Compositio Math. 142 (2006) 529–540
doi:10.1112/S0010437X05001764

On the Poincaré inequality for one-dimensional foliations

Vincent Cavalier and Daniel Lehmann

Abstract

Let d be the degree of an algebraic one-dimensional foliation $\mathcal{F}$ on the complex projective space $\mathbb{P}_n$ (i.e. the degree of the variety of tangencies of the foliation with a generic hyperplane). Let Γ be an algebraic solution of degree δ, and geometrical genus g. We prove, in particular, the inequality $(d-1)\delta + 2 - 2g \geqslant \mathcal{B}(\Gamma)$, where $\mathcal{B}(\Gamma)$ denotes the total number of locally irreducible branches through singular points of Γ when Γ has singularities, and $\mathcal{B}(\Gamma) = 1$ (instead of 0) when Γ is smooth. Equivalently, when $\Gamma = \bigcap_{\lambda=1}^{n-1} S_\lambda$ is the complete intersection of $n-1$ algebraic hypersurfaces S_λ, we get $(d+n-\sum_{\lambda=1}^{n-1}\delta_\lambda)\delta \geqslant \mathcal{B}(\Gamma) - \mathcal{E}(\Gamma)$, where δ_λ denotes the degree of S_λ and $\mathcal{E}(\Gamma) = 2 - 2g + (\sum_\lambda \delta_\lambda - (n+1))\delta$ the correction term in the genus formula. These results are also refined when Γ is reducible.

1. Introduction

In connection with the existence of first integrals, Poincaré raised the question of bounding the degree δ of an algebraic solution Γ for an algebraic differential system $\mathcal{F}$ on the complex projective plane $\mathbb{P}_2$, in terms of the degree d of $\mathcal{F}$. This is not possible without further conditions on $\mathcal{F}$ or on Γ. For example, Lins Neto proved in [Lin02] that the problem has no solution in the presence of dicritical singularities, i.e. of singularities through which there are infinitely many germs of separatrices (see Example 5.6 below).

In fact, the inequality $d + 2 - \delta \geqslant 0$ has been proved by Cerveau and Lins Neto [CL91] (see also [Soa01]) when Γ has only nodal singularities, and by Carnicer [Car94] when the foliation has no dicritical singularity. Moreover, Brunella [Bru97] recovered Carnicer's result by observing that the negativity of the GSV-indices (see [GSV91]) is an obstruction to the above inequality, and proving that these indices are always non-negative in the non-dicritical case. Carnicer and Campillo [CC97] proved also that there exists some non-negative integer a, depending on conditions imposed on $\mathcal{F}$ or on Γ, such that $d + 2 - \delta \geqslant -a$.

In higher dimension (i.e. for one-dimensional algebraic foliations on the complex projective space $\mathbb{P}_n$ leaving invariant an algebraic curve Γ), the inequality $(d+n-\sum_{\lambda=1}^{n-1}\delta_\lambda) \geqslant 1$ has been proved by Soares [Soa00], when Γ is the complete intersection $\bigcap_{\lambda=1}^{n-1} S_\lambda$ of $n-1$ algebraic hypersurfaces S_λ of degree δ_λ, under the further conditions that Γ be smooth, and the restriction of the foliation to Γ has non-degenerate singularities. More generally, in [Soa97, Soa00] he gave a lower bound for the degree of the algebraic foliations leaving invariant a smooth submanifold of $\mathbb{P}_n$, under conditions of non-degeneracy of the foliation. Also, Esteves and Kleiman [EK03] proved the inequality $(d-1)(\delta-1) - 2g \geqslant 1 - r(\Gamma)$, $r(\Gamma)$ denoting the number of globally irreducible components.

In this paper, we consider the case of curves with any kind of singularity, in any dimension. Let $\mathcal{F}$ be the one-dimensional holomorphic foliation on a holomorphic manifold M defined by a

Received 22 November 2004, accepted in final form 18 May 2005.
2000 Mathematics Subject Classification 57R20, 57R25, 19E20.
Keywords: degree of an algebraic foliation, Poincaré inequality, genus formula, GSV-index, residues.
This journal is © Foundation Compositio Mathematica 2006.

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

morphism $\ell : \mathcal{L} \to TM$ of holomorphic vector bundles from a line bundle $\mathcal{L}$ into TM. Let Γ be a curve in M (i.e. a complex analytic set of pure complex dimension one) invariant by $\mathcal{F}$. Assume that Γ is compact and connected. The normal bundle N_{Γ_0} to the non-singular part Γ_0 of Γ in M has a stable class which always admits a natural extension $[N_\Gamma]$ in the Grothendieck group $K^0(\Gamma)$. If Γ is, moreover, a locally complete intersection (LCI) in M, $[N_\Gamma]$ may even be realized as the stable class of a natural bundle N_Γ which is a natural extension of N_{Γ_0} to all of Γ. (See, for instance, [LS95, LSS95] for the LCI case, and [CLS04] in general). Denote by $\Sigma = \mathrm{Sing}\Gamma \cup (\mathrm{Sing}\mathcal{F} \cap \Gamma)$ the union (made of isolated points) of the singular part of Γ with the set of singular points of $\mathcal{F}$ which are in Γ. To each point m_α in Σ, we can associate an integer $\mathrm{GSV}_{m_\alpha}(\mathcal{F}, \Gamma)$, generalizing the index of Gomez-Mont *et al.* [GSV91], such that

$$c_1([TM - \mathcal{L}]|_\Gamma - [N_\Gamma]) \frown [\Gamma] = \sum_\alpha \mathrm{GSV}_{m_\alpha}(\mathcal{F}, \Gamma).$$

In particular, we prove, when Γ is irreducible, the inequality

$$\sum_\alpha \mathrm{GSV}_{m_\alpha}(\mathcal{F}, \Gamma) \geqslant \mathcal{B}(\Gamma) - \mathcal{E}(\Gamma),$$

where $\mathcal{B}(\Gamma)$ denotes the total number of locally irreducible branches through singular points of Γ when Γ has singularities or when M is not a projective space, and $\mathcal{B}(\Gamma) = 1$ (instead of 0) when Γ is smooth and $M = \mathbb{P}_n$; and $\mathcal{E}(\Gamma) = 2 - 2g - c_1(TM|_\Gamma - [N_\Gamma]) \frown [\Gamma]$ denotes the correction term in the genus formula (g being the geometrical genus of Γ).

Equivalently, we get $-c_1(\mathcal{L}) \frown [\Gamma] + 2 - 2g \geqslant \mathcal{B}(\Gamma)$.

When $\mathcal{F}$ is an algebraic foliation of degree d on $\mathbb{P}_n$, and when Γ is no longer assumed to be smooth, but is still the complete intersection $\bigcap_{\lambda=1}^{n-1} S_\lambda$ of $n-1$ algebraic hyper-surfaces S_λ, we get

$$\left(d + n - \sum_{\lambda=1}^{n-1} \delta_\lambda\right)\delta \geqslant \mathcal{B}(\Gamma) - \mathcal{E}(\Gamma),$$

where δ_λ and $\delta = \prod_\lambda \delta_\lambda$ denote the degree of S_λ and of Γ, respectively. More generally, even when Γ is not a LCI, we have

$$(d-1)\delta + 2 - 2g \geqslant \mathcal{B}(\Gamma).$$

A significative example (but others are also given in the last section) arises from irreducible curves in $\mathbb{P}_2$, whose singularities are all 'elementary', i.e. are multiple points with local branches all smooth and having distinct tangents. Let n_r be the number of the r-uple points of Γ ($r \geqslant 2$). We have then $\mathcal{B}(\Gamma) = \sum_r n_r r$ when Γ is singular, and $\mathcal{E}(\Gamma) = \sum_r n_r r(r-1)$. We therefore get

$$(d + 2 - \delta)\delta \geqslant -\sum_r n_r r(r-2).$$

These results will also be refined when Γ is reducible.

In § 2, we fix our notation and recall some basic facts. Most of them are well known. Theorem 2.4, proved in [CLS05], is the main technical tool needed in this paper for computing the various residues. Also, the definition of the normal Chern class of a complex analytic singular compact curve, necessary for writing a genus formula in § 4, is recalled. For that, K-theory is indispensable in the non-LCI case, and enlightening in general. Note, however, that when Γ is the complete intersection $\bigcap_{\lambda=1}^{n-1} S_\lambda$ of $n-1$ algebraic hypersurfaces S_λ as above, it is theoretically sufficient to know[1] that N_Γ is the restriction of $\bigoplus_\lambda \mathcal{O}(\delta_\lambda)$ to Γ with the notation as above; in particular, $N_\Gamma = \mathcal{O}(\delta)|_\Gamma$ for $n = 2$.

In § 3, we bound from below the degree of an algebraic foliation in words of the geometrical genus of a prescribed algebraic invariant curve. This is only a first step: in the case $n = 2$ for instance,

[1] As usual, $\mathcal{O}(k)$ will denote the $|k|$th tensorial power of the tautological line bundle over $\mathbb{P}_n$ (respectively of its dual) if k is a negative (respectively positive) integer.

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

we wish to get a lower-bound expressed as much as possible in words of the degree of this invariant curve and not of its genus.

This is achieved in § 4, giving by the way a K-theoretical interpretation of the GSV-index which may be interesting in itself.

Some examples of applications are given in § 5.

2. Notation and backgrounds (without proof)

2.1 Index of an analytic vector field along a branch of Γ

Let m be a point of Γ, singular or not, and Γ_h a local irreducible branch of Γ through m (we write briefly $m \in h$). Let $(x_1, x_2, \dots, x_n)$ be local coordinates in M near m, such that m has coordinates $(0, 0, \dots, 0)$.

Let $\varphi : \tilde{D} \to M$ be a minimal Puiseux-parametrization of Γ_h such that $\varphi(0) = m$, where $\widetilde{D}$ denotes some open 2-disk in $\mathbb{C}$ centered at 0: to each $t \in \widetilde{D}$, φ associates the point $\varphi(t)$ of local coordinates $x_i = \varphi_i(t)$. After shrinking $\tilde{D}$, we can assume moreover that all points of $\varphi(\widetilde{D})$ are regular points of Γ, except perhaps the point m itself. Let D be a closed disk included into $\widetilde{D}$, with center 0 and boundary the circle ∂D.

For any holomorphic function f on $\widetilde{D}$, and also for meromorphic function with pole at 0, denote by $\nu(f) = (1/2\sqrt{-1}\pi) \int_{\partial D} (df/f)$ the order of f at 0. In particular, denote by $\mathbf{p}_i$ the integer $p_i = \nu(\varphi_i)$: this number is always greater than one if m is a singular point of Γ, and greater or equal to one (one at least of the p_i being equal to 1) if m is a regular point.

An analytic vector field $w = \sum_{i=1}^n a_i (\partial/\partial x_i)$ tangent to Γ_h is a map $w : t \mapsto w(t)$ which, to each $t \in \widetilde{D}$, associates the vector $\sum_{i=1}^n a_i(t) (\partial/\partial x_i)_{\varphi(t)} \in T_{\varphi(t)} M$ such that $w(t)$ be tangent to Γ_h at any point $\varphi(t)$, and such that all components a_i are holomorphic, including at the point $t = 0$. By abuse of notation, we sometimes write $w = \sum_{i=1}^n a_i(t)(\partial/\partial x_i)$. In particular, denote by $w_0 = \sum_{i=1}^n {\varphi'}_i(t)(\partial/\partial x_i)$ the tangent vector field to Γ_h given by the parametrization, and for any integer $r \geqslant 0$, set more generally $w_r = t^r w_0$.

Lemma 2.1. *Let w be an analytic vector field tangent to Γ_h. The meromorphic function σ on $\widetilde{D}$ such that $w = \sigma w_0$, which might have a priori a pole at 0, is in fact holomorphic. In other words, there is a well-defined non-negative integer $r = \nu(\sigma)$ and a unit holomorphic function u of t ($u(0) \neq 0$) such that $w = u w_r$.*

Definition. For any analytic vector field $w = \sigma w_0$ tangent to Γ_h, the non-negative integer $r = \nu(\sigma)$ above will be denoted by

$$\begin{aligned} \mu_m(w, \Gamma_h) &= \nu(\sigma), \\ &= \nu(a_i) - p_i + 1 \end{aligned}$$

and will be called the 'index' of w at m.

In particular, let $v = \sum_{i=1}^n A_i (\partial/\partial x_i)$ be a holomorphic vector field on M, defined on a neighborhood of m in M, and such that $\varphi^{-1} v = \sum_{i=1}^n (A_i \circ \varphi)(\partial/\partial x_i)$ be tangent to Γ_h. The number $\mu_m(\varphi^{-1} v, \Gamma_h)$ does not depend on the choices of the local coordinates, of the Puiseux-parametrization (as far as it is a minimal one), and remains unchanged if v is multiplied by a unit function: we sometimes write this number as $\mu_m(\mathcal{F}, \Gamma_h)$ if v defines the foliation $\mathcal{F}$.

Lemma 2.2. *If v denotes a local holomorphic vector field on M, not identically zero, vanishing at m, and leaving Γ invariant, we get*

$$\mu_m(\varphi^{-1} v, \Gamma_h) \geqslant 1,$$

and $\nu(A_i \circ \varphi) \geqslant p_i$ for all i such that φ_i is not identically zero.

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

2.2 Residues for the relative K-theory

Let $S = \{m_1, m_2, \ldots, m_\alpha, \ldots, m_s\}$ be a finite family of points on the complex analytic curve Γ in the holomorphic manifold M. Let $\pi : \Gamma' \to \Gamma$ be the normalization of Γ, and $\widehat{\pi}$ the composition of π with the inclusion $\Gamma \subset M$. Denote by $(m'_h)_h$ the points in $S' = \pi^{-1}(S)$: we get one point m'_h for any locally irreducible branch Γ_h of Γ through a point m_α of S, and $\pi(m'_h) = m_\alpha$ if and only if Γ_h is a branch through m_α, which will be written $\alpha \in h$. Similarly, the notation $h \subset C$ will mean that the branch Γ_h is included into the irreducible component C.

Separate all points m'_h by open 2-disks $\widetilde{D}_h$ in Γ' centered at m'_h. (The restriction of $\widehat{\pi}$ to $\widetilde{D}_{\alpha,h}$ may be seen as a minimal Puiseux-parametrization of Γ_h, once $\widetilde{D}_h$ identified to a 2-disk $\widetilde{D}$ of center 0 in $\mathbb{C}$ by means of a biholomorphism). Let D_h be a closed 2-disk, still centered at m'_h, and bounded by circles ∂D_h. The excision theorem in K-theory asserts that the family of the restrictions

$$K(\Gamma', \Gamma' \setminus S') \to K(\widetilde{D}_h, \widetilde{D}_h \setminus \{m'_h\}) = K(D_h, \partial D_h)$$

defines an isomorphism $K(\Gamma', \Gamma' \setminus S') \cong \bigoplus_h K(D_h, \partial D_h)$. Recall also (see [Ati64]) that the data of a family $(P_0, P_1, \ldots, P_k)$ of vector bundles over Γ' and of a sequence of bundles exact above $\Gamma' \setminus S'$

$$0 \to P_0|_{\Gamma'\setminus S'} \to P_1|_{\Gamma'\setminus S'} \to \cdots \to P_k|_{\Gamma'\setminus S'} \to 0$$

defines naturally an element $\theta \in K(\Gamma', \Gamma' \setminus S')$, which is a lift of $\sum_\lambda (-1)^{\lambda+1} P_\lambda \in K(\Gamma')$ by the natural map $K(\Gamma', \Gamma' \setminus S') \to K(\Gamma')$.

We call the 'residue' of θ the image $\mathrm{Res}(\theta) = c_1(\theta) \frown [\mathcal{T}, \partial\mathcal{T}]$ of the Chern class $c_1(\theta) \in H^2(\Gamma', \Gamma' \setminus S'; \mathbb{Z})$ by the Alexander duality

$$(\ .\ ) \frown [\mathcal{T}, \partial\mathcal{T}] : H^2(\Gamma', \Gamma' \setminus S'; \mathbb{Z}) \to H_0(S'; \mathbb{Z})$$

(where $[\mathcal{T}, \partial\mathcal{T}] = \sum_h [D_h, \partial D_h]$ denotes the fundamental class of $(\Gamma', \Gamma' \setminus S')$).

Denoting by $(\theta_h)_h \in K(D_h, \partial D_h)$ the components of θ relative to the isomorphism above, we can define in the same way $\mathrm{Res}_h(\theta) = c_1(\theta_h) \frown [D_h, \partial D_h] \in \mathbb{Z}$.

We write $\mathrm{Res}_{m_\alpha}(\theta) = \sum_{h,\alpha\in h} \mathrm{Res}_h(\theta)$ for any $m_\alpha \in S$ so that $\mathrm{Res}(\theta) = \sum_{m_\alpha \in S} \mathrm{Res}_{m_\alpha}(\theta)$.

Similarly, for any irreducible component C of Γ, we define the image θ_C of θ by the restriction $K(\Gamma', \Gamma' \setminus S') \to K(C', C' \setminus (C' \cap S'))$. We set $\mathrm{Res}_C(\theta) = \sum_{h,h\subset C} \mathrm{Res}_h(\theta)$.

Lemma 2.3. *We get*

$$\left[\sum_{\lambda=0}^{k} (-1)^{\lambda+1} c_1(P_\lambda)\right] \frown [\Gamma'] = \sum_{m_\alpha \in S} \mathrm{Res}_{m_\alpha}(\theta),$$

and, for any C,

$$\left[\sum_{\lambda=0}^{k} (-1)^{\lambda+1} c_1(P_\lambda)\right] \frown [C'] = \sum_{h,h\subset C} \mathrm{Res}_h(\theta).$$

2.3 Computation of the residues

Let D be a closed 2-disk centered at a point a in $\mathbb{C}$, and ξ the element in $K(D, \partial D)$ defined by a sequence $(P_\lambda, \beta_\lambda)_\lambda$ of holomorphic complex fibre bundles above D_a whose restriction

$$0 \to P_0|_{D\setminus\{a\}} \xrightarrow{\beta_0} P_1|_{D\setminus\{a\}} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{k-2}} P_{k-1}|_{D\setminus\{a\}} \xrightarrow{\beta_{k-1}} P_k|_{D\setminus\{a\}} \to 0$$

to $D \setminus \{a\}$ is exact.

As D is contractible, one can assume that all bundles P_λ are trivial and the morphism β_λ defined by a matrix with holomorphic coefficients. As the sequence of morphisms is exact

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

above $D\setminus\{a\}$, any of these matrices has a constant rank r_λ above $D\setminus\{a\}$: for any point $m \in D\setminus\{a\}$, the value $(\beta_\lambda)_m$ of β_λ has a square submatrix of size r_λ with non-vanishing determinant. As any holomorphic function defined on D, which vanishes at all points of a sequence in $D\setminus\{a\}$ converging to a, is identically zero, it is possible to choose the same lines and the same columns at any point m in $D\setminus\{a\}$ for the determination of the above square submatrix: more precisely, by induction on λ we can define, a square submatrix ψ_λ of β_λ, of size r_λ, whose determinant Δ_λ does not vanish on $D\setminus\{a\}$, and such that $P_\lambda = K_\lambda \oplus C_\lambda$ when $\psi_\lambda : C_\lambda \to K_{\lambda+1}$ is interpreted as a linear map from a subbundle C_λ of P_λ into a subbundle $K_{\lambda+1}$ of $P_{\lambda+1}$. Denote by $\nu_\lambda = \nu(\Delta_\lambda)$ the order of the determinant Δ_λ at a.

Theorem 2.4 [CLS05]. *The following formula holds:*

$$c_1(\xi) \frown [D, \partial D] = \sum_\lambda (-1)^\lambda\, \nu_\lambda.$$

2.4 Definition of the normal Chern classes to Γ in the non-smooth case

Let Γ be a compact complex analytic curve in a holomorphic manifold M of complex dimension n. Denote by $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M, by $\mathcal{I}$ the sheaf of ideals of the germs of holomorphic functions on M vanishing on Γ, and by $\mathcal{O}_\Gamma$ the quotient sheaf $\mathcal{O}_M/\mathcal{I}$. Let $\pi : \Gamma' \to \Gamma$ denote the normalization of Γ, and $\hat\pi : \Gamma' \to M$ the composition of π with the natural inclusion $\Gamma \subset M$. After [AH61], there exists a $\mathcal{A}$-locally free resolution of length at most 2:

$$0 \to \underline{E}_2^* \xrightarrow{\lambda_2^*} \underline{E}_1^* \xrightarrow{\lambda_1^*} \underline{E}_0^* \xrightarrow{\lambda_0^*} \mathcal{A} \otimes_{\mathcal{O}_{\Gamma'}} \hat\pi^{-1}[(\mathcal{O}_M/\mathcal{I}) \otimes_{\mathcal{O}_M} (\mathcal{I}/\mathcal{I}^2)] \to 0 \qquad (*)$$

of the sheaf $\mathcal{A} \otimes_{\mathcal{O}_{\Gamma'}} \hat\pi^{-1}[(\mathcal{O}_M/\mathcal{I}) \otimes_{\mathcal{O}_M} (\mathcal{I}/\mathcal{I}^2)]$, where $\mathcal{A}$ denotes the sheaf of germs of $\mathbb{R}$-analytical $\mathbb{C}$-valued functions on Γ' and $\underline{E}_j^*$ the sheaf of germs of $\mathbb{R}$-analytical sections of some $\mathbb{R}$-analytical complex vector bundle E_j^* over Γ'. Let E_j denote the dual of E_j^*.

Note that the restriction of $(\mathcal{I}/\mathcal{I}^2)$ to Γ is the conormal sheaf, which coincides on Γ_0 with the sheaf of germs of holomorphic sections of the bundle $N^*_{\Gamma_0}$ dual to N_{Γ_0}. Thus, $[E_0 - E_1 + E_2] \in K^0(\Gamma')$ is an extension to all of Γ' of the stable class $[\pi^{-1}N_{\Gamma_0}]$ of $\pi^{-1}N_{\Gamma_0}$, and it does not depend after [AH61] on the choice of the resolution $(*)$. Moreover, π induces a monomorphism $\pi^* : K^0(\Gamma) \to K^0(\Gamma')$, and $[E_0 - E_1 + E_2]$ belongs to its image, so that we get a well-defined element $[N_\Gamma] \in K^0(\Gamma)$ whose image by π^* is $[E_0 - E_1 + E_2]$. We define the normal Chern classes of Γ in M as being those of $[N_\Gamma]$.

By duality, we get a sequence of $\mathbb{R}$-analytical complex vector bundles above Γ', exact above $\Gamma'_0 = \pi^{-1}(\Gamma_0)$

$$0 \to \pi^{-1}(N_{\Gamma_0}) \xrightarrow{\lambda_0} E_0|_{\Gamma_0} \xrightarrow{\lambda_1} E_1|_{\Gamma_0} \xrightarrow{\lambda_2} E_2|_{\Gamma_0} \to 0. \qquad (*_0)$$

If the length of $(*)$ is smaller than 2, we understand that some of the E_i^* may be of rank 0.

In particular, when Γ is a LCI in M, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is $(\mathcal{O}_M/\mathcal{I})$-locally free, and $(\mathcal{O}_M/\mathcal{I})$-isomorphic to the sheaf of sections of some $(\mathcal{O}_M/\mathcal{I})$-complex vector bundle N^*_Γ, whose dual N_Γ is a natural extension of N_{Γ_0}. Therefore, in this case, we can take $E_0 = \pi^{-1}N^*_\Gamma$, $E_1 = E_2 = 0$. More precisely, near each of its points m, Γ is defined locally by a reduced equation $f = 0$ ($f : U \to \mathbb{C}^{n-1}$ denoting some holomorphic map from an open neighborhood U of m in M), whose components f_i ($1 \leqslant i \leqslant n-1$) generate the ideal $\mathcal{I}|_U$. Moreover, the data of the local reduced equation $f = 0$ for Γ defines a local trivialization of N_Γ, such that $df : TM|_{\Gamma_0 \cap U} \to \mathbb{C}^{n-1}$ corresponds to the natural projection $\varpi : TM|_{\Gamma_0} \to N_{\Gamma_0}$ by this trivialization.

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

3. Bounding from below the degree of an algebraic foliation having a prescribed algebraic solution Γ

Let $\ell : \mathcal{L} \to TM$ be a one-dimensional holomorphic foliation $\mathcal{F}$ on a holomorphic manifold M, and Γ a compact curve in M invariant by $\mathcal{F}$. Let $\pi : \Gamma' \to \Gamma$ be the normalization of Γ. Let Σ be the union of $\mathrm{Sing}(\Gamma)$ and of $\mathrm{Sing}(\mathcal{F}) \cap \Gamma$, and set $\Sigma' = \pi^{-1}(\Sigma)$. Assume that Σ is a finite family of isolated points m_α. For each m_α, write $(\Gamma_h)_{\alpha \in h}$ the family of local branches Γ_h of Γ through m_α, and denote by m'_h the point of Σ' contained in the lift of Γ_h. Denote by $\mathcal{B}(\Gamma)$ the total number of locally irreducible branches of Γ through singular points of Γ, when Γ is singular or is not a projective space, and $\mathcal{B}(\Gamma) = 1$, when Γ is non-singular and $M = \mathbb{P}_n$.

The homomorphism ℓ determines an isomorphism $\pi^{-1}(\ell)$ between the restrictions of $\pi^{-1}(\mathcal{L})$ and $T\Gamma'$ to $\Gamma' \setminus \Sigma'$. Thus, after § 2.1, we get a lift $\zeta = \zeta(\mathcal{F}, \Gamma)$ of $[T\Gamma' - \pi^{-1}(\mathcal{L})]$ in $K(\Gamma', \Gamma' \setminus \Sigma')$.

For any irreducible component C of Γ when Γ is reducible, denote by g_C its geometric genus (i.e. the genus of the connected component $C' = \pi^{-1}(C)$ in Γ'), and by $\mathcal{B}_C(\Gamma)$ the total number of locally irreducible branches included into C through points m_α in $\mathrm{Sing}(\Gamma) \cap C$ (such a branch will be denoted shortly by $h \subset C$). Observe that in the reducible case, $\mathcal{B}_C(\Gamma)$ is always at least 1, even if C is smooth.

Theorem 3.1. *With the notation of § 2, we assume that $\mathcal{F}$ is defined locally near m_α by the vector field $\sum_{i=1}^n A_i(\partial/\partial x_i)$ with respect to local coordinates $(x_1, \ldots, x_n)$. The following equality and inequality hold:*

$$\begin{aligned} 2 - 2g_C - (c_1(\mathcal{L}) \frown [C]) &= \sum_{h, h \subset C} \mu_{m_\alpha}(\varphi^{-1} v, \Gamma_h), \\ &= \sum_{h, h \subset C} \nu(A_i \circ \varphi) - p_i + 1 \\ &\geqslant \mathcal{B}_C(\Gamma), \end{aligned}$$

i being any index such that $\varphi_i(t)$ is not identically zero along Γ_h.

Proof. As v is the image of a trivialization of $\mathcal{L}$ by ℓ, the restriction of $\pi^{-1}(\mathcal{L})$ (respectively $T\Gamma'$) to a small disk D_h around m'_h is trivialized by $\varphi^{-1} v$ (respectively w_0), with the notation of § 2, a minimal Puiseux-parametrization φ of Γ_h being given by the composition of π with a biholomorphism $D \to D_h$. Set $\varphi^{-1} v = \sigma w_0$. The homomorphism $\pi^{-1}(\ell)$ is then defined by the multiplication by σ, and the formula of Lemma 2.3 and Theorem 2.4 applied to ζ give the equality

$$2 - 2g_C - (c_1(\mathcal{L}) \frown [C]) = \sum_{h, h \subset C} \mu_{m_\alpha}(\varphi^{-1} v, \Gamma_h).$$

As $\mu_{m_\alpha}(\varphi^{-1} v, \Gamma_h) \geqslant 1$ after Lemma 2.2, we get the conclusion of the theorem, when Γ is singular. When Γ is non-singular and $M = \mathbb{P}_n$, the foliation has necessarily at least one singular point on Γ, according to an argument of Soares [Soa97, Soa00], and we still have $\mu_{m_\alpha}(\varphi^{-1} v, \Gamma_h) \geqslant 1$ at such a point, hence the wanted formula with $\mathcal{B}_C(\Gamma) = 1$. □

Corollary to Theorem 3.1. *For an algebraic foliation of degree d on $\mathbb{P}_n$, the previous formulae become, denoting by δ the degree of Γ and by δ_C the degree of C,*

$$d \geqslant 1 + \sup_C \frac{2g_C - 2 + \mathcal{B}_C(\Gamma)}{\delta_C},$$

with the sup *running through all irreducible components C of Γ.*

Proof. In fact, for an algebraic foliation of degree d, $\mathcal{L}$ is equal to $\mathcal{O}(1-d)$. The corollary then results from the equality

$$c_1(\mathcal{O}(1-d)) \frown [C] = -(d-1)\delta_C. \qquad \square$$

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

Remarks.

(1) The right-hand term in the inequality of the corollary depends only on the curve Γ.

(2) In [EK03, Corollary 6.2], Esteves and Kleiman proved the inequality

$$d \geqslant 1 + \frac{2g - r(\Gamma) + 1}{\delta - 1},$$

where $r(\Gamma)$ denotes the number of irreducible components of Γ. Their lower-bound for d may be better than ours in some cases: for instance, if Γ is non-singular (in fact, both lower bounds coincide for $n = 2$, and our lower bound coincides with that given in [Soa00] for smooth complete intersections). However, our lower bound is better when Γ is sufficiently singular: for instance, we recover the result of [CL91] for nodal singularities.

4. The GSV-index

Let Σ_1 be the singular part $\Gamma \setminus \Gamma_0$ of a complex analytic curve Γ in the holomorphic manifold M. We still write $\pi : \Gamma' \to \Gamma$ the normalization of Γ, and $\widehat{\pi}$ the composition of π with the inclusion $\Gamma \subset M$. Set $\Sigma_1' = \pi^{-1}(\Sigma_1)$. We get an exact sequence above $\Gamma_0' = \pi^{-1}(\Gamma_0)$

$$0 \to T\Gamma_0' \to \hat{\pi}^{-1}(TM)|_{\Gamma_0'} \xrightarrow{\lambda_0 \circ q} E_0|_{\Gamma_0'} \xrightarrow{\lambda_1} E_1|_{\Gamma_0'} \xrightarrow{\lambda_2} E_2|_{\Gamma_0'} \to 0, \qquad (**)$$

q denoting the natural projection $q : TM|_{\Gamma_0} \to N_{\Gamma_0}$, and the E_i defined as in § 2.4.

Thus, we get a natural lift $\xi = \xi(\Gamma) \in K(\Gamma', \Gamma' \setminus \Sigma_1')$ by the canonical map $K(\Gamma', \Gamma' \setminus \Sigma_1') \to K(\Gamma')$ of $[\widehat{\pi}^{-1}(TM) - \pi^{-1}(N_\Gamma) - T\Gamma']$ in $K(\Gamma')$. We denote by

$$\mathcal{E}(\Gamma) = \sum_h \mathcal{E}_h(\Gamma), \quad \mathcal{E}_{m_\alpha}(\Gamma) = \sum_{\alpha \in h} \mathcal{E}_h(\Gamma) \quad \text{and} \quad \mathcal{E}_C(\Gamma) = \sum_{h \subset C} \mathcal{E}_h(\Gamma)$$

the residues of $-\xi$, according to the notation of § 2.2.

A foliation $\ell : \mathcal{L} \to TM$ leaving Γ invariant being given, again let $\Sigma = \Sigma_1 \cup (\mathrm{Sing}\mathcal{F} \cap \Gamma)$ be the union of the singular part of Γ with the set of singular points of $\mathcal{F}$ which are in Γ. Set $\Sigma' = \pi^{-1}(\Sigma)$. The above definition of $\mathcal{E}_{m_\alpha}(\Gamma)$ and $\mathcal{E}_h(\Gamma)$ make sense for m_α belonging to Γ_0 (in particular, for $m_\alpha \in \Sigma \setminus \Sigma_1$), and Γ_h being a local branch through such a point, but both are then zero, as the exact sequence $(**)$ remains exact at the point m_α. Thus, we can see $\xi(\Gamma) \in K(\Gamma', \Gamma' \setminus \Sigma_1') = \prod_{\pi(m_h') \in \Sigma_1} K(\Gamma', \Gamma' \setminus \{m_h'\})$ as an element still denoted $\xi(\Gamma)$ in

$$K(\Gamma', \Gamma' \setminus \Sigma') = \prod_{\pi(m_h') \in \Sigma} K(\Gamma', \Gamma' \setminus \{m_h'\}),$$

understanding that the h-components of $\xi(\Gamma)$ such that $\pi(m_h')$ belongs to $\Sigma \setminus \Sigma_1$ are zero.

Combining the exact sequence $(*_0)$ of § 2.4 with ℓ and the natural morphism $T\Gamma' \to \widehat{\pi}^{-1}(TM)$, we get the sequence over $\Gamma_0' = \Gamma' \setminus \Sigma'$

$$0 \to \hat{\pi}^{-1}(\mathcal{L}) \xrightarrow{\hat{\pi}^{-1}(\ell)} \hat{\pi}^{-1}(TM) \xrightarrow{\lambda_0 \circ q} E_0 \xrightarrow{\lambda_1} E_1 \xrightarrow{\lambda_2} E_2 \to 0, \qquad (***)$$

q denoting the natural projection $q : TM|_{\Gamma_0} \to N_{\Gamma_0}$. Moreover, as $(***)$ is exact above $\Gamma' \setminus \Sigma'$, and as all bundles in this sequence are defined over all of Γ', we get an element in $K(\Gamma', \Gamma' \setminus \Sigma')$, which is a lift of $\pi^{-1}[[TM - \mathcal{L}]|_\Gamma - [N_\Gamma]]$ by the natural map $K^0(\Gamma', \Gamma' \setminus \Sigma) \to K(\Gamma')$. We denote by $\eta = \eta(\mathcal{F}, \Gamma)$ this element in $K(\Gamma', \Gamma' \setminus \Sigma')$, and denote by $\mathrm{GSV}(\mathcal{F}, \Gamma) = \mathrm{Res}(\eta(\mathcal{F}, \Gamma))$ its residue.

We also set, with the notation of § 2.2,

$$\mathrm{GSV}_{m_\alpha}(\mathcal{F}, \Gamma) = \mathrm{Res}_{m_\alpha}(\eta(\mathcal{F}, \Gamma)) \quad \text{and} \quad \mathrm{GSV}_h(\mathcal{F}, \Gamma) = \mathrm{Res}_h(\eta(\mathcal{F}, \Gamma)).$$

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

Remarks.

(1) As $(***)$ is obtained by composition of the exact sequence $(**)$ with ℓ, we get

$$\eta = \zeta + \xi \quad \text{in } K(\Gamma', \Gamma' \setminus \Sigma').$$

(2) In the case $n = 2$, and if v is a local vector field on M vanishing at m and tangent to each Γ_h through m, $\mathrm{GSV}_m(v, \Gamma)$ is equal to the GSV-index of v at m with respect to Γ such as defined in [GSV91], hence the notation in higher dimension. The case of Γ being a locally complete intersection in M defined by a section (s-LCI) has already been defined in [LSS95] and [LS99]. The K-theoretical definition of GSV has already been given in [CL01] for $n = 2$, and in [CLS05] in the general case.

(3) Be careful not to confuse $\mathrm{GSV}_{m_\alpha,h}(\mathcal{F}, \Gamma)$ (the contribution of the branch Γ_h to the GSV-index at m_α with respect to Γ), with $\mathrm{GSV}_{m_\alpha}(\mathcal{F}, \Gamma_h)$ (the GSV-index at m_α with respect to Γ_h). See, for instance, [Suw95] in the case $n = 2$.

Theorem 4.1. *For any irreducible component C of Γ, the following formulae hold:*

$$(c_1([TM - N_\Gamma]) \frown [C]) + 2g_C - 2 = -\sum_{h \subset C} \mathcal{E}_h(\Gamma)$$

and

$$(c_1([TM - N_\Gamma]) \frown [C]) - (c_1(\mathcal{L}) \frown [C]) = \sum_{h \subset C} \mathrm{GSV}_h(\Gamma).$$

Proof. Apply Lemma 2.3 to ξ and to $\eta = \xi + \zeta$. □

We want now to compute $\mathcal{E}_h(\Gamma)$. We can trivialize $T\Gamma'$ by $\partial/\partial t$, and TM by $(\partial/\partial x_1, \dots, \partial/\partial x_n)$. Define in the same way $\beta_1(t)$ for $\lambda_0 \circ q$, $\beta_2(t)$ for λ_1 and $\beta_3(t)$ for λ_2. With the notation of Theorem 2.4, take for ψ_0 the 1×1 matrix (φ_i'), the index i being chosen such that $\varphi_i(t)$ is not constant near 0. Define by induction, as in Theorem 2.4, $\psi_1 = \psi_{1,i}$, $\psi_2 = \psi_{2,i}$ and $\psi_3 = \psi_{3,i}$ from this choice of i. Denoting by $\nu_{1,i}$ (respectively $\nu_{2,i}$, respectively $\nu_{3,i}$) the order $\nu(\det\, \psi_{1,i}(u))$ (respectively $\nu(\det\, \psi_{2,i}(u))$, respectively $\nu(\det \psi_{3,i}(u))$).

Theorem 4.2. *We have the following.*

(i) *With the notation above, the residue $\mathcal{E}_h(\Gamma)$ is given by the formula*

$$\mathcal{E}_h(\Gamma) = \nu_{1,i} - \nu_{2,i} + \nu_{3,i} - p_i + 1,$$

which is independent of the index i such that $\varphi_i(t)$ is not constant along Γ_h.

(ii) *The following inequality holds:* $\mathrm{GSV}_h(\Gamma) \geqslant p_i - \nu_{1,i} + \nu_{2,i} - \nu_{3,i}$.

Proof. Apply Theorem 2.4 to ξ, and use the equality $\mathrm{GSV}_h(\mathcal{F}, \Gamma) = \mu_{m_\alpha}(\varphi^{-1} v, \Gamma_h) - \mathcal{E}_h(\Gamma)$. □

In particular, if Γ is a LCI locally defined near a point $m_\alpha \in \Gamma_h$ by the reduced equation $f = 0$ with $f = (f_1, \dots, f_{n-1})$, assume $\mathcal{F}$ to be locally defined by the vector field $v = \sum_i A_i(\partial/\partial x_n)$, with respect to local coordinates $(x_1, \dots, x_n)$ near m_α.

Corollary 4.3. *Assume that Γ is a LCI. With the notation above, the residues $\mathcal{E}_h(\Gamma)$ and $\mathrm{GSV}_h(\mathcal{F}, \Gamma)$ are given, for any index i such that $\varphi_i(t)$ (or, equivalently, $A_i \circ \varphi$) is not identically zero along Γ_h, by the formulae*

$$\mathcal{E}_h(\Gamma) = \nu_i(f) - p_i + 1,$$

and

$$\begin{aligned} \mathrm{GSV}_h(\mathcal{F}, \Gamma) &= \nu(A_i \circ \varphi) - \nu_i(f), \\ &\geqslant p_i - \nu_i(f), \end{aligned}$$

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

where $\nu_i(f)$ denotes the order of

$$\left(\det \frac{D(f_1,\ldots,f_{n-1})}{D(x_1,\ldots,\widehat{x_i},\ldots,x_n)}\right)\circ\varphi.$$

Proof. In fact, in this case,

$$\beta_1 = \frac{D(f_1,\ldots,f_{n-1})}{D(x_1,\ldots,x_n)}$$

with obvious local trivializations of the bundles, while $\beta_2=\beta_3=0$. □

Remark. While $[E_0-E_1+E_2-\hat{\pi}^{-1}(TM)+T\Gamma']$ and $[E_0-E_1+E_2]-\hat{\pi}^{-1}[TM-\mathcal{L}]$ do not depend in $K(\Gamma')$ on the resolution $(*)$ (as the same is true for $[E_0-E_1+E_2]$ after [AH61]), we have not been able to prove that it remains true for their lift $-\mathcal{E}(\Gamma)$ and $\mathrm{GSV}(\mathcal{F},\Gamma)$ in $K(\Gamma',\Gamma'\setminus\Sigma')$, unless Γ is a LCI. Thus, all computations of residues in the non-LCI case are relative to a particular resolution. However, the sum of these residues does not depend on the resolution, once mapped into $H_0(\Gamma')$ by the natural map $H_0(\Sigma')\to H_0(\Gamma')$.

5. Examples

5.1 Case $n=2$

In the case of an algebraic foliation on $\mathbb{P}_2$, we get

$$c_1([TM-N_\Gamma-\mathcal{L})\frown[C] \;=\; (d+2-\delta)\delta_C.$$

Proposition 5.1. *If Γ has an irreducible component C such that all singularities of $\mathcal{F}$ which are in C are non-dicritical, then $d+2-\delta\geqslant 0$.*

Proof. In fact, the indices $\mathrm{GSV}_{m_\alpha}(\mathcal{F},\Gamma)$ at a non-dicritical singularity m_α are all non-negatives after [Bru97], and the same is true for $\mathrm{GSV}_h(\mathcal{F},\Gamma)$ after [CL01] if Γ_h is included into C, for C as in the statement of the proposition. □

In particular, we recover the result of [Bru97, Car94] when Γ has only non-dicritical singularities.

Case of smooth branches. Assume that Γ_h is a smooth branch of Γ through a singular point m_α. Then, we can choose local coordinates (x,y) such that Γ_h has $y=0$ for an equation and parametrize Γ_h by the map $\varphi: t\mapsto(x(t)=t, y(t)\equiv 0)$. There exists some holomorphic function $g\to U\to\mathbb{C}$, such that $f(x,y)=yg(x,y)$ and $g(x,0)$ is not identically zero.

Lemma 5.2. *The following formula holds: $\mathcal{E}_h(\Gamma)=\nu(g\circ\varphi)$.*

Proof. Use Corollary 4.3 after observing that $f'_y(x,0)=g(x,0)$. □

In particular, call any singular point m_α with exactly r local branches, all smooth, and having distinct tangents an 'elementary' r-multiple point of Γ $(r\geqslant 2)$. We then get the following.

Proposition 5.3. *If m_α is an elementary r-multiple point of Γ then, for any local branch through m_α and for any foliation $\mathcal{F}$, the following formulae hold:*

$$\mathcal{E}_h(\Gamma)=r-1 \quad\textit{and}\quad \mathrm{GSV}_h(\mathcal{F},\Gamma)\geqslant-(r-2).$$
$$\mathcal{E}_{m_\alpha}(\Gamma)=r(r-1) \quad\textit{and}\quad \mathrm{GSV}_{m_\alpha}(\mathcal{F},\Gamma)\geqslant-r(r-2).$$

Proof. Under the given assumptions, the order of the function g above Γ_h is $r-1$, hence the result, using Lemma 5.2 and Corollary 4.3. □

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

Theorem 5.4. *Let $\mathcal{F}$ be a foliation of degree d, leaving invariant an irreducible curve Γ of degree δ in $\mathbb{P}_2$. Assume that Γ has only elementary singularities, and let n_r be the number of r-multiple points. Then, the following formulae hold:*

$$(d-1)\delta + 2 - 2g \geqslant \sum_{r\geqslant 2} n_r r, \quad \text{and} \quad (d+2-\delta)\delta \geqslant -\sum_{r\geqslant 2} n_r r(r-2).$$

More generally, if Γ is reducible and has only elementary singularities, denote by $n_r(C)$ the total number of local branches included into some irreducible component C through singular r-multiple points of Γ. Then, the following formulae hold:

$$(d-1)\delta_C + 2 - 2g_C \geqslant \sum_{r\geqslant 2} n_r(C), \quad \text{and} \quad (d+2-\delta)\delta_C \geqslant -\sum_{r\geqslant 2} n_r(C)(r-2).$$

Proof. Combine Proposition 5.3 above with Theorems 3.1 and 4.1. □

Corollary 5.5. *If a curve Γ has an irreducible component C such that all singularities of Γ which are in C are nodal points of Γ, then the inequality $d+2-\delta \geqslant 0$ still holds.*

In particular, we recover the result of [CL91, Soa01], when Γ has only nodal singularities.

Example 5.6 (Reducible case). Let Γ_1 be a non-degenerate conic in $\mathbb{P}_2$, m_0 be a point of Γ_1, and $(\Delta_i)_i$ a family of $s-1$ projective straight lines Δ_i through m_0 in the plane ($s \geqslant 3$), none of them being tangent to Γ_1. Taking for Γ the union of this conic and of these straight lines, all singularities of Γ are nodal (double point), except m_0 which is an s-uple point, and the degree of the curve is $s+1$. After Theorem 4.1, we get

$$\begin{aligned} d+1-s &\geqslant -(s-2) \quad \text{with respect to } \Delta_i, \\ (d+1-s)2 &\geqslant -(s-2) \quad \text{with respect to the conic.} \end{aligned}$$

The strongest of these inequalities is of course the second, hence $d \geqslant s/2$.

Other examples arise from [Lin02]. For instance, with homogeneous coordinates (X, Y, Z) in $\mathbb{P}_2$, the set Γ of the equation $(X^3-Z^3)(Y^3-Z^3)(X^3-Y^3) = 0$ is the union of nine straight lines with 12 singularities (which are all elementary triple points): thus, we get $d \geqslant 3$; this proves that the family $\mathcal{F}^4_\alpha$ which leaves Γ invariant does not have the minimal degree allowed by our formula. The image of these foliations by the map S defined, for $Z \neq 0$, by $(x, y) \mapsto (x+y, xy)$ (with $x = X/Z, y = Y/Z$) is a family $\mathcal{F}^3_\alpha$ of foliations of degree 3. The image $S(\Gamma)$, preserved by the foliations $\mathcal{F}^3_\alpha$, is the union of two conics C_1 and C_2 (which are bitangent) and of three straight lines (which are tangent to C_1 and which intersect on C_2). The corollary of Theorem 3.1 then gives $d \geqslant 2$ when applied to one of the lines and $d \geqslant 3$ when applied to one of the conics; thus, $\mathcal{F}^3_\alpha$ here has the minimal degree authorized by our inequality. On these two examples, the quantity $d+2-\delta$ takes, respectively, the strictly negative values -3 and -2: this proves *a priori* that the foliations $\mathcal{F}^4_\alpha$ and $\mathcal{F}^3_\alpha$ do have dicritical singularities.

5.2 Higher dimension

Theorem 5.7. *Assume $\mathcal{F}$ to be an algebraic foliation of degree d on $\mathbb{P}_n$, leaving invariant the complete intersection $\Gamma = \bigcap_{\lambda=1}^{n-1} S_\lambda$ of $n-1$ algebraic hypersurfaces S_λ. For any irreducible component C of Γ, we get*

$$\left(d + n - \sum_{\lambda=1}^{n-1} \delta_\lambda\right)\delta_C \geqslant \mathcal{B}_C(\Gamma) - \mathcal{E}_C(\Gamma),$$

where δ_λ and δ_C denote the degree of S_λ and of C, respectively.

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

Proof. Use Theorem 4.1, with $N_\Gamma = \bigoplus_\lambda \mathcal{O}(\delta_\lambda)$. □

Corollary 5.8. *Under the assumptions of Theorem 5.7, assume moreover that* Γ *has a smooth irreducible component* C*. Then we have the inequality*

$$\Big(d + n - \sum_{\lambda=1}^{n-1} \delta_\lambda\Big)\delta_C \geqslant \mathcal{B}_C(\Gamma).$$

Proof. In fact, in this case, $\mathcal{E}_C(\Gamma) = 0$. As $\mathcal{B}_C(\Gamma) \geqslant 1$, we recover in particular the formula given in [Soa00], when Γ is smooth. □

Proposition 5.9. *Assume that* m_α *has four branches, all smooth, with distinct tangents, and such that three of them are never coplanar. If* Γ *is a LCI, then we have*

$$\mathcal{E}_h(\Gamma) = 2, \quad \textit{and} \quad \mathrm{GSV}_h(\mathcal{F}, \Gamma) \geqslant -1.$$

Proof. Under the assumption, there exist local coordinates (x, y, z) near m_α, such that one branch is given by the local equations $(x = 0, y = 0)$, a second by $(x = 0, z = 0)$ and a third by $(y = 0, z = 0)$. The fourth branch, which may always be parametrized by z, has local equations $x - \varphi(z) = 0, y - \psi(z) = 0$, with $\varphi(0) = \psi(0) = 0$, $\varphi'(0) \neq 0$, $\psi'(0) \neq 0$. The curve Γ is then locally defined by the equations $f = 0$, with $f_1(x, y, z) = x(y - \psi(z))$ and $f_2(x, y, z) = y(x - \varphi(z)) = 0$. The jacobian matrix is

$$\frac{D(f_1, f_2)}{D(x, y, z)} = \begin{pmatrix} y - \psi(z) & x & -x\psi'(z) \\ y & x - \varphi(z) & -y\varphi'(z) \end{pmatrix}.$$

Using Theorem 4.2 along the branch $y = z = 0$, we get $\mathcal{E}_h(\Gamma) = 2$. All branches in fact playing the same role, the same formula is true for the three other branches. □

Example 5.10 (Not a LCI). Let Γ be the rational quintic parametrized by the map

$$[u, v] \mapsto [X(u, v) = u^3v^2, Y(u, v) = u^4v, Z(u, v) = u^5, T(u, v) = v^5]$$

from $\mathbb{P}_1$ (with homogeneous coordinates $[u, v]$) into $\mathbb{P}_3$ (with homogeneous coordinates $[X, Y, Z, T]$). It has only the origin for singular point with one local branch at this point, hence $\mathcal{B}(\Gamma) = 1$. According to [CLS04], $c_1(N_\Gamma) \frown [\Gamma] = 21$, hence $\mathcal{E}(\Gamma) = 3$. We get $d \geqslant 1$ (we can also use Theorem 3.1, with $g = 0$). In fact, Γ is invariant by the foliation of degree 1 defined by the vector field $3x(\partial/\partial x) + 4y(\partial/\partial y) + 5z(\partial/\partial z)$ in $\mathbb{C}^3$: thus, the lower bound for d is reached. Note that this foliation has dicritical singularities. It can be shown that the minimal degree of the foliations without dicritical singularities leaving this quintic invariant is 2.

Acknowledgement

We thank M. Soares for helpful correspondence, as well as the referee for his suggestions.

References

Ati64 M. F. Atiyah, *K-theory*, Notes by D. W. Anderson (Harvard University, Cambridge, MA, 1964).

AH61 M. F. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1961), 25–45.

Bru97 M. Brunella, *Some remarks on indices of holomorphic vector fields*, Publ. Mat. **41** (1997), 527–544.

CC97 A. Campillo and M. Carnicer, *Proximity inequalities and bounds for the degree of invariant curves by foliations of* $\mathbb{P}^2_{\mathbb{C}}$, Trans. Amer. Math. Soc. **349** (1997), 2211–2228.

Car94 M. Carnicer, *The Poincaré problem in the nondicritical case*, Ann. of Math. (2) **140** (1994), 289–294.

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press

CL01 V. Cavalier and D. Lehmann, *Localisation des résidus de Baum-Bott, courbes généralisées et K-théorie. Première partie : Feuilletages dans* $\mathbb{C}^2$, Comment. Math. Helv. **76** (2001), 1–19.

CLS04 V. Cavalier, D. Lehmann and M. G. Soares, *Classes de Chern des ensembles analytiques*, C. R. Acad. Sci. Paris Ser. I **338** (2004), 879–884.

CLS05 V. Cavalier, D. Lehmann and M. G. Soares, *Chern classes of analytical sets and applications*, K-theory **34** (2005), 35–69.

CL91 D. Cerveau and A. Lins Neto, *Holomorphic foliations in* CP(2) *having an invariant algebraic curve*, Ann. Inst. Fourier (Grenoble) **41** (1991), 883–903.

EK03 E. Esteves and S. Kleiman, *Bounds on leaves of one-dimensional foliations*, Bull. Braz. Mat. Soc. (NS) **34** (2003), 145–169.

GSV91 X. Gomez-Mont, J. Seade and A. Verjovski, *The index of a holomorphic flow with an isolated singularity*, Math. Ann. **291** (1991), 737–751.

LS95 D. Lehmann and T. Suwa, *Residues of holomorphic vector fields relative to singular invariant subvarieties*, J. Differential Geom. **42** (1995), 367–384.

LS99 D. Lehmann and T. Suwa, *Generalization of variations and Baum-Bott residues for holomorphic foliations on singular varieties*, Internat. J. Math. **10** (1999), 367–384.

LSS95 D. Lehmann, M. G. Soares and T. Suwa, *On the index of a holomorphic vector field tangent to a singular variety*, Bol. Soc. Bras. Mat. **26** (1995), 183–199.

Lin02 A. Lins Neto, *Some examples for Poincaré and Painlevé problems*, Ann. Sci. Ecole Norm. Sup. (4) **35** (2002), 231–266.

Soa97 M. G. Soares, *The Poincaré problem for hypersurfaces invariant by one-dimensional foliations*, Invent. Math. **128** (1997), 495–500.

Soa00 M. G. Soares, *Projective varieties invariant by one-dimensional foliations*, Ann. of Math. (2) **152** (2000), 369–382.

Soa01 M. G. Soares, *On the geometry of Poincaré's problem for one-dimensional projective foliations*, An. Acad. Brasil. Ciênc. **73** (2001), 475–482.

Suw95 T. Suwa, *Indices of holomorphic vector fields relative to invariant curves on surfaces*, Proc. Amer. Math. Soc. **123** (1995), 2989–2997.

Vincent Cavalier cavalier@math.univ-montp2.fr
Département des Sciences Mathématiques, CP 051, Université de Montpellier II, Place Eugène Bataillon, F-34095 Montpellier cedex 5, France

Daniel Lehmann lehmann@math.univ-montp2.fr
Département des Sciences Mathématiques, CP 051, Université de Montpellier II, Place Eugène Bataillon, F-34095 Montpellier cedex 5, France

https://doi.org/10.1112/S0010437X05001764 Published online by Cambridge University Press