Canad. Math. Bull. Vol. **46** (4), 2003 pp. 575–587

Optimization of Polynomial Functions

M. Marshall

Abstract. This paper develops a refinement of Lasserre's algorithm for optimizing a polynomial on a basic closed semialgebraic set via semidefinite programming and addresses an open question concerning the duality gap. It is shown that, under certain natural stability assumptions, the problem of optimization on a basic closed set reduces to the compact case.

Recently progress has been made in the development of algorithms for optimizing polynomials. The main idea being stressed is that of reducing the problem to an easier problem involving semidefinite programming [18]. It seems that in many cases the method dramatically outperforms other existing methods. The idea traces back to work of Shor [16] [17] and is further developed by Parrilo [10] and by Parrilo and Sturmfels [11] and by Lasserre [7] [8].

In [7] [8] Lasserre describes an extension of the method to minimizing a polynomial on an arbitrary basic closed semialgebraic set and uses a result due to Putinar [13] to prove that the method produces the exact minimum in the compact case. In the general case it produces a lower bound for the minimum.

The ideas involved come from three branches of mathematics: algebraic geometry (positive polynomials), functional analysis (the moment problem) and optimization. This makes the area an attractive one not only from the computational but also from the theoretical point of view.

In Section 1 we define three lower bounds for a polynomial and point out relationships between them. In Section 2 we outline Lasserre's method. In Sections 3 and 4 we describe a refinement of Lasserre's method in the empty interior case and address questions left open in [7] [8] concerning the duality gap. In Section 5 we show that, in the presence of certain stability assumptions, the problem of minimization of a polynomial on a basic closed semialgebraic set reduces naturally to the compact case (so can be handled using Lasserre's method, yielding exact results).

1 Lower Bounds for a Polynomial

Denote the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$ by $\mathbb{R}[\underline{x}]$ for short. Fix a finite subset $S = \{g_1, \dots, g_s\}$ of $\mathbb{R}[\underline{x}]$. We consider the problem of minimizing a polynomial f on the basic closed semialgebraic set

$$K_S := \{p \in \mathbb{R}^n : g_i(p) \geq 0, i = 1, \dots, s\}.$$

Received by the editors April 16, 2002; revised April 2, 2003.
AMS subject classification: Primary: 14P10; secondary: 46L05, 90C22.

Denote by M_S the *quadratic module* in $\mathbb{R}[\underline{x}]$ generated by S. By definition, M_S is the set of all finite sums of the form

$$\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s,$$

where each σ_i a sum of squares in $\mathbb{R}[\underline{x}]$, $i = 0, \dots, s$.

The main focus of Lasserre's work in [7] [8] is on the case where K_S is compact but other cases are not excluded. In particular, the case $S = \varnothing$ (global minimization) investigated earlier by Shor [16] [17] and more recently by Parrilo and Sturmfels [10] [11] is not excluded. Note that if $S = \varnothing$, then $K_S = \mathbb{R}^n$ and $M_S =$ the set of sums of squares in $\mathbb{R}[\underline{x}]$.

Fix $f \in \mathbb{R}[\underline{x}]$. Various sorts of lower bounds for f on K_S are considered. Define:

$$\begin{aligned} f^* &= \inf\{f(p) \mid p \in K_S\} \\ \bar{f}^{\mathrm{sos}} &= \inf\{L(f) \mid L \in \bar{K}_S\} \\ f^{\mathrm{sos}} &= \sup\{\lambda \in \mathbb{R} \mid f - \lambda \in M_S\} \end{aligned}$$

where $\bar{K}_S$ is the convex cone in the dual space of $\mathbb{R}[\underline{x}]$ defined by

$$\bar{K}_S := \{L \colon \mathbb{R}[\underline{x}] \to \mathbb{R} \mid L \text{ is linear}, L(1) = 1, L(M_S) \geq 0\}.$$

Note that $\bar{K}_S$ depends on S (the particular presentation) not just on K_S.

Proposition 1.1 $f^* \geq \bar{f}^{\mathrm{sos}} \geq f^{\mathrm{sos}}$.

Proof The inequality $f^* \geq \bar{f}^{\mathrm{sos}}$ is a consequence of the embedding $K_S \hookrightarrow \bar{K}_S$, $p \mapsto L_p$, where L_p denotes evaluation at p, *i.e.*, $L_p(g) = g(p)$. Suppose $p \in K_S$. Then $f(p) = L_p(f) \geq \bar{f}^{\mathrm{sos}}$. Thus $f^* \geq \bar{f}^{\mathrm{sos}}$. The inequality $\bar{f}^{\mathrm{sos}} \geq f^{\mathrm{sos}}$ is obvious. If $f - \lambda \in M_S$ then, $\forall L \in \bar{K}_S$, $L(f - \lambda) \geq 0$. Since L is linear and $L(1) = 1$ this implies $L(f) - \lambda \geq 0$, *i.e.*, $L(f) \geq \lambda$. This proves $\bar{f}^{\mathrm{sos}} \geq \lambda$ for any such λ. This proves $\bar{f}^{\mathrm{sos}} \geq f^{\mathrm{sos}}$. ∎

The following result is due to Putinar [13]. Jacobi gives another proof in [4] based on an extension of the Kadison-Dubois Theorem.

Theorem 1.2 **[13]** *Suppose K_S is compact and $r^2 - \|\underline{x}\|^2 \in M_S$ for some real number r. Then, for any $f \in \mathbb{R}[\underline{x}]$, $f > 0$ on $K_S \Rightarrow f \in M_S$.*

If K_S is compact then K_S is completely inside some big ball centered at the origin, with radius r say. In this case, we can add $r^2 - \|\underline{x}\|^2$ to our set S without changing K_S. Thus there is no harm in assuming, to begin with, that $r^2 - \|\underline{x}\|^2 \in M_S$.

At the same time, the condition that $r^2 - \|\underline{x}\|^2$ belongs to M_S for sufficiently large r is automatically satisfied in many cases. It is automatically satisfied if M_S is closed under multiplication [14, Corollary 3]. According to results of Jacobi and Prestel [5] it is automatically satisfied in a variety of other cases as well. For example, it is automatically satisfied if $s = |S| \leq 2$ or, more generally, if M_S is 'partially closed

https://doi.org/10.4153/CMB-2003-054-7 Published online by Cambridge University Press

under multiplication' (see [5, Theorem 4.4]) or if $\dim(K_S) = 1$, provided $\mathfrak{a} \subseteq M_S$ where $\mathfrak{a}$ denotes the ideal of polynomials vanishing on K_S.

Corollary 1.3 *If K_S is compact and $r^2 - \|\underline{x}\|^2 \in M_S$ for some real number r, then $f^{\text{sos}} = \bar{f}^{\text{sos}} = f^*$.*

Proof By definition of f^*, $f - f^* \geq 0$ on K_S. Suppose $\epsilon > 0$ is given. Then $f - f^* + \epsilon > 0$ on K_S so, by Putinar's Theorem, $f - f^* + \epsilon \in M_S$, *i.e.*, $f^{\text{sos}} \geq f^* - \epsilon$. The result follows from this, using Proposition 1.1. ∎

Following standard terminology [6] [9] [12] [14] [15] we say that *the moment problem holds for M_S* if for each $L \in \bar{K}_S$ there exists a positive Borel measure μ on K_S such that $\forall f \in \mathbb{R}[\underline{x}]$, $L(f) = \int_{K_S} f \, d\mu$.

Proposition 1.4 *The following are equivalent:*

(1) *The moment problem holds for M_S.*
(2) $\forall f \in \mathbb{R}[\underline{x}]$, $\bar{f}^{\text{sos}} = f^*$.

Proof (1) $\Rightarrow$ (2). Let $f \in \mathbb{R}[\underline{x}]$. Suppose $\lambda \in \mathbb{R}$, $f^* \geq \lambda$. Then $f - \lambda \geq 0$ on K_S so, by (1), for any $L \in \bar{K}_S$, $L(f - \lambda) = \int_{K_S} (f - \lambda) \, d\mu \geq 0$ where μ is a positive Borel measure associated to L, *i.e.*, $L(f) \geq \lambda$. This proves $\bar{f}^{\text{sos}} \geq \lambda$ for any such λ so $\bar{f}^{\text{sos}} \geq f^*$. Thus $\bar{f}^{\text{sos}} = f^*$. (2) $\Rightarrow$ (1). Let $f \in \mathbb{R}[\underline{x}]$, $f \geq 0$ on K_S (so $\bar{f}^{\text{sos}} = f^* \geq 0$). Then, for any $L \in \bar{K}_S$, $L(f) \geq \bar{f}^{\text{sos}} \geq 0$, so by Haviland's Theorem [2] [3], there is a positive Borel measure μ on K_S corresponding to L. ∎

The moment problem holds for M_S in the compact case discussed above, but also in a large number of non-compact cases [6] [9] [12] [15]. At the same time, the moment problem is known to fail for M_S in a great many cases, *e.g.*, in [6, Corollary 3.10] it is shown that it fails whenever K_S contains a 2-dimensional cone.

$\mathbb{R}[\underline{x}]$ comes equipped with its unique finest locally convex topology [1] [12]. The closure of the quadratic module M_S is equal to

$$\bar{M}_S = \bigcap_{L \in \bar{K}_S} \{g \in \mathbb{R}[\underline{x}] \mid L(g) \geq 0\},$$

e.g., see [6, Lemma 3.3].

Proposition 1.5 *M_S closed $\Rightarrow \forall f \in \mathbb{R}[\underline{x}]$, $f^{\text{sos}} = \bar{f}^{\text{sos}}$.*

Proof Suppose $\lambda \in \mathbb{R}$, $\bar{f}^{\text{sos}} \geq \lambda$. Suppose $L \in \bar{K}_S$. Thus $L(f) \geq \lambda$, *i.e.*, $L(f - \lambda) \geq 0$. Since M_S is closed, this implies $f - \lambda \in M_S$ for all such λ, *i.e.*, $f^{\text{sos}} \geq \bar{f}^{\text{sos}}$. ∎

Note: If M_S is closed and $K_S \neq \varnothing$ then 'sup' can be replaced by 'max' in the definition of f^{sos} (provided, of course, that $\{\lambda \in \mathbb{R} \mid f - \lambda \in M_S\} \neq \varnothing$).

In the compact case M_S is almost never closed, *e.g.*, M_S is never closed if $\dim(K_S) \geq 3$ [6, Theorem 3.8], but it is closed in certain non-compact cases [1]

[6] [12]. According to [1, Theorem 3], M_S is closed if $S = \varnothing$. More generally, M_S is closed if K_S contains an n-dimensional cone [6, Theorem 3.5].

Examples 1.6 (1) Suppose $n = 1$. If K_S is compact then $f^* = \bar{f}^{\mathrm{sos}} = f^{\mathrm{sos}}$ by Corollary 1.3 (using [5, Remark 4.7]). The same is true if $S = \varnothing$. In all other cases M_S is closed [6, Theorem 3.5] so $\bar{f}^{\mathrm{sos}} = f^{\mathrm{sos}}$ by Proposition 1.5. $f^* = \bar{f}^{\mathrm{sos}}$ may or may not hold, depending on the presentation of K_S. It follows from [6, Theorem 2.2] that $f^* = \bar{f}^{\mathrm{sos}}$ holds when S contains the standard generators of K_S (up to scaling) and M_S is closed under multiplication. In all other cases $f^* \neq \bar{f}^{\mathrm{sos}}$ for appropriate f.

(2) Take $n = 1$, $S = \{x^3\}$ (so $K_S = [0, \infty)$ and S does not contain the standard generator x). Take $f = x$. Then $f^* = 0$, $\bar{f}^{\mathrm{sos}} = f^{\mathrm{sos}} = -\infty$. Take $f = x^2 + 2ax$, $a > 0$. Then $f^* = 0$, $\bar{f}^{\mathrm{sos}} = f^{\mathrm{sos}} = -a^2$.

(3) Take $n = 1$, $S = \{x^3(x+1)\}$ (so $K_S = (-\infty, -1] \cup [0, \infty)$ but S does not contain the standard generator $x(x+1)$), $f = x(x+1)$. Then $f^* = 0$, $\bar{f}^{\mathrm{sos}} = f^{\mathrm{sos}} = -1/4$.

(4) Take $n = 2$, $S = \varnothing$, $f = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$ (the Motzkin polynomial). Then $f^* = 0$, $\bar{f}^{\mathrm{sos}} = f^{\mathrm{sos}} = -\infty$.

(5) Take $n = 2$, $S = \{x^3(1-x)^3\}$ (so K_S is the infinite strip $[0,1] \times \mathbb{R}$), $f = xy^2$. Then $f^* = \bar{f}^{\mathrm{sos}} = 0$, $f^{\mathrm{sos}} = -\infty$ [6, Ex. 5.2].

(6) Take $n = 2$, $S = \{x^3, 1-x, y^3\}$ (so K_S is the infinite half strip $[0,1] \times [0, \infty)$). Take $f = x(y^2 + ay)$, $a > 0$. Then $f^* = 0$, $\bar{f}^{\mathrm{sos}} = -a^2$, $f^{\mathrm{sos}} = -\infty$.

(7) Take $n = 2$, $S = \{x - 1/2, y - 1/2, 1 - xy\}$ (the Jacobi-Prestel example). Then K_S is compact but $\forall r \in \mathbb{R}$, $r^2 - (x^2 + y^2) \notin M_S$ [5, Ex. 4.6] so Corollary 1.3 does not apply. Take $f = -(x^2 + y^2)$. Then $f^* = -17/4$, $f^{\mathrm{sos}} = -\infty$.

2 Lasserre's Method

The computational method described by Lasserre in [7] [8] involves looking at certain finite dimensional analogs of $\mathbb{R}[\underline{x}]$, M_S, $\bar{K}_S$ and of f^{sos} and $\bar{f}^{\mathrm{sos}}$.

For a fixed positive integer d, denote by P_d the vector space consisting of all polynomials in $\mathbb{R}[\underline{x}]$ of degree $\leq 2d$. P_d is finite dimensional with basis consisting of all monomials $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha_1 + \cdots + \alpha_n \leq 2d$. Define $g_0 = 1$. Let $v_i = \deg(g_i)$, $i = 0, \ldots, s$ (so $v_0 = 0$). Define M_d to be the set of all elements of P_d of the form $\sum_{i=0}^{s} \sigma_i g_i$ where σ_i is a sum of squares of polynomials in $\mathbb{R}[\underline{x}]$ of degree $\leq 2d - v_i$, $i = 0, \ldots, s$. M_d is a subcone of the cone $M_S \cap P_d$ in P_d considered in [6]. Define K_d to be the set of all linear mappings $L\colon P_d \to \mathbb{R}$ satisfying $L(1) = 1$ and $L(M_d) \geq 0$. Finally, for $f \in \mathbb{R}[\underline{x}]$ of degree $\leq 2d$, define

$$f_{(d)} = \sup\{\lambda \in \mathbb{R} \mid f - \lambda \in M_d\}, \quad \bar{f}_{(d)} = \inf\{L(f) \mid L \in K_d\}.$$

It follows from results in [7] [8] that computation of $f_{(d)}$ is a semidefinite programming problem and that computation of $\bar{f}_{(d)}$ is the dual problem. See [7] [8] for details. Also see Section 4 below. In the case $S = \varnothing$ (global optimization) the semidefinite program simplifies; see [7] [10] [11] and Proposition 2.3 below.

Proposition 2.1

(1) $f_{(d)} \le \bar{f}_{(d)}$.
(2) *The sequences* $\{f_{(d)}\}$, $\{\bar{f}_{(d)}\}$ *are increasing.*
(3) $\lim_{d\to\infty} f_{(d)} = f^{\text{sos}}$.
(4) $\bar{f}_{(d)} \le \bar{f}^{\text{sos}}$.

Proof (1) If $f - \lambda \in M_d$ and $L \in K_d$ then $L(f - \lambda) \ge 0$, *i.e.*, $L(f) \ge \lambda$. It follows that $\bar{f}_{(d)} \ge \lambda$ for any such λ so $\bar{f}_{(d)} \ge f_{(d)}$.

(2) If $d_1 \le d_2$ then $M_{d_1} \subseteq M_{d_2}$ and the restriction map $L \mapsto L|_{P_{d_1}}$ maps K_{d_2} into K_{d_1}. The inequalities $f_{d_1} \le f_{d_2}$ and $\bar{f}_{d_1} \le \bar{f}_{d_2}$ are clear from these two facts.

(3) Clear from $M_S = \bigcup_{d\ge 1} M_d$.

(4) Clear: The restriction map $L \mapsto L|_{P_d}$ maps $\bar{K}_S$ into K_d. ∎

Corollary 2.2 [7] [**8**] *If* K_S *is compact and* $r^2 - \|\underline{x}\|^2 \in M_S$ *for some real number* r*, then*

$$\lim_{d\to\infty} f_{(d)} = \lim_{d\to\infty} \bar{f}_{(d)} = f^*.$$

Proof Combine Corollary 1.3 with Proposition 2.1. ∎

A major shortcoming of Lasserre's method is the lack of control over the degree, *i.e.*, how large does one have to take d, in general, for $f_{(d)}$ and $\bar{f}_{(d)}$ to be close to f^*? A major positive feature of Lasserre's method is that even if $f_{(d)}$ and $\bar{f}_{(d)}$ are not close to f^*, they do provide reliable lower bounds for f^*. From a practical point of view this can be useful.

Unfortunately, the only cases where bounds on the degree are known are cases where K_S is not compact. For example, we have the following:

Proposition 2.3 *If* K_S *contains an n-dimensional cone then* $f_{(d)} = \bar{f}_{(d)} = f^{\text{sos}} = \bar{f}^{\text{sos}}$ *whenever* $2d \ge \deg(f)$.

Note: This applies in particular in the case $S = \varnothing$ (global optimization).

Proof If $f - \lambda \in M_S$ and $2d \ge \deg(f)$ then, by degree considerations (see the proof of [6, Theorem 3.5]), $f - \lambda$ has a presentation $f - \lambda = \sigma_0 g_0 + \cdots + \sigma_s g_s$, σ_i a sum of squares of degree $\le 2d - v_i$, $i = 0, \dots, s$. This implies $f_{(d)} \ge f^{\text{sos}}$. Since M_S is closed, the rest is clear. ∎

3 A Refinement of Lasserre's Method

Denote by $\operatorname{int}(K_S)$ the interior of K_S in $\mathbb{R}^n$ in the Euclidean topology. The new results in this section are all in the case $\operatorname{int}(K_S) = \varnothing$. The case $\operatorname{int}(K_S) \ne \varnothing$ is already covered in [7] [8].

Fix an ideal $\mathfrak{a}$ in $\mathbb{R}[\underline{x}]$ consisting of polynomials which vanish on K_S. Then $K_S \subseteq V$ where $V \subseteq \mathbb{R}^n$ denotes the zero set of $\mathfrak{a}$. If $\operatorname{int}(K_S) \ne \varnothing$, then necessarily $\mathfrak{a} = \{0\}$ and $V = \mathbb{R}^n$.

https://doi.org/10.4153/CMB-2003-054-7 Published online by Cambridge University Press

To keep the notation as simple as possible we assume always that $\mathfrak{a} \subseteq M_S$. If this is not the case to begin with, it can be achieved simply by adding the elements $h_1, -h_1, \dots, h_t, -h_t$ to S where $h_1, \dots, h_t$ is a set of generators for the ideal $\mathfrak{a}$, using the identity

$$\sum r_i h_i = \sum \frac{1}{4}\left((r_i+1)^2 - (r_i-1)^2\right) h_i = \sum \frac{1}{4}(r_i+1)^2 h_i + \sum \frac{1}{4}(r_i-1)^2(-h_i).$$

Of course, adding $h_1, -h_1, \dots, h_t, -h_t$ to S does not change K_S.

We define new objects P'_d, M'_d, K'_d, $f'_{(d)}$ and $\bar{f}'_{(d)}$ (depending on $\mathfrak{a}$) which are in some sense more appropriate than P_d, M_d, K_d, $f_{(d)}$ and $\bar{f}_{(d)}$. In case $\mathfrak{a} = \{0\}$ these coincide with the objects P_d, M_d, K_d, $f_{(d)}$ and $\bar{f}_{(d)}$ defined in Section 2.

Denote by P'_d the image of P_d in $\mathbb{R}[\underline{x}]/\mathfrak{a}$ and by M'_d the image of M_d in $\mathbb{R}[\underline{x}]/\mathfrak{a}$. Denote by K'_d the set of all linear maps $L\colon P'_d \to \mathbb{R}$ satisfying $L(1) = 1$ and $L(M'_d) \geq 0$. For $f \in \mathbb{R}[\underline{x}]$, denote by f' the image of f in $\mathbb{R}[\underline{x}]/\mathfrak{a}$, *i.e.*, $f' = f + \mathfrak{a}$. For $f \in \mathbb{R}[\underline{x}]$ of degree $\leq 2d$ define

$$f'_{(d)} = \sup\{\lambda \in \mathbb{R} \mid (f-\lambda)' \in M'_d\}, \quad \bar{f}'_{(d)} = \inf\{L(f') \mid L \in K'_d\}.$$

Parts (1)–(4) of Proposition 2.1 carry over immediately with $f_{(d)}, \bar{f}_{(d)}$ replaced by $f'_{(d)}, \bar{f}'_{(d)}$ (assuming $\mathfrak{a} \subseteq M_S$). Using the natural surjection $M_d \to M'_d$ and the natural injection $K'_d \hookrightarrow K_d$ we see that $f'_{(d)} \geq f_{(d)}$ and $\bar{f}'_{(d)} \geq \bar{f}_{(d)}$.

For the remainder of the section we assume that $\mathfrak{a}$ is the ideal of all polynomials vanishing on K_S. We also assume that S has been adjusted, if necessary, so that $\mathfrak{a} \subseteq M_S$.

Theorem 3.1

(1) *M'_d is closed in P'_d (in the Euclidean topology).*
(2) *$M'_d = \bigcap_{L \in K'_d} \{f' \in P'_d \mid L(f') \geq 0\}$.*

Proof If $K_S = \varnothing$ then $\mathfrak{a} = (1)$, $M'_d = P'_d = \{0\}$ and $K'_d = \varnothing$. The result is clear in this case. Thus we may assume $K_S \neq \varnothing$.

We prove M'_d is closed in P'_d by a simple modification of the proof of [12, Proposition 2.6]. Choose a set of monomials x^α, $\alpha \in \Lambda(d)$ whose cosets modulo $\mathfrak{a}$ form a basis for P'_d. For later use we insist that $0 \in \Lambda(d)$. Let

$$I := \left\{ i \mid i \in \{0, \dots, s\}, v_i \leq 2d, g'_i \neq 0 \right\}.$$

Clearly $0 \in I$. For $i \in I$, let

$$Q_i := \{g' \mid g \in \mathbb{R}[\underline{x}] \text{ has degree } \leq d - v_i/2\}.$$

Choose a set of monomials x^α, $\alpha \in \Lambda_i$ of degree $\leq d - v_i/2$ whose cosets modulo $\mathfrak{a}$ form a basis of Q_i modulo $Q_i \cap \operatorname{Ann}(g'_i)$. Each element of M_d is represented modulo $\mathfrak{a}$ by a sum of the form

$$\sum_{i \in I} \sum_{j=1}^{\ell_i} h_{ij}^2 g_i$$

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where h_{ij} is some linear combination of the monomials x^α, $\alpha \in \Lambda_i$. Using the Gram matrix description of sums of squares, we can (and we do) choose $\ell_i = |\Lambda_i|$. Consider the map

$$\Phi\colon \prod_{i\in I}\Big(\frac{Q_i}{Q_i \cap \operatorname{Ann}(g_i')}\Big)^{\ell_i} \to P_d'$$

defined by

$$(h_{i1},\dots,h_{i\ell_i})_{i\in I} \mapsto \sum_{i\in I}\sum_{j=1}^{\ell_i} h_{ij}^2 g_i.$$

Coordinatizing in terms of the coefficients, one checks that $\Phi\colon \mathbb{R}^M \to \mathbb{R}^N$ where $M = \sum_i \ell_i^2$, $N = |\Lambda(d)|$ is homogeneous of degree 2, *i.e.*, $\Phi(z) = \big(\Phi_1(z),\dots,\Phi_N(z)\big)$ where each Φ_i is homogeneous of degree 2. As in the proof of [12, Proposition 2.6] we see that $\Phi^{-1}(0) = 0$. By [12, Lemma 2.7] Φ is a closed map (even a proper map). In particular, $\Phi(R^M)$ is closed. Since $\Phi(R^M)$ is identified with M_d' this proves that M_d' is closed in P_d'.

The rest of the proof is standard. Suppose $f \in p_d'$, $f \notin M_d'$. We want to construct $L \in K_d'$ such that $L(f) < 0$. Consider the hyperplane H in P_d' through f_0 perpendicular to $f - f_0$ where f_0 is a point on M_d' closest to f. Since M_d' is convex f_0 is unique and M_d' lies on the opposite side of H from f. Since M_d' is a cone, $0 \in H$. Thus we have a linear map $L_0\colon P_d' \to \mathbb{R}$ with $L_0(f) < 0$, $L_0(M_d') \geq 0$. If $L_0(1) \neq 0$ then $L_0(1) > 0$ and we take $L = \lambda L_0$ for suitable $\lambda > 0$. Suppose $L_0(1) = 0$. In this case fix $L_1 \in K_d'$ (*e.g.*, fix a point $p_1 \in K_S$ and define $L_1 \in K_d'$ by $L_1(g) := g(p_1)$) and take $L = L_1 + \lambda L_0$, λ sufficiently large. ■

Corollary 3.2 *If $f \in \mathbb{R}[\underline{x}]$ has degree $\leq 2d$ then $\bar f_{(d)}' = f_{(d)}'$.*

Proof For $\lambda \in \mathbb{R}$, there are only two possibilities: If $f' - \lambda \in M_d'$ then $f_{(d)}' \geq \lambda$. If $f' - \lambda \notin M_{(d)}'$ then, by Theorem 3.1, there exists $L \in K_{(d)}'$ such that $L(f' - \lambda) < 0$, *i.e.*, $L(f') < \lambda$, so $\bar f_{(d)}' < \lambda$. Coupled with the fact that $f_{(d)}' \leq \bar f_{(d)}'$, this implies $f_{(d)}' = \bar f_{(d)}'$. ■

Notes 3.3 (1) In [7] [8] this same result is proved, but only in the case $\operatorname{int}(K_S) \neq \varnothing$.

(2) Theorem 3.1 implies that 'sup' can be replaced by 'max' in the definition of $f_{(d)}'$ (provided, of course, that $\{\lambda \in \mathbb{R} \mid (f-\lambda)' \in M_d'\} \neq \varnothing$).

(3) Suppose $f^{\text{sos}} \in \mathbb{R}$. If $f - f^{\text{sos}} \in M_S$ then $f_{(d)}' = f^{\text{sos}}$ for d sufficiently large. Conversely, if $f_{(d)}' = f^{\text{sos}}$ then $f - f^{\text{sos}} \in M_S$.

We also note the following strengthening of Proposition 2.1.

Corollary 3.4

(1) *For any $f \in \mathbb{R}[\underline{x}]$ of degree $\leq 2d$, $\bar f_{(d)} \leq f^{\text{sos}}$.*
(2) $\lim_{d\to\infty} f_{(d)} = \lim_{d\to\infty} \bar f_{(d)} = f^{\text{sos}}$.

Proof (1) Since $\bar f_{(d)} \leq \bar f_{(d)}'$ and $f_{(d)}' \leq f^{\text{sos}}$ this is immediate from Corollary 3.2.

(2) Combine (1) and Proposition 2.1. ■

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4 Computation of $f'_{(d)}$ and $\bar{f}'_{(d)}$

We indicate briefly how $f'_{(d)}$ and $\bar{f}'_{(d)}$ can be computed. In case $\mathfrak{a} = \{0\}$ this is precisely the computation of $f_{(d)}$ and $\bar{f}_{(d)}$ described in [7] [8]. We use notation from the proof of Theorem 3.1. P'_d is identified with $\mathbb{R}^N$ where $N = |\Lambda(d)|$. For each $i \in I$, $\ell_i := |\Lambda_i|$. Denote by $\mathbb{S}^{\ell_i}$ the vector space of symmetric $\ell_i \times \ell_i$ matrices and define the linear map $\Psi\colon \prod_{i\in I} \mathbb{S}^{\ell_i} \to P'_d$ by

$$A = (A^{(i)})_{i\in I} \mapsto \sum_{i\in I} \sum_{\alpha,\beta\in\Lambda_i} x^\alpha A^{(i)}_{\alpha\beta} x^\beta g_i.$$

An element f' in P'_d belongs to M'_d iff $f' = \Psi(A)$ with each $A^{(i)}$, $i \in I$ positive semidefinite (PSD for short). Thus to compute f'_d we must maximize $\lambda \in \mathbb{R}$ subject to the constraint

$$f' - \lambda = \Psi(A) \text{ and each } A^{(i)}, i \in I \text{ is PSD}.$$

Decompose $P'_d = \mathbb{R}^N$ as $\mathbb{R} \times \mathbb{R}^{N-1}$ and let $\Psi = (\Psi_0, -\Psi_1)$ be the corresponding decomposition of Ψ. Thus $\Psi_0(A) = a_0 - \lambda$ where a_0 is the coefficient of x^0 in f'. Consequently, to compute $f'_{(d)}$ we must

$$(1) \qquad \begin{cases} \text{minimize: } \Psi_0(A) \\ \text{subject to the constraints: } \Psi_1(A) = p \text{ and each } A^{(i)},\ i \in I \text{ is PSD} \end{cases}$$

where p is the projection of $-f'$ onto $\mathbb{R}^{N-1}$. This is a semidefinite programming problem [18].

Consider the dual map $\Psi^*\colon P'^*_d \to (\prod_{i\in I} \mathbb{S}^{\ell_i})^*$. $(\prod_{i\in I} \mathbb{S}^{\ell_i})^*$ is identified with $\prod_{i\in I} \mathbb{S}^{\ell_i}$ via the scalar product $\langle A, B\rangle := \sum_{i\in I} \operatorname{Tr}(A^{(i)}B^{(i)})$. $L \in P'^*_d$ belongs to K'_d iff $L(1) = 1$ and $\Psi^*(L)$ is PSD. Computing $\bar{f}_{(d)}$ amounts to minimizing $L(f)$ subject to the constraints that $L(1) = 1$ and $\Psi^*(L)$ is PSD. Coordinatizing P'^*_d using the dual basis and decomposing L as $L = (1, y)$, $y \in \mathbb{R}^{N-1}$ we see that $a_0 - L(f) = y^T p$ and $\Psi^*(L) = \Psi^*_0(1) - \Psi^*_1(y)$. Consequently, to compute $\bar{f}'_{(d)}$ we must

$$(2) \qquad \begin{cases} \text{maximize: } y^T p \\ \text{subject to the constraint: } \Psi^*_0(1) - \Psi^*_1(y) \text{ is PSD}. \end{cases}$$

This is the dual problem to (1) [18].[1]

The computation can be implemented on a computer if a Gröbner basis for $\mathfrak{a}$ is known. In doing this it would seem that there are important advantages in choosing $\mathfrak{a}$ as large as possible. Not only would one expect the approximations $f'_{(d)}$ and $\bar{f}'_{(d)}$ to be better but also the matrix size is reduced, allowing one to attempt previously inaccessible problems.

[1] Added March 15, 2003. In recent independent work this refinement of Lasserre's algorithm is considered in more detail in cases where the algebra $\mathbb{R}[\underline{x}]/\mathfrak{a}$ is zero dimensional. See the preprint 'Semidefinite representation for finite varieties' by M. Laurent.

It is also worth noting the relationship between the map Ψ and the map

$$\Phi\colon \prod_{i\in I}\Big(\frac{Q_i}{Q_i\cap \operatorname{Ann}(g_i')}\Big)^{\ell_i} \to P_d'$$

in the proof of Theorem 3.1. Instead of identifying $\prod_{i\in I}(\frac{Q_i}{Q_i\cap \operatorname{Ann}(g_i')})^{\ell_i}$ with $\mathbb{R}^M$ where $M=\sum_{i\in I}\ell_i^2$, we can identify it with $\prod_{i\in I}\mathbb{R}^{\ell_i\times\ell_i}$. Φ is just the composite map

$$\prod_{i\in I}\Big(\frac{Q_i}{Q_i\cap \operatorname{Ann}(g_i')}\Big)^{\ell_i} \to \prod_{i\in I}\mathbb{R}^{\ell_i\times\ell_i} \to \prod_{i\in I}\mathbb{S}^{\ell_i} \xrightarrow{\Psi} P_d'$$

where the middle map is given by $(B_i)_{i\in I}\mapsto (B_i^TB_i)_{i\in I}$.

5 Stability Assumptions

Suppose $S=\{g_1,\dots,g_s\}$, $g_i\in\mathbb{R}[\underline{x}]$, $\deg(g_i)=\nu_i$, $i=1,\dots,s$. We say K_S is *stably compact* if K_S remains compact for all sufficiently small perturbations of the coefficients of the $g_1,\dots,g_s$, *i.e.*, if there exists real $\epsilon>0$ such that for all polynomials $\delta_1,\dots,\delta_s\in\mathbb{R}[\underline{x}]$, if $\deg(\delta_i)\le\nu_i$ and the coefficients of δ_i are $\le\epsilon$ in absolute value for each i, then $K_{\{g_1+\delta_1,\dots,g_s+\delta_s\}}$ is compact. We say $f\in\mathbb{R}[\underline{x}]$, $\deg(f)=\nu$, is *stably bounded from below* on K_S if f remains bounded from below on K_S for all sufficiently small perturbations of the coefficients of $f,g_1,\dots,g_s$, *i.e.*, if there exists real $\epsilon>0$ such that for all $\delta,\delta_1,\dots,\delta_s\in\mathbb{R}[\underline{x}]$, if $\deg(\delta)\le\nu$ and $\deg(\delta_i)\le\nu_i$ and the coefficients of δ and the δ_i are $\le\epsilon$ in absolute value, then $f+\delta$ is bounded below on $K_{\{g_1+\delta_1,\dots,g_s+\delta_s\}}$. These two concepts are closely related; see Theorem 5.3 below.

Suppose $f\in\mathbb{R}[\underline{x}]$. A presentation

$$f-\lambda=\sigma_0+\sigma_1g_1+\cdots+\sigma_sg_s,$$

σ_i a sum of squares in $\mathbb{R}[\underline{x}]$, $i=0,\dots,s$, witnesses the fact that λ is a lower bound for f on K_S. In practice, because semidefinite programming computations are done using floating point arithmetic, and also because the f and $g_1,\dots,g_s$ may not be known exactly, there is an error term e:

$$f-\lambda=\sigma_0+\sigma_1g_1+\cdots+\sigma_sg_s+e.$$

If f attains its minimum value at p^*, all we can safely say is that $\lambda+e(p^*)$ is a lower bound for f on K_S. If we have no a priori knowledge of p^* then this is not satisfactory. Even if the individual coefficients of e are small, if $\|p^*\|$ is large, $|e(p^*)|$ could be large.

If K_S is stably compact then the situation is better, for then we have an upper bound r_ϵ for $\|p^*\|$ given by Theorem 5.1 (2) below and, consequently, we also have an upper bound for $|e(p^*)|$. We will show that this remains true whenever f is stably bounded from below on K_S; see Corollary 5.4 below.

We begin by considering stable compactness. Stable compactness is easier to check than compactness. Let $b_{i\gamma}$ denote the coefficient of x^γ in g_i. Decompose g_i as $g_i=$

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$\sum_{j=0}^{\nu_i} g_{ij}$ where $g_{ij} = \sum_{|\gamma|=j} b_{i\gamma} x^\gamma$, the homogeneous component of g_i of degree j. By definition, $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

Theorem 5.1

(1) *K_S is stably compact if and only if the function* $\max\{-g_{1\nu_1}, \ldots, -g_{s\nu_s}\}$ *is strictly positive on the unit sphere.*
(2) *If $\epsilon > 0$ is a lower bound for the function* $\max\{-g_{1\nu_1}, \ldots, -g_{s\nu_s}\}$ *on the unit sphere, then K_S lies in the ball centered at the origin with radius*

$$r_\epsilon = \max\Big\{ 1, \sum_{|\gamma|<\nu_i} |b_{i\gamma}|/\epsilon : i = 1, \ldots, s \Big\}.$$

Notes 5.2 (1) The computation of a lower bound $\epsilon > 0$ for $\max\{-g_{1\nu_1}, \ldots, -g_{s\nu_s}\}$ on the unit sphere is itself a problem of polynomial optimization on a compact semi-algebraic set: Just take $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_s\}$ where, for each $i \in \{1, \ldots, s\}$, ϵ_i is a positive lower bound for $-g_{i\nu_i}$ on the unit sphere subject to the constraints $g_{j\nu_j} - g_{i\nu_i} \geq 0$ for $j \neq i$. There are obvious problems with this if s is too large.

(2) One way to ensure stable compactness is to include $r^2 - \|\underline{x}\|^2$ in the set S to begin with for some $r \in \mathbb{R}$.

Proof of Theorem 5.1 Assume K_S is stably compact. For p on the unit sphere (*i.e.*, $\|p\| = 1$) consider the one variable polynomials

$$g_i(tp) = g_{i0} + g_{i1}(p)t + \cdots + g_{i\nu_i}(p)t^{\nu_i},$$

$i = 1, \ldots, s$ and the corresponding intersection of K_S with the half line $\{tp : t \in \mathbb{R}, t \geq 0\}$. Let $\bar{S} = \{\bar{g}_1, \ldots, \bar{g}_s\}$, $\bar{g}_i = g_i + \epsilon(\delta_1 x_1 + \cdots + \delta_n x_n)^{\nu_i}$, where

$$\delta_i := \begin{cases} 1 & \text{if } p_i \geq 0 \\ -1 & \text{if } p_i < 0 \end{cases}.$$

Then

$$\bar{g}_{i\nu_i}(p) = g_{i\nu_i}(p) + \epsilon(|p_1| + \cdots + |p_n|)^{\nu_i} \geq g_{i\nu_i}(p) + \epsilon\|p\|^{\nu_i} = g_{i\nu_i}(p) + \epsilon.$$

Choose ϵ sufficiently small so that the set $K_{\bar{S}}$ is compact. Then the intersection of $K_{\bar{S}}$ with the half line $\{tp : t \in \mathbb{R}, t \geq 0\}$ is compact so $\bar{g}_{i\nu_i}(p) \leq 0$ for some i, *i.e.*, $g_{i\nu_i}(p) \leq -\epsilon$ for some i. This proves that the function $\max\{-g_{1\nu_1}, \ldots, -g_{s\nu_s}\}$ is strictly positive on the unit sphere. Conversely, if $\max\{-g_{1\nu_1}, \ldots, -g_{s\nu_s}\}$ is strictly positive on the unit sphere, then using the fact that $f \mapsto f(p)$ is a continuous function of the coefficients, this will remain true for any sufficiently small perturbation of the coefficients of the g_i. Thus to complete the proof of (1) it suffices to prove assertion (2).

To prove (2), we make use of the standard fact that the real roots of a polynomial $t^n + a_1 t^{n-1} + \cdots + a_n$ are bounded by $\max\{1, \sum_{i=1}^n |a_i|\}$. Fix p on the unit sphere,

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fix i such that $g_{iv_i}(p) \leq -\epsilon$, and consider the largest non-negative root of $g_i(tp)$ (assuming it has a non-negative root). For $t \geq 0$, $t >$ this largest root, $g_i(tp)$ is strictly negative, so tp is not in K_S. We know that

$$\max\left\{1, \sum_{j=0}^{v_i-1} \frac{|g_{ij}(p)|}{|g_{iv_i}(p)|}\right\}$$

is an upper bound for this largest non-negative root. Now $|g_{iv_i}(p)| \geq \epsilon$. Also, for $j < v_i$, $|g_{ij}(p)| = |\sum_{|\gamma|=j} b_{i\gamma} p^\gamma| \leq \sum_{|\gamma|=j} |b_{i\gamma}||p^\gamma| \leq \sum_{|\gamma|=j} |b_{i\gamma}|$. Thus $\sum_{j=0}^{v_i-1} |g_{ij}(p)| \leq \sum_{|\gamma|<v_i} |b_{i\gamma}|$, and the points of K_S on the half line $\{tp : t \in \mathbb{R}, t \geq 0\}$ are contained in the interval $tp : 0 \leq t \leq \max\{1, \sum_{|\gamma|<v_i} |b_{i\gamma}|/\epsilon\}$. Letting p vary now on the unit sphere we see that K_S is contained in the ball centered at the origin with radius $r_\epsilon = \max\{1, \sum_{|\gamma|<v_i} |b_{i\gamma}|/\epsilon : i = 1, \dots, s\}$. ∎

Theorem 5.3 *The following are equivalent:*

(1) *f is stably bounded from below on K_S.*
(2) *$K_{S\cup\{-f\}}$ is stably compact.*

Proof (1) $\Rightarrow$ (2). Fix p on the unit sphere. As in the proof of Theorem 5.1, let $\bar{S} = \{\bar{g}_1, \dots, \bar{g}_s\}$, $\bar{g}_i = g_i + \epsilon(\delta_1 x_1 + \cdots + \delta_n x_n)^{v_i}$, where

$$\delta_i := \begin{cases} 1 & \text{if } p_i \geq 0 \\ -1 & \text{if } p_i < 0 \end{cases}.$$

Also, let $f = \sum_{j=0}^{v} f_j$ be the decomposition of f into homogeneous parts, and let $\bar{f} = f - \epsilon(\delta_1 x_1 + \cdots + \delta_n x_n)^v$. Choose $\epsilon > 0$ so small that $\bar{f}$ is stably bounded on $K_{\bar{S}}$. Then, looking at the intersection of $K_{\bar{S}}$ with the half line $\{tp : t \geq 0\}$, we see that either $\bar{f}_v(p) \geq 0$ or $\bar{g}_{iv_i}(p) \leq 0$ for some $i \in \{1, \dots, s\}$, *i.e.*, either $f_v(p) \geq \epsilon$ or $g_{iv_i}(p) \leq -\epsilon$ for some $i \in \{1, \dots, s\}$. It follows from Theorem 5.1 (1) that $K_{S\cup\{-f\}}$ is stably compact.

(2) $\Rightarrow$ (1). If $K_{S\cup\{-f\}}$ is stably compact then this remains the case for small perturbations of the coefficients of $f, g_1, \dots, s$. Thus it suffices to show that $K_{S\cup\{-f\}}$ compact implies f is bounded below on K_S. But this is clear. On $K_{S\cup\{-f\}}$ f is bounded below by some λ. On $K_{S\cup\{f\}}$, f is bounded below by 0. Thus, on K_S, f is bounded below by the minimum of λ and 0. ∎

Corollary 5.4 *Suppose f is stably bounded from below on K_S with $K_S \neq \varnothing$. Fix a lower bound $\epsilon > 0$ for $\max\{f_v, -g_{1v_1}, \dots, -g_{sv_s}\}$ on the unit sphere. Normalize so that $0 \in K_S$ and $f(0) = 0$. Denote by a_γ the coefficient of x^γ in f. Then minimizing f on K_S is equivalent to minimizing f on the compact set $K_{S\cup\{\rho_\epsilon^2 - \|\underline{x}\|^2\}}$ where*

$$\rho_\epsilon = \max\left\{1, \sum_{|\gamma|<v} |a_\gamma|/\epsilon, \sum_{|\gamma|<v_i} |b_{i\gamma}|/\epsilon : i = 1, \dots, s\right\}.$$

Proof According to Theorem 5.1(2), $K_{S\cup\{-f\}}$ lies in the closed ball centered at the origin with radius ρ_ϵ. ∎

https://doi.org/10.4153/CMB-2003-054-7 Published online by Cambridge University Press

Conclusions 5.5 (1) One can proceed as follows: First compute $\epsilon > 0$, a positive lower bound for the function $\max\{f_\nu, -g_{1\nu_1}, \dots, -g_{s\nu_s}\}$ on the unit sphere. This serves to confirm that f is indeed stably bounded from below on K_S. Then minimize f on the compact set $K_{S\cup\{\rho_\epsilon^2 - \|\underline{x}\|^2\}}$. In this way, it is possible to apply Lasserre's method with ensured exact results even in cases where K_S is not compact.

(2) The method applies in particular to global optimization, *i.e.*, $S = \varnothing$: First compute a positive lower bound ϵ for f_ν on the unit sphere. This serves to confirm that f is indeed stably bounded from below on $\mathbb{R}^n$. Then minimize f on the closed ball centered at the origin with radius

$$\rho_\epsilon = \max\Big\{ 1, \sum_{|\gamma|<\nu} |a_\gamma|/\epsilon \Big\}.$$

(3) In the test examples considered by Parrilo and Sturmfels in [11], f is stably bounded below on $\mathbb{R}^n$. In all these examples exact results are obtained without make the reduction to the compact case described above. This raises the question of when such reduction is actually necessary.

(4) In cases where f is not stably bounded from below on K_S, any procedure for approximating f^* using floating point computations involving the coefficients is necessarily somewhat suspect.

References

[1] C. Berg, J. P. R. Christensen and C. U. Jenson, *A remark on the multidimensional moment problem*. Math. Ann. **243**(1979), 163–169.

[2] E. K. Haviland, *On the momentum problem for distribution functions in more than one dimension*. Amer. J. Math. **57**(1935), 562–572.

[3] ———, *On the momentum problem for distribution functions in more than one dimension II*. Amer. J. Math. **58**(1936), 164–168.

[4] T. Jacobi, *A representation theorem for certain partially ordered commutative rings*. Math. Z. **23**(2001).

[5] T. Jacobi and A. Prestel, *Distinguished representations of strictly positive polynomials*. J. Reine Angew. Math. **532**(2001), 223–235.

[6] S. Kuhlmann and M. Marshall, *Positivity, sums of squares and the multidimensional moment problem*. Trans. Amer. Math. Soc. **354**(2002), 4285–4301.

[7] J. B. Lasserre, *Optimization globale et théorie des moments*. C. R. Acad. Sci. Paris Sér. I Math. **331**(2000), 929–934.

[8] ———, *Global optimization with polynomials and the problem of moments*. SIAM J. Optim. (3) **11**(2000/01), 796–817 (electronic).

[9] M. Marshall, *Approximating positive polynomials using sums of squares*. to appear.

[10] P. A. Parrilo, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. Ph.D. thesis, California Institute of Technology, May, 2000.

[11] P. A. Parrilo and B. Sturmfels, *Minimizing polynomial functions*. DIMACS Series in Discrete Math. Theor. Comput. Sci., to appear.

[12] V. Powers and C. Scheiderer, *The moment problem for non-compact semialgebraic sets*. Adv. in Geom. **1**(2001), 71–88.

[13] M. Putinar, *Positive polynomials on compact semialgebraic sets*. Indiana Univ. Math. J. (3) **43**(1993), 969–984.

[14] K. Schmüdgen, *The K-moment problem for compact semialgebraic sets*. Math. Ann. **289**(1991), 203–206.

[15] ———, *On the moment problem of closed semi-algebraic sets*. J. Reine Angew. Math. **558**(2003), 225–234.

[16] N. Z. Shor, *A class of global minimum bounds of polynomial functions*. Cybernetics (6) **23**(1987), 731–734.

[17] N. Z. Shor and P. I. Stetsyuk, *The use of a modification of the r-algorithm for finding the global minimum of polynomial functions*. Cybernet. Systems Anal. **33**(1997), 482–497.

[18] H. Wolkowicz, R. Saigal and L. Vandenberghe (eds), *Handbook of Semidefinite programming*. Kluwer, 2000.

Algebra and Logic Research Unit
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 5E6