S. Itatsu and H. Kaneta
Nagoya Math. J.
Vol. 79 (1980), 23–32

SPECTRAL PROPERTIES OF FIRST ORDER ORDINARY DIFFERENTIAL OPERATORS WITH SHORT RANGE POTENTIALS

S. ITATSU AND H. KANETA

§1. Introduction and main theorem

The purpose of the present paper is to give a complete proof of the theorem which will be used in a paper of the second author [4].

We will discuss certain spectral properties of selfadjoint ordinary differential operators of the form $iA(d/dx) + V$ acting in $L^2(\boldsymbol{R})_n = \sum \oplus L^2(\boldsymbol{R})$, where A is a real diagonal constant matrix and V an Hermitian matrix valued function on $\boldsymbol{R}$ which satisfies some conditions to be stated in the sequel.

According to [1, p. 156] a function v in $L^2_{\text{loc}}(\boldsymbol{R})$ is said to belong to the class SR if, for some $\varepsilon > 0$, the multiplication map: $u(x) \to (1 + |x|)^{1+\varepsilon} v(x) u(x)$ is a compact operator from the Sobolev space $H_1(\boldsymbol{R})$ into $L^2(\boldsymbol{R})$ (the square integrable functions on $\boldsymbol{R}$). For a selfadjoint operator L in a Hilbert space $\mathfrak{H}$, let L_{p}, L_{c} and L_{ac} stand, respectively, for the restriction of L to the subspace $\mathfrak{H}_{\text{p}}$ spanned by all eigenvectors, $\mathfrak{H}_{\text{p}}^{\perp}$ (the orthogonal complement of $\mathfrak{H}_{\text{p}}$) and the absolutely continuous subspace $\mathfrak{H}_{\text{ac}}$([5], p. 516). Thus we have $L = L_{\text{p}} \oplus L_{\text{c}}$ and $L_{\text{c}} \supset L_{\text{ac}}$. Let A be a real diagonal $n \times n$-matrix with (j, j)-component a_j and V an Hermitian $n \times n$-matrix valued function on $\boldsymbol{R}$ with the (j, k)-component V_{jk} in $L^2_{\text{loc}}(\boldsymbol{R})$. As will be shown in Lemma 1, the symmetric operator $\dot{L} = iA(d/dx) + V$ with domain $C_0^\infty(\boldsymbol{R})_n = \sum \oplus C_0^\infty(\boldsymbol{R})$ is essentially selfadjoint in the Hilbert space $L^2(\boldsymbol{R})_n = \sum \oplus L^2(\boldsymbol{R})$, we denote the selfadjoint extension of $\dot{L}$ by L. Then our main result is the following.

THEOREM. (i) *Assume that $a_1 \cdots a_n \neq 0$. If each matrix element V_{jk} of V belongs to the class SR, then $L = L_{\text{ac}}$. Under the additional assumption that for some $\varepsilon > 0$ and $0 < \theta < 1/2$ each V_{jk} satisfies*

Received September 8, 1978.

$$(*) \qquad \sup_{x\in \boldsymbol{R}}\left[(1+|x|)^{2+2\varepsilon}\int_{|x-y|\leqslant 1}|V_{jk}(y)|^2|y-x|^{1-2\theta}dy\right]<\infty\,,$$

L_{ac} is unitarily equivalent to the selfadjoint multiplication operator M in $L^2(\boldsymbol{R})_n$ defined by $Mf(\lambda)=\lambda f(\lambda)$. Note that the condition (*) is satisfied if $V_{jk}(x)=O(|x|^{-1-\varepsilon})$ as $|x|\to\infty$.

(ii) *Assume that $a_1=\cdots=a_m=0$ and $a_{m+1}\cdots a_n\neq 0$ for some $0<m<n$ and that*

$$(**) \qquad \begin{cases} V_{jk}=0 \qquad \text{for } j,\,k=1,\cdots,m\,,\\ V_{jk} \text{ is bounded for } j=1,\cdots,m \text{ and } k=m+1,\cdots,n\,,\\ V_{jk} \text{ and } W_{jk}=\sum_{1\leqslant \ell\leqslant m}V_{j\ell}V_{\ell k} \text{ are of the class } SR \text{ for } j,k\\ \qquad\qquad =m+1,\cdots,n\,. \end{cases}$$

Then L has no eigenvalues differing from zero and $L_c=L_{ac}$. In addition, if each V_{jk} belongs to $C^1(\boldsymbol{R})$ and satisfies

$$V_{jk}(x)=O(|x|^{-1-\varepsilon}) \qquad as\ |x|\longrightarrow\infty$$

for some $\varepsilon>0$, then L_c is unitarily equivalent to the selfadjoint multiplication operator M in $L^2(\boldsymbol{R})_{n-m}$ defined by $Mf(\lambda)=\lambda f(\lambda)$.

In § 3 a sufficient condition for L to have no eigenvalues will be found.

§ 2. Proof of the theorem

We proceed as Agmon [1]. To begin with, we explain our notations. The real and complex numbers will be denoted by $\boldsymbol{R}$ and $\boldsymbol{C}$ respectively. As usual, $\boldsymbol{C}_\pm=\{z\in\boldsymbol{C}\colon \pm\operatorname{Im} z>0\}$ and $\boldsymbol{R}^*=\boldsymbol{R}\backslash\{0\}$.

$$L^p_{loc}(\boldsymbol{R})=\left\{u(x)\colon \int_K|u(x)|^p dx<\infty \text{ for any compact set } K \text{ in } \boldsymbol{R}\right\}.$$

$L^2(\boldsymbol{R})=$ the square integrable functions with the usual norm $\|\ \|$. For real s,

$$L^{2,s}(\boldsymbol{R})=\{u(x)\colon (1+x^2)^{s/2}u\in L^2(\boldsymbol{R})\} \text{ with the norm } \|\ \|_{0,s}\colon$$
$$\|u\|_{0,s}=\|(1+x^2)^{s/2}u\|\,.$$

For any integer $m\geq 0$ and real s, we define the weighted Sobolev space $H_{m,s}(\boldsymbol{R})$ by

$$H_{m,s}(\boldsymbol{R})=\{u(x)\colon D^\alpha u\in L^{2,s}(\boldsymbol{R}),\ 0\leqslant m\} \text{ with the norm } \|\ \|_{m,s}\colon$$
$$\|u\|_{m,s}=\left(\sum_{0\leqslant\alpha\leqslant m}\|D^\alpha u\|^2_{0,s}\right)^{1/2}, \text{ where } D=-i\frac{d}{dx}\,.$$

For real m, the Sobolev space $H_m(\boldsymbol{R})$ of order m is defined as the completion of $C_o^\infty(\boldsymbol{R})$ under the norm

$$\|u\|_m = \int |\hat{u}(\lambda)|^2(1 + \lambda^2)^m d\lambda .$$

Here $\hat{u}$ stands for the Fourier transform of u, namely,

$$\hat{u}(\lambda) = (2\pi)^{-1/2} \int u(x)e^{-ix\lambda}dx .$$

Thus $H_{m,0}(\boldsymbol{R}) = H_m(\boldsymbol{R})$ for non-negative integer m. The continuous functions and continuously differentiable functions on $\boldsymbol{R}$ will be denoted by $C(\boldsymbol{R})$ and $C^1(\boldsymbol{R})$ respectively. For any $0 < \theta < 1$ and real s we denote by $C^{\theta,s}(\boldsymbol{R})$ the continuous functions such that

$$|||u|||_{\theta,s} = \sup_{x\in \boldsymbol{R}} (1 + |x|)^s|u(x)| + \sup_{\substack{x,y\\ 0<|x-y|<1}} \left[(1 + |x|)^s \frac{|u(x) - u(y)|}{|x - y|^\theta}\right] < \infty .$$

$\boldsymbol{C}^n$-valued functions on $\boldsymbol{R}$ whose components lie in $L^2(\boldsymbol{R})$, for example, will be denoted by $L^2(\boldsymbol{R})_n$.

Finally,

A: a real diagonal matrix with the (j,j)-component a_j.

V: an Hermitian matrix valued function on $\boldsymbol{R}$ whose (j, k)-component is V_{jk}.

$\tilde{V}$: an Hermitian matrix valued function on $\boldsymbol{R}$ whose (j, k)-component is $V_{jk}(m < j, k \leqslant n)$.

W: an Hermitian matrix valued function on $\boldsymbol{R}$ whose (j, k)-component is $W_{jk} = \sum_{1\leqslant \ell\leqslant m} V_{j\ell}V_{\ell k}(m < j, k \leqslant n)$.

D_L: the domain of the operator $L = iA(d/dx) + V$.

LEMMA 1. *The operator $\dot{L} = iA(d/dx) + V$ with domain $C_o^\infty(\boldsymbol{R})_n$ is essentially selfadjoint in $L^2(\boldsymbol{R})_n$.*

Proof. Recall that $V_{jk} \in L^2_{\text{loc}}(\boldsymbol{R})$. Obviously $\dot{L}$ is symmetric. Assume first that the diagonal matrix is non-degenerate. It remains only to show that the range of $\dot{L} - z$ is dense for any $z \in \boldsymbol{C}_\pm$. To this end, suppose that a $g \in L^2(\boldsymbol{R})_n$ satisfies

$$((\dot{L} - z)f, g) = 0 \qquad \text{for any } f \in C_o^\infty(\boldsymbol{R})_n . \tag{1}$$

Since $V_{jk} \in L^2_{\text{loc}}(\boldsymbol{R})$, (1) implies that g is absolutely continuous and that

$$iAg' + (V - z)g = 0 . \tag{2}$$

Thus it follows easily that

(3) $$(Ag(x), g(x))' = -2 \ \mathrm{Im}\, z(g(x), g(x)) .$$

Since a monotone function in $L^1(\boldsymbol{R})$ is zero, the function $(Ag(x), g(x))$ is zero. Now from (3) it follows that $g = 0$. Next assume that $a_1 = \cdots = a_m = 0$ and $a_{m+1} \cdots a_n \neq 0$. Then (1) implies that components $g_j (m < j \leqslant n)$ are absolutely continuous. The rest of the proof is the same as that in the case where $\det A \neq 0$. Q.E.D.

Remark. The domain D_L is $H_1(\boldsymbol{R})_n$ in the case (i) and $L^2(\boldsymbol{R})_m \oplus H_1(\boldsymbol{R})_{n-m}$ in the case (ii) of our theorem. In order to verify this, recalling the theorem 4.3 of [5, p. 287], it suffices to show that there exist some constants $0 \leqslant a$ and $0 \leqslant b < 1$ such that

$$\|vf\|^2 \leqslant a^2\|f\|^2 + b^2\|f\|_1^2$$

for a function v belonging to class SR and for any $f \in H_1(\boldsymbol{R})$. To this end, note first that the following inequality holds for some constant c.

$$\|(1 + |x|)^{1+\varepsilon} vf\|^2 \leqslant c\|f\|_1^2 .$$

Hence there exists positive constant r such that

$$\int_{|x| \geqslant r} |vf|^2 dx \leqslant \|f\|_1^2/4 .$$

Since $\|f\|_\infty \leqslant c\|f\|_1^2$ for some constant, taking N large enough, we have

$$\int_{|x|<r} |vf|^2 dx = \left(\int_{|x|<r, |v| \leqslant N} + \int_{|x|<r, |v|>N}\right) |vf|^2 dx \leqslant N^2\|f\|^2 + \|f\|_1^2/4 .$$

2.1. Eigenvalues. The following lemma, together with Proposition 3 in § 3, implies that L has no eigenvalues in the case (i) and that L has no eigenvalues differing from zero in the case (ii).

LEMMA 2. *If v belongs to the class SR, then v is integrable.*

Proof. Assume that for a positive ε the map $u \to (1 + |x|)^{1+\varepsilon} vu$ is a compact operator from $H_1(\boldsymbol{R})$ into $L^2(\boldsymbol{R})$. Then $(1 + |x|)^{1+\varepsilon}|v||u|^2$ is integrable for any $u \in H_1(\boldsymbol{R})$, in particular, for $u = (1 + x^2)^{-(1+\varepsilon)/4}$. Q.E.D.

2.2. The limiting absorbtion principle.

Case (i). Let $R_0(z)$ be the resolvent $(iA(d/dx) - z)^{-1}$ for $z \in \boldsymbol{C}_\pm$. We note that the theorem 4.1 of [1] holds for $R_0(z)$, hence the boundary value

https://doi.org/10.1017/S0027763000018900 Published online by Cambridge University Press

$R_o^{\pm}(\lambda)$ is a well defined bounded operator in $B(L^{2,s}(\boldsymbol{R})_n,\ H_{1,-s}(\boldsymbol{R})_n)$ for any $s > 1/2$.

DEFINITION. A function $u \in H_1^{\mathrm{loc}}(\boldsymbol{R})_n$ will be called a λ-outgoing function (resp. λ-incoming function) if for $\lambda \in \boldsymbol{R}$ the relation holds:

$$u = R_o^+(\lambda)f \quad (\text{resp. } u = R_o^-(\lambda)f) \text{ for some } f \in L^{2,s}(\boldsymbol{R})_n$$

with some $s > 1/2$. Among several steps to prove the limiting absorption principle (cf. Theorem 4.2, [1]), Lemma 4.2 of [1] is the only one whose proof needs new idea. A difficulty arises because A is not necessarily definite. Therefore, we confine ourselves to the proof of the following

LEMMA 3 (cf. Lemma 4.2, [1]). *Let $u \in H_1^{\mathrm{loc}}(\boldsymbol{R})_n$ be a λ-outgoing (λ-incoming) function satisfying a differential equation in the distribution sense:*

$$\left(iA\frac{d}{dx} + V - \lambda\right)u = 0\,, \tag{5}$$

where the matrix element of V are of class SR. Then u belongs to $H_{1,s}(\boldsymbol{R})_n$ for all real s.

Proof. We shall prove the lemma for u outgoing, the proof for u incoming is similar. By the assumption, $u = R_o^+(\lambda)\, f$ for some $f \in L^{2,s_0}(\boldsymbol{R})_n$, $s_o > 1/2$. This implies

$$\begin{aligned} u_j(x) &= ia_j^{-1}\int_x^\infty e^{-ia_j^{-1}(x-y)\lambda} f_j(y)dy \qquad \text{for } j \in J_+ = \{j\colon a_j > 0\} \\ &= -ia_j^{-1}\int_{-\infty}^x e^{-ia_j^{-1}(x-y)\lambda} f_j(y)dy \qquad \text{for } j \in J_- = \{j\colon a_j < 0\}\,. \end{aligned} \tag{6}$$

Since f is integrable, it follows that $u_j(\infty) = 0$ (resp. $u_j(-\infty) = 0$) for $j \in J_+$(resp. $j \in J_-$) and that u is absolutely continuous. Thus (5) holds in the ordinary sense, which yields, setting $\operatorname{Im} z = 0$ in (3), the function $(Au(x), u(x))$ is constant. Thus we have

$$0 \geqslant \lim_{x\to-\infty} \sum_{j\in J_-} a_j|u_j(x)|^2 = \sum_{1\leqslant j\leqslant n} a_j u_j(x)^2 = \lim_{x\to\infty} \sum_{j\in J_+} a_j|u_j(x)|^2 \geqslant 0\,.$$

From this and (6) follows that $\hat{f}_j(-\lambda a_j^{-1}) = 0$. From now, the reasoning in the proof of Lemma 4.2 of [1] is applicable. Q.E.D.

Case (ii). Let $R(z)$ be the resolvent $(iA(d/dx) + V - z)^{-1}$ for $z \in \boldsymbol{C}_\pm$, I_+ the injection $(f_{m+1}, \cdots, f_n)^t \to (0, \cdots 0, f_{m+1}, \cdots, f_n)^t$ and P_+(resp. P_o) the

https://doi.org/10.1017/S0027763000018900 Published online by Cambridge University Press

projection $(f_1, \cdots, f_n)^t \to (f_{m+1}, \cdots, f_n)^t$ (resp. $(f_1, \cdots, f_m)^t$). For $z \in \boldsymbol{C}_\pm$ we consider an operator $\tilde{L}(z)$ with domain $H_1(\boldsymbol{R})_{n-m}$:

(7)
$$\tilde{L}(z) = i\tilde{A}\frac{d}{dx} + \tilde{V} + z^{-1}W - z\,.$$

First of all, note that the inverse $\tilde{R}(z)$ of $\tilde{L}(z)$ exists and that it satisfies

(8)
$$R(z) = z^{-1}(-P_o + P_o V I_+ \tilde{R}(z) P_+) \oplus I_+ \tilde{R}(z) P_+\,.$$

In fact, given an $f \in L^2(\boldsymbol{R})_n$, the equation $(L - z)u = f$ has a unique solution $u = R(z)f \in L^2(\boldsymbol{R})_m \oplus H_1(\boldsymbol{R})_{n-m}$. As one sees easily, $u = R(z)f$ if and only if

(9)
$$\left(i\tilde{A}\frac{d}{dx} + \tilde{V} + z^{-1}W - z\right)P_+u = P_+f + z^{-1}P_+VP_of\,,$$
$$P_ou = z^{-1}(-P_of + P_oVu)\,.$$

Since V_{jk} $(m<j\leqslant n,\ 1\leqslant k\leqslant m)$ is bounded, the range $(P_+ + z^{-1}P_+VP_o)(L^2(\boldsymbol{R})_n)$ is equal to $L^2(\boldsymbol{R})_{n+m}$. Now assume that for a given $f_+ \in L^2(\boldsymbol{R})_{n-m}$ the equation $\tilde{L}(z)u_+ = f_+$ admits two different solutions $u_+^{(j)}$ $(j = 1, 2)$. Then, from the preceding observation, the equation $(L - z)u = I_+f_+$ has two distinct solutions, which is a contradiction. The existence of $\tilde{R}(z)$ has been proved. Now (8) follows from (9). We will show that $\tilde{R}(z)$ is a $\boldsymbol{B}((L^{2,s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})$-valued continuous function on $\boldsymbol{C}_\pm$ which has a continuous extension on $\boldsymbol{C}_\pm \cup \boldsymbol{R}^*(s > 1/2)$. To this end, note that

(10)
$$\tilde{R}(z) + \tilde{R}_o(Z)(\tilde{V} + z^{-1}W)\tilde{R}(z) = \tilde{R}_o(z) \qquad \text{for } z \in \boldsymbol{C}_\pm\,,$$

where $\tilde{R}_o(z)$ denotes the resolvent $(i\tilde{A}(d/dx) - z)^{-1}$. Since $\tilde{V}$ as well as W belongs to *SR* class by the assumption (**), repeating the argument in the proof of Theorem 4.2 [1], together with Lemma 3, we see that a $\boldsymbol{B}(H_{1,-s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})_{n-m})$-valued function $\tilde{T}(z) = \tilde{R}_o(z)$ $(\tilde{V} + z^{-1}W)$ has continuous extensions on $\boldsymbol{C}_\pm \cup \boldsymbol{R}^*$ and that $I + \tilde{T}^\pm(z)$ $(z \in \boldsymbol{C}_\pm \cup \boldsymbol{R}^*)$ is invertible if and only if z is not an eigenvalue of L. Since L has no nonzero eigenvaues, $\tilde{R}(z)$ has the boundary values $\tilde{R}^\pm(\lambda) = (I + T^\pm(\lambda))^{-1}R_o^\pm(\lambda)$, which is automatically continuous in $\lambda \in \boldsymbol{R}^*$:

$$\lim_{\substack{z\to\lambda \\ \pm \operatorname{Im} z>0}} \tilde{R}(z) = R^\pm(\lambda) \qquad \text{in } \boldsymbol{B}(L^{2,s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})_{n-m})\,.$$

In view of (8), $R(z)$ is a $\boldsymbol{B}(L^{2,s}(\boldsymbol{R})_m, L^{2,s}(\boldsymbol{R})_m) \oplus \boldsymbol{B}(L^{2,s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})_{n-m})$-valued function which admits continuous extensions $R^\pm(z)$ on $\boldsymbol{C}_\pm \cup \boldsymbol{R}^*$. Now the absolute continuity of the spectrum of L on $\boldsymbol{R}^*$ follows.

2.3. The multiplicity of L_{ac}.

Case (i). We assume the condition (*). In our case Theorem 5.1 of [1] runs as follows.

PROPOSITION 1. *There exist two families $\varphi_\pm(x, \lambda)$ of generalized eigenfunctions of L defined for any $\lambda \in \boldsymbol{R}$ having the following properties (recall that L has no eigenvalues).*

(i) *As a function of x and λ, $\varphi_\pm(x, \lambda)$ is a measurable matrix valued function of class $L^2_{\mathrm{loc}}(\boldsymbol{R}\times\boldsymbol{R})$.*

(ii) *For every fixed λ the function $\varphi_\pm(x, \lambda)$ belongs to $C(\boldsymbol{R}) \cap H_1^{\mathrm{loc}}(\boldsymbol{R})$ and satisfies the differential equation $(iA(d/dx) + V - \lambda)\varphi_\pm(x, \lambda) = 0$.*

(iii) *For any vector g in $\boldsymbol{C}^m$, put $\varphi_o(x, \lambda) = e^{(iA)^{-1}x\lambda}|A|^{-1/2}$ and*

$$\varphi^g_\pm(x, \lambda) = \varphi_\pm(x, \lambda)g,\ \varphi^g_o(x, \lambda) = \varphi_o(x, \lambda)g\,.$$

Here $|A|^{-1/2}$ denotes the diagonal matrix with (j, j) component $|a_j|^{-1/2}$. Then for a fixed $\lambda \in \boldsymbol{R}$, the function $\varphi^g_\pm(x, \lambda)$ has the representation

$$\varphi^g_\pm(x, \lambda) = \varphi^g_o(x, \lambda) - R^\mp(\lambda)[V(.)\varphi^g_o(. , \lambda)](x)\,,$$

where $R^\mp(\lambda)$ are boundary values of the resolvent $R(z)$ of L. In particular $\varphi^g_\pm(x, \lambda)$ lies in $C^{\theta,-s}(\boldsymbol{R})_n \cap H_{1,-s}(\boldsymbol{R})_n$ for any $s > 1/2$ and satisfies the differential equation (5).

Therefore we can verify the eigenfunction expansion theorem for L along the line of the proof of Theorem 6.2 [1]. Namely, define bounded linear maps $F_\pm : L^2(\boldsymbol{R})_n \to L^2(\boldsymbol{R})_n$ by

$$F_\pm f(\lambda) = (2\pi)^{-1/2} \lim_{N\to\infty} \int_{|x|<N} \varphi^*_\pm(x, \lambda)f(x)dx \qquad \text{in } L^2(\boldsymbol{R})_n\,,$$

Then $F_\pm$ unitarily transforms L into the selfadjoint multiplication operator M defined by $Mf(\lambda) = \lambda f(\lambda)$.

Case (ii). We first note

PROPOSITION 2. *Let the potential V be of class $C^1(\boldsymbol{R})$ and satisfy the first condition of the conditions (**). Then the multiplicity of $L^\perp$ (the restriction of L to the orthogonal complement of the space $\mathfrak{H}_0$ spanned by eigenvectors for eigenvalue zero) is at most $n - m$.*

Proof. We shall show that L has an $(n - m)\times(n - m)$-matrix valued

https://doi.org/10.1017/S0027763000018900 Published online by Cambridge University Press

spectral matrix ρ. The proof follows the same development as that of Theorem 3.1 in Chapter 10 of [3]. However, in connection with the proof of Parseval equality we should note that the image $L(C_0^\infty(\boldsymbol{R})_n)$ is dense in the orthogonal complement $\mathfrak{H}_0^\perp$, that it is a subset of D_L because V is smooth and that, making use of notations in Chapter 10 [3], we have

$$\int_{\varepsilon<|\lambda|<1} \lambda^2 |g(\lambda)|^2 d\rho_\delta \leqslant \int_{\varepsilon<|\lambda|<1} |g|^2 d\rho_\delta \leqslant \int_{\boldsymbol{R}} |Lf(x)|^2 dx\,,$$
$$\int_{1<|\lambda|<\mu} \lambda^2 |g(\lambda)|^2 d\rho_\delta \leqslant \mu^{-2} \int_{1<|\lambda|<\mu} \lambda^4 |g(\lambda)|^2 d\rho_\delta \leqslant \mu^{-2} \int_{\boldsymbol{R}} |L^2 f(x)|^2 dx\,.$$

The lemma below completes the proof of our theorem.

LEMMA 4. *Let L_0 be the selfadjoint operator $iA(d/dx)$ in $L(\boldsymbol{R})_n$. For any $f \in C_0^\infty(\boldsymbol{R})_n$ of the form $f = (0, \cdots, 0, f_{m+1}, \cdots, f_n)^t$, $e^{itL}e^{-itL_0}f$ converges strongly as $t \to \infty$.*

Proof. As is well known ([5], Theorem 3.7 in Chapter X), the convergence follows from the fact that $\|Ve^{-itL_0}f\|$ is integrable on some interval (t_0, ∞). By the assumption (***) there exist positive constants ε, K and r (>1) such that $|V_{jk}(x)| \leqslant K|x|^{-1-\varepsilon}$ for $|x| > r$. Since $(e^{-itL_0}f)_j(x) = f_j(x + a_j t)$, assuming that a finite interval $(-c, c)$ includes the support of f and denoting $\min_{m<j} |a_j|$ (resp. $\sup_{j,x} |f_j(x)|$) by a (resp. s), we have the following inequality:

$$\|Ve^{-itL_0}f\|^2 \leqslant 2cK^2 s^2 n^3 |c + at|^{-2-2\varepsilon}\,,$$

which yields the desired integrability of $\|Ve^{-itL_0}f\|$. Q.E.D.

Proposition 2 and Lemma 4 imply that L_{ac} is unitary equivalent to the multiplication operator in $L^2(\boldsymbol{R})_{n-m}$. Since we have shown that $L_c = L_{ac}$ (see 2.2), the last assertion of our theorem has been proved.

§ 3. Sufficient condition for L to have no eigenvalues

As stated in § 1, A denotes a real diagonal matrix, while V stands for an Hermitian matrix valued function of class $L^2_{loc}(\boldsymbol{R})$.

PROPOSITION 3. (i) *Assume that* $\det A \neq 0$. *If A is positive (or negative) definite or if V is integrable on a half line, then L has no eigenvalue.*

(ii) *Assume that $a_1 = \cdots = a_m = 0$ and $a_{m+1} \cdots a_n \neq 0$ for some $0 < m < n$ and that*

$V_{jk} = 0 \qquad \textit{for } 1 \leqslant j,\ k \leqslant m\,,$

V_{jk} *and* $W_{jk} = \sum_{1 \leq l \leq m} V_{jl} V_{lk}$ *are integrable on a common half line*

for $m < j,\ k \leq n\,,$

then L has no eigenvalues differing from zero.

Proof. Suppose $u \in D_L$ satisfies the following equation for a real λ.

$$\left(iA\frac{d}{dx} + V - \lambda\right)u = 0\,. \tag{11}$$

We shall show that $u=0$. Note that u_j are absolutely continuous in the case (i) and that u_j $(j > m)$ are also absolutely continuous in the case (ii) (cf. the proof of Lemma 1). If A is definite, (3) implies that the function $(Au(x), u(x))$ is constant, thus $u=0$. If V is integrable, say on $(0, \infty)$, define $v \in L^2(\boldsymbol{R})_n$ by the formula $u = e^{(iA)^{-1}x\lambda}v$. Then v satisfies

$$v'(x) = e^{-(iA)^{-1}x\lambda} V(x) e^{(iA)^{-1}x\lambda} v(x).$$

Since v has a non-zero limit as $x \to \infty$, provided $v \neq 0$ ([3], problem 6 in Chapter 3), we conclude that $u=0$. In the case (ii) we must show $u=0$, assuming that $\lambda \neq 0$. We rewrite (11) in the form (9) with $f=0$ and $z=\lambda$. Since the Hermitian matrix valued function $\tilde{V} + \lambda^{-1}W$ is integrable on a half line, it follows that $P_+u=0$ via the same reasoning for the case (i). From the second equality of (9), $P_0 u=0$. Thus $u=0$. Q.E.D.

References

[1] S. Agmon, Spectral properties of Schrodinger operators and scattering theory, Annali della Scuol Normal Superiore di Pisa, series **4**, vol. **2** (1975), 51–218.

[2] E. Angelopoulos, Reduction on the Lorentz subgroup of UIR's of the Poincaré group induced by a semisimple little group, Math. Phys. vol. **15** (1974), 155–165.

[3] E. A. Coddington, N. Levinson, Theory of ordinary differential equations, McGraw-Hill, 1955.

[4] H. Kaneta, Irreducibility of some unitary representations of the Poincaré group with respect to the Poincaré subsemigroup, I, Nagoya Math. J. vol. **78** (1980), 113–136.

[5] T. Kato, Perturbation theory for linear operators, Springer, 1966.

[6] V. V. Martynov, Conditions for discreteness and continuity of the spectrum of a selfadjoint operator of first order differential equations, Dokl. Acad. Nauk, SSSR, **165** (1965), 986–991.

[7] K. Mochizuki, Spectral and scattering theory for symmetric hyperbolic system in an exterior domain, Pub. RIMS, Kyoto Univ., **5** (1969), 219–258.

[8] K. Yajima, The limiting absorption principle for uniformly propagative systems, J. Fac. Sci. Univ. Tokyo Sec. 1A, **21** (1974), 119–131. Eigenfunction expansions associated with uniformly propagative systems and their applications to scattering theory, J. Fac. Sci. Univ. Tokyo, Sec. 1A, **22** (1975), 121–151.

Department of Mathematics
Faculty of Science
University of Shizuoka

Department of Mathematics
Faculty of Science
University of Nagoya

https://doi.org/10.1017/S0027763000018900 Published online by Cambridge University Press