \leq \tau \max_{-\infty < t < \infty} |f(t)|, -\infty < X < \infty \), and the identity

\[
f(X + iY) - f(X) = i \int_{0}^{\gamma} f'(X + iu) du.
\]

However, the example \( f(z) = e^{-\epsilon z}, 0 < \epsilon \leq \tau \), shows that, in (3), the inequality may not hold if \( h_{1}(\pi/2) > 0 \) (and \( \eta \neq 0 \)).

For other recent results obtained with that method of approximation see [4, Theorem 1] and [3].

2. Some lemmas. Given \( f \in B \), let

\[
f_{h}(X) := \sum_{K=0}^{\infty} \left( \frac{\sin \pi (hX + K)}{\pi (hX + K)} \right)^{2} f\left( X + \frac{K}{h} \right), -\infty < X < \infty, h > 0.
\]

We shall use the

Lemma 1. [5] The functions \( f_{h} \) defined by (6) are trigonometric polynomials with period \( 1/h \) and degree less than \( N := 1 + [\pi/2\pi h] \). When \( X \) is real we have \( |f_{h}(X)| \leq 1 \) whenever \( \max_{-\infty < t < \infty} |f(t)| \leq 1 \), and \( f_{h}(z) \to f(z) \) uniformly in every bounded set when \( h \to 0 \).

In view of Lemma 1 we may write

\[
f_{h}(X) = \sum_{m=-N}^{N} C_{m}(h) e^{2\pi i m X}
\]

where

\[
C_{m}(h) = h^{1/h} f_{h}(X) e^{-2\pi i m X} dX.
\]

We have also the

Lemma 2. (See [4, proof of Theorem 1] or [3, Lemma 2]). If \( h_{1}(\pi/2) \leq 0 \) then \( C_{m}(h) = 0 \) for \(-N \leq m \leq -1\).
Theorem 2 is in fact an extension of an inequality on algebraic polynomials which is a consequence of the following interpolation formula:

**Lemma 3.** Let \( P(z) := \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( \leq n \). If \( p > 0 \) is given, then, for any number \( R \geq (p^n + p^{-n})/(p^{n-1} + p^{-(n-1)}) \) and any real \( \gamma \), we have

\[
e^{i\gamma}[P(R e^{i\theta}) - P(e^{i\theta})] + p^{-n}(P(R p^2 e^{i\theta}) - P(p^2 e^{i\theta}))\]

\[
= \frac{1}{n} \sum_{k=1}^{2n} (-1)^k A_k(R, \gamma, p) P(e^{i(\theta + (k \pi + \gamma)/n)})
\]

for all real \( \theta \), where

\[
A_k(R, \gamma, p) := R^n - 1 + \sum_{j=1}^{n-1} (R^{n-j} - 1) (p^j + p^{-j}) \cos j \left( \frac{(K \pi + \gamma)}{n} \right).
\]

The coefficients \( A_k(R, \gamma, p) \) are non negative and

\[
\frac{1}{2n} \sum_{k=1}^{2n} A_k(R, \gamma, p) = R^n - 1.
\]

**Proof.** Substituting for \( P(p e^{i(\theta + (k \pi + \gamma)/n)}) \) and \( A_k(R, \gamma, p) \) we have

\[
\frac{1}{n} \sum_{k=1}^{2n} (-1)^k A_k(R, \gamma, p) P(p e^{i(\theta + (k \pi + \gamma)/n)})
\]

\[
= \frac{(R^n - 1)}{n} \sum_{k=1}^{2n} A_k(R, \gamma, p) e^{i(\theta + (k \pi + \gamma)/n)}
\]

\[
+ \frac{1}{n} \sum_{k=1}^{2n} \sum_{j=1}^{n-1} \sum_{\ell=0}^{n} (-1)^{k} (R^{n-j} - 1) (p^j + p^{-j}) \cos j \left( \frac{(k \pi + \gamma)}{n} \right) a_k p^\ell e^{i(\theta + (k \pi + \gamma)/n)}.
\]

Interchanging the order of summation, replacing \( \cos j(k \pi + \gamma)/n \) by \( (e^{ij(k \pi + \gamma)/n} + e^{-ij(k \pi + \gamma)/n})/2 \) and using the identity

\[
\sum_{k=1}^{2n} e^{i(mk \pi /n)} = \begin{cases} 
2n & \text{if } m \equiv 0 \pmod{2n} \\
0 & \text{if } m \not\equiv 0 \pmod{2n}
\end{cases}
\]

three times (with an appropriate integer \( m \)), we obtain

\[
\frac{1}{n} \sum_{k=1}^{2n} (-1)^k A_k(R, \gamma, p) P(p e^{i(\theta + (k \pi + \gamma)/n)})
\]

\[
= 2 (R^n - 1) a_n \rho^n e^{i\theta + i\gamma} + \sum_{j=1}^{n-1} \sum_{\ell=0}^{n} (R^{n-j} - 1) a_\ell (p^j + p^{-j}) e^{i(\theta + i(j + \ell) \pi /n)}
\]

\[
= \sum_{j=1}^{n} (R^j - 1) a_j (p^j + p^{j-n}) e^{i\theta + i\gamma} = e^{i\gamma}[\rho^n(P(R e^{i\theta}) - P(e^{i\theta})) + \rho^{-n}
\]

\[
\times (P(R p^2 e^{i\theta}) - P(p^2 e^{i\theta})).
\]
The identity (9) follows from (8) if we set \( P(z) = z^n \). To show that the coefficients \( A_j(R, \gamma, \rho) \) are non negative we may use a result of Rogosinski and Szegö [9, p. 75] according to which

\[
\lambda_0 + 2 \sum_{j=1}^{n} \lambda_j \cos j \theta \geq 0 \quad (0 \in \mathbb{R})
\]

if \( \lambda_n \geq 0, \lambda_{n-1} - 2 \lambda_n \geq 0 \) and \( \lambda_{j-1} - 2 \lambda_j + \lambda_{j+1} \geq 0 \) for \( 0 < j < n \). In order to verify the third condition we are led to show that

\[
R^{n-j+1} \left\{ \rho^{j-1} \left( 1 - \frac{\rho}{R} \right)^2 + \frac{1}{\rho^{j-1}} \left( 1 - \frac{1}{\rho R} \right)^2 \right\} \geq \rho^{j-1} (1 - \rho)^2 + \frac{1}{\rho^{j-1}} \left( 1 - \frac{1}{\rho} \right)^2,
\]

\( 0 < j < n \),

for \( R \geq (\rho^n + \rho^{-n})/(\rho^{n-1} + \rho^{-n+1}) \) \((\geq 1, \rho > 0)\). But the function \( \phi(R) := \rho^{j-1} (1 - \rho/R)^2 + (1/\rho^{j-1}) [1 - (1/\rho R)]^2 \) is increasing for \( R \geq (\rho^n + \rho^{-n})/(\rho^{n-1} + \rho^{-n+1}) \) so that (10) is satisfied with \( \lambda_j := (R^{n-j} - 1)(\rho^j + \rho^{-j})/2 \). This completes the proof of the lemma.

It follows from Lemma 3 that if \( P \) is an algebraic polynomial of degree \( \leq n \) such that \( \max_{1 \leq k \leq 2n} |P(e^{i(\gamma+\theta)/2})| \leq 1 \) then, for all real \( \theta \),

\[
|\rho^n \left( P \left( \frac{Re^{i\theta}}{\rho} \right) - P\left( \frac{e^{i\theta}}{\rho} \right) \right) + \rho^{-n} \left( P(R\rho e^{i\theta}) - P(\rho e^{i\theta}) \right) | \leq 2(R^n - 1),
\]

\( R \geq \frac{\rho^n + \rho^{-n}}{\rho^{n-1} + \rho^{-(n-1)}}, \rho > 0 \).

**Remarks.**

1. Szegö [10] had proved that the inequality

\[
|P(Re^{i\theta}) - P(e^{i\theta})| \leq (R^n - 1), \quad R \geq 1, \quad \theta \in \mathbb{R}
\]

holds for all polynomials \( P \) such that \( \max_{1 \leq k \leq 2n} |P(e^{i(\gamma+\theta)/2})| \leq 1 \). Using (12) we deduce that the left member of (11) is less or equal to \( \rho^n (R^n - 1) M_P(1/\rho) + \rho^{-n} (R^n - 1) M_P(\rho) \) where \( M_P(\rho) := \max_{|z|=\rho} |P(z)| \). But, in view of Hadamard’s three circles theorem, \( 2M_P(1) \leq \rho^n M_P(1/\rho) + \rho^{-n} M_P(\rho) \) for all \( \rho > 0 \), so that (11) is effectively a refinement of (12).

2. If \( S(\theta) := \sum_{m=-n}^{n} b_m e^{im\theta} \) is a trigonometric polynomial of degree \( \leq n \) then for \( \theta, \gamma \in \mathbb{R}, \rho > 0 \) and \( R \geq (\rho^n + \rho^{-n})/(\rho^{n-1} + \rho^{-n+1}) \) we have

\[
e^{i\gamma} \sum_{m=1}^{n} (R^{m-1} - 1) b_m (\rho^n + \rho^{2m-n}) e^{im\theta} + e^{-i\gamma} \sum_{m=-n}^{-1} (R^{-m} - 1) \\
\times b_m (\rho^{2m+n} + \rho^{-n}) e^{im\theta} = \frac{1}{n} \sum_{k=1}^{2n} (-1)^k A_k(R, \gamma, \rho) \\
\times S\left( \theta + \frac{k\pi + \gamma}{n} - i \log \rho \right).
\]
Thus, the inequality
\[
\left| \sum_{m=1}^{n} (R^m - 1) b_m (p^n + p^{2m-n}) e^{i(m-n)\theta} + \sum_{m=-n}^{-1} (R^{-m} - 1) b_m (p^{2m+n} + p^{-n}) e^{i(m+n)\theta} \right| \leq 2(R^n - 1) \max_{1 \leq k \leq 2n} \left| S \left( \frac{k\pi}{n} - i \log \rho \right) \right|
\]
is satisfied for
\[
R \geq \frac{\rho^n + \rho^{-n}}{\rho^{n-1} + \rho^{-(n-1)}}, \quad \rho > 0, \quad \theta \in \mathbb{R}.
\]

3. **Proof of the theorems.** As mentioned in the introduction we will prove Theorem 1 with an additional hypothesis on the roots of the function \(f\). If \(f\) has some kind of zero at infinity it may happen that the approximating functions (7) have a sequence of zeros in \(\text{Im}(z) > 0\) (which tends to infinity). In that case, since the transformation \(z \mapsto e^{i\theta}\) maps any rectangle of the form \(\{z \in \mathbb{C} : -X_0 < \text{Re}(z) < X_0, 0 < \text{Im}(z) < Y_0\}\), \(X_0, Y_0 > 0\), on a domain contained in \(\{z \in \mathbb{C} : |z| < 1\}/\{0\}\), the following argument seems not to apply easily.

**Proof of Theorem 1.** We shall prove the result by requiring that there is no curve \(C\) in the upper half-plane for which \(f(z)\) approaches zero as \(|z| \rightarrow \infty\), \(z \in C\).

Since \(h_{\theta} (\pi/2) = 0\) we have (by Lemma 2) \(f_{\theta}(z) = P_{\theta}(e^{i\pi/h_{\theta}})\) where \(P_{\theta}\) is an algebraic polynomial of degree \(< N\). If \(f(z) \neq 0\) in \(\text{Im}(z) \geq 0\) then \(f_{\theta}(z) \neq 0\) in \(\text{Im}(z) > 0\) whenever \(h\) is made sufficiently small. The polynomials \(P_{\theta}(z)\) are thus \(\neq 0\) in \(|z| < 1\).

Applying (2) we obtain that
\[
(\xi - z) P_{\theta}(z) + NP_{\theta}(z) \neq 0, \quad |z| < 1, \quad |\xi| < 1,
\]
or, equivalently
\[
(\xi - e^{i\theta}) P_{\theta}(e^{i\theta}) + NP_{\theta}(e^{i\theta}) \neq 0, \quad \text{Im}(z) > 0, \quad |\xi| < 1,
\]
that is
\[
(\xi - e^{i\theta}) f_{\theta}' \left( \frac{z}{2\pi h} \right) + 2\pi hNi f_{\theta} \left( \frac{z}{2\pi h} \right) \neq 0, \quad \text{Im}(z) > 0, \quad |\xi| < 1.
\]
If we change \(z\) to \(2\pi h z\), in (17), we obtain that the functions \(g_{\theta}(z) := (\xi - e^{i\pi/h_{\theta}}) f_{\theta}(z) + 2\pi hNi e^{i\pi/h_{\theta}} f_{\theta}(z)\) have no zero in \(\text{Im}(z) > 0\). In view of Hurwitz’s theorem we conclude that \(g(z) := \lim_{\theta \rightarrow 0} g_{\theta}(z)\) is different from 0 in \(\text{Im}(z) > 0\), or \(g(z) \equiv 0\). But (using Lemma 1) \(g(z) = (\xi - 1) f'/(1 - i\tau f(z) \equiv 0\) if and only if \(f(z) = c e^{i\tau z/(1 - \xi)}\) for some constant \(c\) (a function of that form is in \(B\) if \(|\xi - 1| \geq 1\)) which are not admissible functions in (1). Thus, \((\xi - 1) f'(z) + i\tau f(z) \neq 0\) for \(\text{Im}(z) > 0\) and \(|\xi| < 1\).

Finally, if \(f(z) \neq 0\) only in \(\text{Im}(z) > 0\) then we may apply the result just proved to a function of the form \(f(z + \epsilon i), \epsilon > 0\), and the result follows.
REMARKS. It is possible to find many \( f \in B \), such that \( h_f(\pi/2) = 0 \), \( f(z) \neq 0 \) in \( \text{Im}(z) > 0 \), but \( \lim_{t \to \infty} f(ri) = 0 \); an example is \( f(z) = (e^{iz} - 1)/z \). Also, in the case \( h_f(\pi/2) < 0 \), we have necessarily \( \lim_{t \to \infty} f(ri) = 0 \) (if we let \( R > 0 \) such that \( h_f(\pi/2) = -\delta \) then \( |f(ri)| < e^{-\delta r^2/2}, r \to \infty \)); in that case it is known \([8]\) that (1) may not hold.

It is also to be noted that if the polynomials \( P_h(z) \) happens to have a zero of multiplicity \( k \) at \( z = 0 \), with \( \lim_{t \to 0} \frac{h}{k} = 0 \), then the preceding argument may be used; we need only to observe that (2) may then be applied to the polynomials \( P_h(z)/z^k \) instead of \( P_h(z) \).

PROOF OF THEOREM 2. If \( h_f(\pi/2) \leq 0 \) then (by Lemma 2) \( f_h(z/(2\pi h)) = P_h(e^{i\theta}) \) where \( P_h \) is an algebraic polynomial of degree \( < N \). Applying (11) and Lemma 1 we readily obtain (we may assume that \( \max_{-\infty < \theta < \infty} |f(\theta)| \leq 1 \):

\[
|\rho^N (f_h \left( \frac{Re^{i\theta}}{\rho} \right) - P_h \left( \frac{e^{i\theta}}{\rho} \right) + \rho^{-N} (P_h(Re^{i\theta}) - P_h(\rho e^{i\theta})) | \leq 2(R^N - 1), R \geq \frac{\rho^N + \rho^{-N}}{\rho^{N-1} + \rho^{-(N-1)}}, \quad \theta \in \mathbb{R}.
\]

That inequality may be written in the form

\[
|\rho^N \left( f_h \left( \frac{\theta - i \log R/\rho}{2\pi h} \right) - f_h \left( \frac{\theta + i \log R/\rho}{2\pi h} \right) \right) + \rho^{-N} \left( f_h \left( \frac{\theta - i \log R\rho}{2\pi h} \right) - f_h \left( \frac{\theta + i \log R\rho}{2\pi h} \right) \right) | \leq 2(R^N - 1), R \geq \frac{\rho^N + \rho^{-N}}{\rho^{N-1} + \rho^{-(N-1)}}, \quad \theta \in \mathbb{R}.
\]

Put \( \theta = 2\pi hX \) and change \( R \) to \( R^{2\pi h} \), \( \rho \) to \( \rho^{2\pi h} \), we obtain that

\[
|\rho^{2\pi hN} \left( f_h(X - i \log R + i \log \rho) - f_h(X + i \log \rho) \right) + \rho^{-2\pi hN} \left( f_h(X - i \log R - i \log \rho) - f_h(X - i \log \rho) \right) | \leq 2(R^{2\pi hN} - 1),
\]

\[
R^{2\pi h} \geq \frac{\rho^{2\pi hN} + \rho^{-2\pi hN}}{\rho^{2\pi h(N-1)} + \rho^{-2\pi h(N-1)}}, \quad -\infty < X < \infty.
\]

The condition

\[
R^{2\pi h} \geq \frac{\rho^{2\pi hN} + \rho^{-2\pi hN}}{\rho^{2\pi h(N-1)} + \rho^{-2\pi h(N-1)}},
\]

is certainly satisfied if \( R \geq \rho \) in the case \( \rho \geq 1 \) and if \( R \geq 1/\rho \) in the case \( \rho \leq 1 \). Thus, letting \( h \to 0 \) in (20) (and using Lemma 1) we are led to the inequality

\[
|\rho^\eta (f(X + iY + i \log \rho) - f(X + i \log \rho)) + \rho^{-\eta} (f(X + iY - i \log \rho) - f(X - i \log \rho)) | \leq 2(e^{-\eta Y} - 1), \quad -\infty < X < \infty, \quad \rho > 0,
\]

where \( e^{-\eta} := R \geq e^{\log \rho} \), from which (3) follows (with \( \eta := \log \rho \)).
REFERENCES


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