Canad. Math. Bull. Vol. **29** (3), 1986

REMARKS ON ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. A classical result of Laguerre says that if P is a polynomial of degree n such that $P(z) \neq 0$ for $|z| < 1$ then $(\xi - z)P'(z) + nP(z) \neq 0$ for $|z| < 1$ and $|\xi| < 1$. Rahman and Schmeisser have obtained an extension of that result to entire functions of exponential type: if f is an entire function of exponential type τ, bounded on $\mathbb{R}$, such that $h_f(\pi/2) = 0$ then $(\xi - 1)f'(z) + i\tau f(z) \neq 0$ for $\text{Im}(z) > 0$ and $|\xi| < 1$, whenever $f(z) \neq 0$ if $\text{Im}(z) > 0$. We obtain a new proof of that result. We also obtain a generalization, to entire functions of exponential type, of a result of Szegö according to which the inequality $|P(Rz) - P(z)| \leq R^n - 1$, $|z| \leq 1$, $R \geq 1$, holds for all polynomials P, of degree $\leq n$, such that $|P(z)| \leq 1$ for $|z| \leq 1$.

1. **Statement of the results**. Let B_τ denote the class of entire functions of exponential type $\tau > 0$ bounded on the real axis. The Phragmén–Lindelöf indicator function of $f \in B_\tau$ is defined as

$$h_f(\theta) := \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r}, \ 0 \leq \theta \leq 2\pi.$$

Rahman and Schmeisser [8] have proved the following result:

THEOREM 1. *Let $f \in B_\tau$ such that $h_f(\pi/2) = 0$ and $f(z) \neq 0$ in* $\text{Im}(z) > 0$. *Then, for $|\xi| < 1$ and* $\text{Im}(z) > 0$,

$$(1) \qquad (\xi - 1)f'(z) + i\tau f(z) \neq 0.$$

This theorem represents an interesting generalization of a classical result of Laguerre (see [2] or [7, vol. II, chap. 2]) according to which

$$(2) \qquad (\xi - z)\,P'(z) + nP(z) \neq 0, \ |\xi| < 1, \ |z| < 1,$$

for all polynomials $P(z) := \sum_{\ell=0}^{n} a_\ell z^\ell$ such that $P(z) \neq 0$ in $|z| < 1$. In [8] the theorem is proved by using a property of B-operators. Here, we prove (1) with a method of approximation due to Lewitan [6] in a form given by Hörmander [5]; it will be done by adding only one hypothesis on the roots of the function f. Using that method, we will also prove the

Received by the editors January 29, 1985.
AMS Subject Classification (1980): 30D15, 30A64.
Research supported by "La Fondation du Prêt d'Honneur Inc."

THEOREM 2. *Let $f \in B_\tau$ such that $h_f(\pi/2) \le 0$. If $-\infty < X < \infty$, $-\infty < \eta < \infty$ and $Y \le -|\eta|$ then*:

$$|e^{-\tau\eta}(f(X+iY-i\eta) - f(X-i\eta)) + e^{\tau\eta}(f(X+iY+i\eta) - f(X+i\eta))| \le 2(e^{-\tau Y}-1)\max_{-\infty<t<\infty}|f(t)|. \tag{3}$$

If $\eta = 0$ in Theorem 2 we obtain the inequality

$$|f(X+iY) - f(X)| \le (e^{-\tau Y}-1)\max_{-\infty<t<\infty}|f(t)|,\ -\infty < X < \infty,\ Y \le 0. \tag{4}$$

In that inequality the hypothesis $h_f(\pi/2) \le 0$ is not necessary. In fact, it may be deduced from the classical inequality of Bernstein [1], $|f'(X)| \le \tau \max_{-\infty<t<\infty}|f(t)|$, $-\infty < X < \infty$, and the identity

$$f(X+iY) - f(X) = i\int_0^Y f'(X+iu)\mathrm{d}u. \tag{5}$$

However, the example $f(z) = e^{-i\epsilon z}$, $0 < \epsilon \le \tau$, shows that, in (3), the inequality may not hold if $h_f(\pi/2) > 0$ (and $\eta \ne 0$).

For other recent results obtained with that method of approximation see [4, Theorem 1] and [3].

2. **Some lemmas**. Given $f \in B_\tau$, let

$$f_h(X) := \sum_{K=-\infty}^{\infty}\left(\frac{\sin\pi(hX+K)}{\pi(hX+K)}\right)^2 f\left(X+\frac{K}{h}\right),\ -\infty < X < \infty,\ h > 0. \tag{6}$$

We shall use the

LEMMA 1. [5] *The functions f_h defined by (6) are trigonometric polynomials with period $1/h$ and degree less than $N := 1 + [\tau/2\pi h]$. When X is real we have $|f_h(X)| \le 1$ whenever* $\max_{-\infty<t<\infty}|f(t)| \le 1$, *and $f_h(z) \to f(z)$ uniformly in every bounded set when $h \to 0$.*

In view of Lemma 1 we may write

$$f_h(X) = \sum_{m=-N}^{N} C_m(h)\,\mathrm{e}^{2\pi i h m X} \tag{7}$$

where

$$C_m(h) = h\int_0^{1/h} f_h(X)\,e^{-2\pi i h m X}\mathrm{d}X.$$

We have also the

LEMMA 2. (See [4, proof of Theorem 1] or [3, Lemma 2]). *If $h_f(\pi/2) \le 0$ then $C_m(h) = 0$ for $-N \le m \le -1$.*

https://doi.org/10.4153/CMB-1986-056-4 Published online by Cambridge University Press

Theorem 2 is in fact an extension of an inequality on algebraic polynomials which is a consequence of the following interpolation formula:

LEMMA 3. *Let* $P(z) := \sum_{\ell=0}^{n} a_\ell z^\ell$ *be a polynomial of degree* $\leq n$. *If* $\rho > 0$ *is given then, for any number* $R \geq (\rho^n + \rho^{-n})/(\rho^{n-1} + \rho^{-(n-1)})$ *and any real* γ, *we have*

$$e^{i\gamma}[\rho^n(P(Re^{i\theta}) - P(e^{i\theta})) + \rho^{-n}(P(R\rho^2 e^{i\theta}) - P(\rho^2 e^{i\theta}))] = \frac{1}{n}\sum_{K=1}^{2n} (-1)^K A_K(R,\gamma,\rho)\, P(e^{i(\theta+(K\pi+\gamma)/n)}), \tag{8}$$

for all real θ, *where*

$$A_K(R,\gamma,\rho) := R^n - 1 + \sum_{j=1}^{n-1} (R^{n-j} - 1)(\rho^j + \rho^{-j}) \cos j\frac{(K\pi + \gamma)}{n}.$$

The coefficients $A_K(R,\gamma,\rho)$ *are non negative and*

$$\frac{1}{2n}\sum_{k=1}^{2n} A_k(R,\gamma,\rho) = R^n - 1. \tag{9}$$

PROOF. Substituting for $P(\rho e^{i(\theta+(k\pi+\gamma)/n)})$ and $A_k(R,\gamma,\rho)$ we have

$$\frac{1}{n}\sum_{k=1}^{2n} (-1)^k A_k(R,\gamma,\rho)\, P(\rho e^{i(\theta+(k\pi+\gamma)/n)})$$

$$= \frac{(R^n - 1)}{n}\sum_{k=1}^{2n}\sum_{\ell=0}^{n} (-1)^k a_\ell \rho^\ell e^{i\ell(\theta+(k\pi+\gamma)/n)}$$

$$+ \frac{1}{n}\sum_{k=1}^{2n}\sum_{j=1}^{n-1}\sum_{\ell=0}^{n} (-1)^k (R^{n-j} - 1)(\rho^j + \rho^{-j}) \cos j\frac{(k\pi+\gamma)}{n}\; a_\ell \rho^\ell e^{i\ell(\theta+(k\pi+\gamma)/n)}.$$

Interchanging the order of summation, replacing $\cos j(k\pi + \gamma)/n$ by $(e^{ij(k\pi+\gamma)/n} + e^{-ij(k\pi+\gamma)/n})/2$ and using the identity

$$\sum_{k=1}^{2n} e^{(mk\pi i)/n} = \begin{cases} 2n \text{ if } m \equiv 0 \pmod{2n} \\ 0 \text{ if } m \not\equiv 0 \pmod{2n} \end{cases}$$

three times (with an appropriate integer m), we obtain

$$\frac{1}{n}\sum_{k=1}^{2n} (-1)^k A_k(R,\gamma,\rho)\, P(\rho e^{i(\theta+(k\pi+\gamma)/n)})$$

$$= 2(R^n - 1)\, a_n \rho^n e^{in\theta + i\gamma} + \sum_{\substack{j=1 \\ j+\ell=n}}^{n-1}\sum_{\ell=0}^{n} (R^{n-j} - 1)\, a_\ell(\rho^j + \rho^{-j})\, e^{i\ell\theta + i(j+\ell)\frac{\gamma}{n}}$$

$$= \sum_{\ell=1}^{n} (R^\ell - 1)\, a_\ell(\rho^n + \rho^{2\ell-n})\, e^{i\ell\theta+i\gamma} = e^{i\gamma}[\rho^n(P(Re^{i\theta}) - P(e^{i\theta})) + \rho^{-n} \times (P(R\rho^2 e^{i\theta}) - P(\rho^2 e^{i\theta}))].$$

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The identity (9) follows from (8) if we set $P(z) = z^n$. To show that the coefficients $A_k(R, \gamma, \rho)$ are non negative we may use a result of Rogosinski and Szegö [9, p. 75] according to which

$$\lambda_0 + 2 \sum_{j=1}^{n} \lambda_j \cos j\theta \geq 0 \ (\theta \in \mathbb{R}) \tag{10}$$

if $\lambda_n \geq 0$, $\lambda_{n-1} - 2\lambda_n \geq 0$ and $\lambda_{j-1} - 2\lambda_j + \lambda_{j+1} \geq 0$ for $0 < j < n$. In order to verify the third condition we are led to show that

$$R^{n-j+1} \left\{ \rho^{j-1} \left(1 - \frac{\rho}{R}\right)^2 + \frac{1}{\rho^{j-1}} \left(1 - \frac{1}{\rho R}\right)^2 \right\} \geq \rho^{j-1} (1 - \rho)^2 + \frac{1}{\rho^{j-1}} \left(1 - \frac{1}{\rho}\right)^2,$$
$$0 < j < n,$$

for $R \geq (\rho^n + \rho^{-n})/(\rho^{n-1} + \rho^{-(n-1)})$ $(\geq 1, \rho > 0)$. But the function $\phi(R) := \rho^{j-1} (1 - \rho/R)^2 + (1/\rho^{j-1}) [1 - (1/\rho R)]^2$ is increasing for $R \geq (\rho^n + \rho^{-n})/(\rho^{n-1} + \rho^{-(n-1)})$ so that (10) is satisfied with $\lambda_j := (R^{n-j} - 1) (\rho^j + \rho^{-j})/2$. This completes the proof of the lemma.

It follows from Lemma 3 that if P is an algebraic polynomial of degree $\leq n$ such that $\max_{1 \leq k \leq 2n} |P(e^{(k\pi i)/n})| \leq 1$ then, for all real θ,

$$\left| \rho^n \left(P\left(\frac{Re^{i\theta}}{\rho}\right) - P\left(\frac{e^{i\theta}}{\rho}\right)\right) + \rho^{-n} (P(R\rho\, e^{i\theta}) - P(\rho\, e^{i\theta})) \right| \leq 2(R^n - 1), \tag{11}$$
$$R \geq \frac{\rho^n + \rho^{-n}}{\rho^{n-1} + \rho^{-(n-1)}}, \ \rho > 0.$$

REMARKS.

1. Szegö [10] had proved that the inequality

$$|P(Re^{i\theta}) - P(e^{i\theta})| \leq (R^n - 1), \ R \geq 1, \ \theta \in \mathbb{R} \tag{12}$$

holds for all polynomials P such that $\max_{1 \leq k \leq 2n} |P(e^{(k\pi i)/n})| \leq 1$. Using (12) we deduce that the left member of (11) is less or equal to $\rho^n (R^n - 1) M_P(1/\rho) + \rho^{-n} (R^n - 1) M_P(\rho)$ where $M_P(\rho) := \max_{|z| = \rho} |P(z)|$. But, in view of Hadamard's three circles theorem, $2M_P(1) \leq \rho^n M_P(1/\rho) + \rho^{-n} M_P(\rho)$ for all $\rho > 0$, so that (11) is effectively a refinement of (12).

2. If $S(\theta) := \sum_{m=-n}^{n} b_m e^{im\theta}$ is a trigonometric polynomial of degree $\leq n$ then for $\theta, \gamma \in \mathbb{R}$, $\rho > 0$ and $R \geq (\rho^n + \rho^{-n})/(\rho^{n-1} + \rho^{-(n-1)})$ we have

$$e^{i\gamma} \sum_{m=1}^{n} (R^m - 1)\, b_m (\rho^n + \rho^{2m-n})\, e^{im\theta} + e^{-i\gamma} \sum_{m=-n}^{-1} (R^{-m} - 1)$$
$$\times\, b_m (\rho^{2m+n} + \rho^{-n})\, e^{im\theta} = \frac{1}{n} \sum_{k=1}^{2n} (-1)^k A_k(R, \gamma, \rho)$$
$$\times\, S\left(\theta + \frac{k\pi + \gamma}{n} - i \log \rho\right). \tag{13}$$

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Thus, the inequality

$$(14)\quad \left| \sum_{m=1}^{n} (R^m - 1)\, b_m(\rho^n + \rho^{2m-n})\, e^{i(m-n)\theta} + \sum_{m=-n}^{-1} (R^{-m} - 1)\, b_m(\rho^{2m+n} + \rho^{-n}) \right.$$

$$\left. e^{i(m+n)\theta} \right| \leq 2(R^n - 1) \max_{1 \leq k \leq 2n} \left| S\left(\frac{k\pi}{n} - i \log \rho\right) \right|$$

is satisfied for

$$R \geq \frac{\rho^n + \rho^{-n}}{\rho^{n-1} + \rho^{-(n-1)}}, \quad \rho > 0, \quad \theta \in \mathbb{R}.$$

3. **Proof of the theorems**. As mentioned in the introduction we will prove Theorem 1 with an additional hypothesis on the roots of the function f. If f has some kind of zero at infinity it may happen that the approximating functions (7) have a sequence of zeros in $\mathrm{Im}(z) > 0$ (which tends to infinity). In that case, since the transformation $z \mapsto e^{iz}$ maps any rectangle of the form $\{z \in \mathbb{C} : -X_0 \leq Re(z) \leq X_0, 0 \leq \mathrm{Im}(z) \leq Y_0\}$, $X_0, Y_0 > 0$, on a domain contained in $\{z \in \mathbb{C} : |z| \leq 1\}/\{0\}$, the following argument seems not to apply easily.

PROOF OF THEOREM 1. We shall prove the result by requiring that there is no curve C in the upper half-plane for which $f(z)$ approaches zero as $|z| \to \infty$, $z \in C$.

Since $h_f(\pi/2) = 0$ we have (by Lemma 2) $f_h(z) = P_h(e^{2\pi h i z})$ where P_h is an algebraic polynomial of degree $< N$. If $f(z) \neq 0$ in $\mathrm{Im}(z) \geq 0$ then $f_h(z) \neq 0$ in $\mathrm{Im}(z) > 0$ whenever h is made sufficiently small. The polynomials $P_h(z)$ are thus $\neq 0$ in $|z| < 1$. Applying (2) we obtain that

$$(15)\qquad (\xi - z)\, P_h'(z) + NP_h(z) \neq 0, \quad |z| < 1, \quad |\xi| < 1,$$

or, equivalently

$$(16)\qquad (\xi - e^{iz})\, P_h'(e^{iz}) + NP_h(e^{iz}) \neq 0, \quad \mathrm{Im}(z) > 0, \quad |\xi| < 1,$$

that is

$$(17)\quad (\xi - e^{iz})\, f_h'\left(\frac{z}{2\pi h}\right) + 2\pi h N i\, f_h\left(\frac{z}{2\pi h}\right) \neq 0, \quad \mathrm{Im}(z) > 0, \quad |\xi| < 1.$$

If we change z to $2\pi h z$, in (17), we obtain that the functions $g_h(z) := (\xi - e^{2\pi h i z})\, f_h'(z) + 2\pi h N i\, e^{2\pi h i z} f_h(z)$ have no zero in $\mathrm{Im}(z) > 0$. In view of Hurwitz's theorem we conclude that $g(z) := \lim_{h \to 0} g_h(z)$ is different from 0 in $\mathrm{Im}(z) > 0$, or $g(z) \equiv 0$. But (using Lemma 1) $g(z) = (\xi - 1) f'(z) + i\tau f(z) \equiv 0$ if and only if $f(z) = c\, e^{i\tau z/(1-\xi)}$ for some constant c (a function of that form is in B_τ if $|\xi - 1| \geq 1$) which are not admissible functions in (1). Thus, $(\xi - 1)\, f'(z) + i\tau f(z) \neq 0$ for $\mathrm{Im}(z) > 0$ and $|\xi| < 1$.

Finally, if $f(z) \neq 0$ only in $\mathrm{Im}(z) > 0$ then we may apply the result just proved to a function of the form $f(z + \epsilon i)$, $\epsilon > 0$, and the result follows.

REMARKS. It is possible to find many $f \in B_\tau$ such that $h_f(\pi/2) = 0$, $f(z) \neq 0$ in $\mathrm{Im}(z) > 0$, but $\lim_{r\to\infty} f(ri) = 0$; an example is $f(z) = (e^{i\tau z} - 1)/z$. Also, in the case $h_f(\pi/2) < 0$, we have necessarily $\lim_{r\to\infty} f(ri) = 0$ (if we let $\delta > 0$ such that $h_f(\pi/2) = -\delta$ then $|f(ri)| < e^{-(\delta\tau)/2}$, $r \to \infty$); in that case it is known [8] that (1) may not hold.

It is also to be noted that if the polynomials $P_h(z)$ happens to have a zero of multiplicity k at $z = 0$, with $\lim_{h\to 0} kh = 0$, then the preceeding argument may be used; we need only to observe that (2) may then be applied to the polynomials $P_h(z)/z^k$ instead of $P_h(z)$.

PROOF OF THEOREM 2. If $h_f(\pi/2) \leq 0$ then (by Lemma 2) $f_h(z/(2\pi h)) = P_h(e^{iz})$ where P_h is an algebraic polynomial of degree $< N$. Applying (11) and Lemma 1 we readily obtain (we may assume that $\max_{-\infty<t<\infty} |f(t)| \leq 1$):

$$(18)\qquad \left|\rho^N \left(P_h\left(\frac{Re^{i\theta}}{\rho}\right) - P_h\left(\frac{e^{i\theta}}{\rho}\right)\right) + \rho^{-N}\left(P_h(R\rho\, e^{i\theta}) - P_h(\rho\, e^{i\theta})\right)\right| \leq 2\,(R^N - 1),\ R \geq \frac{\rho^N + \rho^{-N}}{\rho^{N-1} + \rho^{-(N-1)}},\quad \theta \in \mathbb{R}.$$

That inequality may be written in the form

$$(19)\qquad \left|\rho^N \left(f_h\left(\frac{\theta - i\log R/\rho}{2\pi h}\right) - f_h\left(\frac{\theta + i\log \rho}{2\pi h}\right)\right) + \rho^{-N}\left(f_h\left(\frac{\theta - i\log R\rho}{2\pi h}\right) - f_h\left(\frac{\theta - i\log\rho}{2\pi h}\right)\right)\right| \leq 2(R^N - 1),\ R \geq \frac{\rho^N + \rho^{-N}}{\rho^{N-1} + \rho^{-(N-1)}},\qquad \theta \in \mathbb{R}.$$

Put $\theta = 2\pi hX$ and change R to $R^{2\pi h}$, ρ to $\rho^{2\pi h}$; we obtain that

$$(20)\qquad \begin{aligned} &|\rho^{2\pi hN}(f_h(X - i\log R + i\log\rho) - f_h(X + i\log\rho)) \\ &\quad + \rho^{-2\pi hN}(f_h(X - i\log R - i\log\rho) \\ &\quad - f_h(X - i\log\rho))| \leq 2(R^{2\pi hN} - 1), \\ &R^{2\pi h} \geq \frac{\rho^{2\pi hN} + \rho^{-2\pi hN}}{\rho^{2\pi h(N-1)} + \rho^{-2\pi h(N-1)}},\ -\infty < X < \infty. \end{aligned}$$

The condition

$$R^{2\pi h} \geq \frac{\rho^{2\pi hN} + \rho^{-2\pi hN}}{\rho^{2\pi h(N-1)} + \rho^{-2\pi h(N-1)}}$$

is certainly satisfied if $R \geq \rho$ in the case $\rho \geq 1$ and if $R \geq 1/\rho$ in the case $\rho \leq 1$. Thus, letting $h \to 0$ in (20) (and using Lemma 1) we are led to the inequality

$$(21)\qquad |\rho^\tau (f(X + iY + i\log\rho) - f(X + i\log\rho)) + \rho^{-\tau}(f(X + iY - i\log\rho) - f(X - i\log\rho))| \leq 2(e^{-\tau Y} - 1),\ -\infty < X < \infty,\ \rho > 0,$$

where $e^{-Y} := R \geq e^{|\log\rho|}$, from which (3) follows (with $\eta := \log\rho$).

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