UNIQUE FACTORIZATION IN RINGS WITH RIGHT ACC_1

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If R is an integral domain with maximum condition for principal right ideals—right ACC_1—every nonzero non-unit in R has irreducible factors, but is not necessarily a product of such factors. Using additional basic factors—called infinite primes in [**1**]—results about unique factorization in principal right ideal domains have been obtained in [**1**], [**2**], and [**5**].

In this paper we will define generalized atoms, more exactly α – atoms, α an ordinal, for any integral domain R with right ACC_1. The 1-atoms are just the irreducible elements. Every nonzero non-unit in R can be written as a finite product of generalized atoms. If R is a domain with modular factor lattice, i.e. $V(a)=\{bR : a \in bR\}$ is a modular lattice with respect to inclusion for every nonzero element a in R, any two factorizations of a nonzero element into products of generalized atoms contain the same number of α-atoms for a fixed ordinal α, provided no β-atom with $\beta<\alpha$ precedes an α-atom in these factorizations (Theorem 1). Sharper results are obtained in case R is a weak Bezout domain or even a local weak Bezout domain.

Let R be an integral domain with right ACC_1. We write R^* for the multiplicative semigroup of nonzero elements of R. We define for every ordinal α a subsemigroup S_α of R^* as follows:

S_0 is the group of units of R.

$S_\alpha = \cup S_\beta$, $\beta<\alpha$, for α a limit ordinal.

If $\beta=\alpha-1$ exists we say an element a in R^* with a not in S_β is an α – atom if aR is maximal among the principal right ideals bR with b not in S_β. S_α is then defined as the subsemigroup of R^* generated by S_β and the set of α-atoms.

Under the above assumption there exists an ordinal α_0 minimal with the property that $R^*=S_{\alpha_0}$. We refer to α-atoms for any α as generalized atoms. The next result follows immediately.

LEMMA 1. *Let R be an integral domain with right ACC_1. Then every nonzero non-unit a in R can be written as a product of generalized atoms.*

Let R be an integral domain with modular factor lattice and right ACC_1. It then follows that the intersection of any two principal right ideals aR, bR of R is again a principal right ideal: $aR \cap bR = vR$. Further, any two elements a, b in R with $aR \cap bR \neq 0$ have a greatest common left divisor d with $aR \sqcup bR = dR$. We write $[a, b]=v=ab'$ if $aR \cap bR = vR \neq 0$, $a^{-1}[a, b]$ for b' and $(a, b)=d$ if $aR \sqcup bR = dR$. We write $[aR, bR]$ for the sublattice $\{cR ; aR \leq cR \leq bR\}$ of $V(a)$ if $aR \leq bR$. We need the following definition.

DEFINITION. An element a' is related to an element a through d if there exists an element b with $(a, b)=d$ and $b^{-1}[a, b]=a'$.

In case $aR+bR=aR \sqcup bR=dR$ holds, it is clear that a' related to a through d

implies that a' is similar to a_1 with $a = da_1$. We recall that two elements r, r' in R are called similar if R/rR and $R/r'R$ are isomorphic as R-modules (see [4] for details).

LEMMA 2. *Let b, $a = a_1a_2$ be nonzero elements in an integral domain R with right* ACC$_1$ *and modular factor lattice. Then $b^{-1}[b, a_1a_2] = a_1'a_2'$ where a_1' is related to a_1 through d_1 and a_2' is related to a_2 through d_2 with d_1 a left factor of $d = (b, a)$ and d_2 related to d through d_3 with $d_3R = a_1R \sqcup dR$.*

Proof. We have $[b, a_1a_2] = [b, a_1]r$ for some r in R; $[a_1^{-1}[b, a_1], a_2] = a_1^{-1}[b, a_1]r$ follows. This leads to $r = c^{-1}[c, a_2]$ with $c = a_1^{-1}[b, a_1]$ and the desired equation with $a_1' = b^{-1}[b, a_1]$ and $a_2' = c^{-1}[c, a_2]$. We see that $(b, a_1) = d_1$ is a left factor of d. It remains to consider $cR \sqcup a_2R = d_2R$. This leads to $a_1d_2R = (bR \cap a_1R) \sqcup aR = a_1R \cap (aR \sqcup bR) = a_1R \cap dR$, the claimed relation.

We observe that the lattices $V(d_1)$, $V(d_2)$ are isomorphic to lattices $V(t_1)$, $V(t_2)$ respectively where t_1 and t_2 are factors of d. The statement is obvious for $V(d_1)$, since we can take $t_1 = d_1$. To see the second part assume $d = d_3d_4$ for some element d_4. It follows that the lattices

$$V(d_4) = [d_4R, R] \cong [dR, d_3R] \cong [a_1d_2R, a_1R] \cong [d_2R, R] = V(d_2)$$

are isomorphic, since $V(a_1d_2)$ is modular and $d_3R = a_1R \sqcup dR$, $a_1d_2R = a_1R \cap dR$ holds.

COROLLARY 1. *Let $r = a_1 \dots a_n = b_1 \dots b_m$ be two factorizations of an element r. Then either a_1 is a left factor of b_1 or there exists an index $j (2 \le j \le n)$ and an element c in R such that*

$$r = a_1b_1' \dots b_{j-1}'cb_{j+1} \dots b_m$$

and $a_1'c = b_j$. The element a_1' is related to a_1 through d_0 and b_i' is related to b_i through d_i for $i = 1, \dots, j-1$. The lattice $V(d_i)$ is isomorphic to $V(t_i)$ for $i = 0, \dots, j-1$ where t_i is a factor of d defined as $d = (a_1, b_1 \dots b_{j-1})$ with $dR \supset a_1R$.

For a proof it is only necessary to choose j such that a_1 is not a left factor of $b_1 \dots b_{j-1}$, but a left factor of $b_1 \dots b_j$ which then will equal to $[a_1, b_1 \dots b_{j-1}]c$ for some c in R, defining the c above. The rest follows from Lemma 2.

LEMMA 3. *Let R be an integral domain with right* ACC$_1$ *and modular factor lattice. Assume $R = S_{\alpha_0}$ and that β is an ordinal with $0 \le \beta < \alpha_0$, $\gamma = \beta + 1$. Then*

(*i*) *ab is in S_β if and only if a, b are in S_β for elements a, b in R.*

(*ii*) *If x is a γ-atom, $x = dy$ for d in S_β, y in R then y is a γ-atom.*

(*iii*) *If $a = n_1x_1n_2x_2 \dots n_kx_kn_{k+1} = m_1y_1m_2y_2 \dots m_ty_tr$ is an element in S_γ with x_i, y_j γ-atoms and n_i, m_j elements in S_β for all i, j, and r is in R then $t \le k$.*

Proof. We prove these statements simultaneously by transfinite induction on β. Let $\beta = 0$. Then (*i*) is obvious, (*ii*) is true since d will be a unit and $y = d^{-1}x$ is a 1-atom if and only if x is. Finally, (*iii*) follows from the Jordan-Hölder Theorem for modular lattices.

We now assume that the statements are true for all $\alpha < \beta$, β a non-limit ordinal. (The arguments in case β is a limit ordinal are only slightly different and will be omitted).

To prove (i) let r be an element in R, not contained in S_β. Given any natural number q, there exist $\beta-$atoms $x_1, \ldots, x_q$ with $r = x_1 \ldots x_q r_q$ for some element r_q in R. This together with (iii) applied to S_β proves (i).

(ii) Let x be a γ-atom, d in S_β, $x = dy$. The element y is not contained in S_β, $y = zr$ for some γ-atom z with some r in R and $x = dzr$ follows. Using (i) we know that dz is not in S_β, and since x is a γ-atom, we conclude that r must be a unit in R and y is a γ-atom.

(iii) We observe that x' is a γ-atom if x' is related to a γ-atom x through d with $dR \supset xR$. This is obvious if we apply (ii) with $x = dy$ to obtain that y is a γ-atom and recall that $V(x')$ and $V(y)$ are isomorphic lattices. Similarly, if a' is related to a through d with a in S_β it follows that a' is in S_β.

If we apply Corollary 1 to the two factorizations

$$a = n_1 x_1 n_2 \ldots n_k x_k n_{k+1} = m_1 y_1 \ldots m_t y_t r$$

we see that we can assume that $n_1 = 1$. A second application of this Corollary shows that either x_1 is a left factor of m_1 or, in case m_1 equals 1, of y_1, or

$$a = x_1 n_2 \ldots n_k x_k n_{k+1} = x_1 m_1' y_1' \ldots m_j' c_1 m_{j+1} y_{j+1} \ldots r$$

for some $j \leq t$, or $a = x_1 m_1' y_1' \ldots m_t' y_t' c_2$. The elements m_i' are still in S_β, the y_i' are still γ-atoms, $x_1' c_1 = y_j$ implies c_1 is a unit in R and $x_1' c_2 = r$ holds in the second case for some element c_2 in R.

The statement (iii) follows by induction on k after cancelling x_1.

Corollary 2. *Let x be an α-atom, y a β-atom with $\alpha < \beta$ in a ring R satisfying the assumptions of Lemma 3. Then $xy = y'$ or $xy = y'x'$ for a β-atom y' and an α-atom x'.*

Proof. We have $xy = \bar{y}r$ for some β-atom $\bar{y}$ and an element r in R. If $\bar{y} = xy_1$ for some element y_1 in R we obtain that y_1 is a β-atom, $y = y_1 r$ and r must be a unit; $xy = \bar{y}r = y'$ with y' a β-atom. If $\bar{y}$ is not contained in xR we have $xR \cap \bar{y}R = xy_1R = \bar{y}x_1R$ for elements y_1 and x_1 in R. This leads to $xy_1c = \bar{y}x_1c = xy = \bar{y}r$ for some element c in R. Since y_1 is a β-atom, c must be a unit in R. The element x_1 is an α-atom and $xy = y'x'$ follows with $y' = \bar{y}$, $x' = x_1 c$.

Theorem 1. *Let R be an integral domain with modular factor lattice and right* ACC_1. *Then every nonzero non-unit a in R can be written as*

$$a = x_1^{(\alpha_1)} \ldots x_{n_1}^{(\alpha_1)} x_1^{(\alpha_2)} \ldots x_{n_2}^{(\alpha_2)} \ldots x_1^{(\alpha_k)} \ldots x_{n_k}^{(\alpha_k)} \qquad (*)$$

where the $x_i^{(\alpha_j)}$ are α_j-atoms in R with $\alpha_1 > \alpha_2 > \ldots > \alpha_k$. If

$$a = y_1^{(\beta_1)} \ldots y_{m_1}^{(\beta_1)} y_1^{(\beta_2)} \ldots y_{m_2}^{(\beta_2)} \ldots y_1^{(\beta_t)} \ldots y_{m_t}^{(\beta_t)}$$

is another such factorization with β_j-atoms $y_i^{(\beta_j)}$ and $\beta_1 > \beta_2 > \ldots > \beta_t$ then we have $t = k$, $n_i = m_i$, $\alpha_i = \beta_i$ for $i = 1, \ldots, k$ and there exist units $\varepsilon_i (i = 0, \ldots, k)$ with $\varepsilon_0 = \varepsilon_k = 1$ and

$$\varepsilon_{i-1}^{-1} y_1^{(\alpha_i)} \ldots y_{n_i}^{(\alpha_i)} \varepsilon_i = x_1^{(\alpha_i)} \ldots x_{n_i}^{(\alpha_i)}$$

for $i = 1, \ldots, k$.

Proof. It follows from Lemma 1 and Corollary 2 that a factorization (*) exists for every nonzero non-unit a in R. The ordinal α_1 is determined by a as the ordinal α minimal with the property that a is contained in S_α. We obtain $\alpha_1 = \beta_1$ and $n_1 = m_1$ using (*iii*) in Lemma 3. That $x_1^{(\alpha_1)} \ldots x_n^{(\alpha_1)} = y_1^{(\beta_1)} \ldots y_n^{(\beta_1)} \varepsilon_1$ for some unit ε_1 in R follows from a repeated application of Corollary 1. We have to observe that a β-atom cannot have a factor which is related to an α-atom through d for d in S_γ and $\gamma < \alpha, \beta < \alpha$. An easy induction finishes the proof of the theorem.

The above results can be applied to weak Bezout domains with right ACC_1. We recall that an integral domain in which the sum and the intersection of any two principal right ideals is principal whenever the intersection is nonzero is called a *weak Bezout domain.* We make the following definition. Let R be a weak Bezout domain with right ACC_1. Two β-atoms x and y are called *linked* if there exist β-atoms $x = z_0, z_1, z_2, \ldots, z_n = y$ in R such that either z_i is similar to z_{i+1} or that $z_i = d_i z_{i+1}$ or that $d_i z_i = z_{i+1}$ for d_i in S_α with $\alpha < \beta$ for $i = 0, \ldots, n-1$. With this notation we formulate the following addition to Theorem 1.

COROLLARY. *Let R be a weak Bezout domain with right* ACC_1. *Let $a = x_1 \ldots x_n = y_1 \ldots y_n$ be two factorizations of an element a in R into β-atoms x_i, y_j respectively. Then there exists a permutation σ of $\{1, \ldots, n\}$ such that x_i and $y_{\sigma(i)}$ are linked for $i = 1, \ldots, n$.*

The best possible result is obtained for local weak Bezout domains. Here we say a ring R is local if the non-units form an ideal in R.

THEOREM 2. *Let R be a local weak Bezout domain with right* ACC_1.

(*i*) *Let $a = x_1^{(\alpha_1)} x_2^{(\alpha_2)} \ldots x_n^{(\alpha_n)} = y_1^{(\beta_1)} \ldots y_m^{(\beta_m)}$ be two factorizations of an element a in R into α_i-atoms $x_i^{(\alpha_i)}$ and β_j-atoms $y_j^{(\beta_j)}$, respectively, with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_m$. Then $n = m$, $\alpha_i = \beta_i$ and there exist units $\varepsilon_i (i = 0, \ldots, n)$ with $1 = \varepsilon_0 = \varepsilon_n$ and $x_i^{(\alpha_i)} = \varepsilon_{i-1}^{-1} y_i^{(\alpha_i)} \varepsilon_i$.*

(*ii*) *Let x be an α-atom, y a β-atom in R with $\alpha < \beta$. Then xy is a β-atom y' in R.*

Proof. (*i*) We know that $n = m$ and $\alpha_i = \beta_i$ from Theorem 1. Assume x and y are α-atoms with $xR \cap yR \neq 0$. It follows that $xR + yR = dR$ for some d in R, and $x_1 R + y_1 R = R$ where x_1, y_1 are defined through $x = dx_1$, $y = dy_1$. If $dR \neq xR$ and $dR \neq yR$ we see that x_1 and y_1 are still α-atoms and a contradiction would be reached. This leaves us with $xR = yR = dR$. This comment applied to $x_1^{(\alpha_1)}$ and $y_1^{(\alpha_1)}$ leads to $x_1^{(\alpha_1)} = y_1^{(\alpha_1)} \varepsilon_1$ for some unit ε_1 in R. Cancellation of $x_1^{(\alpha_1)}$ and induction after n ends the proof of (*i*). To prove (*ii*) let x be an α-atom, y a β-atom with $\alpha < \beta$. We have $xy = \bar{y}r$ for some β-atom $\bar{y}$ in R (Corollary 2) and as in the argument above $xR + \bar{y}R = xR$ must follow since R is a local weak Bezout domain. This implies that only the first case of the two cases treated in the proof of Corollary 2 can happen and proves the result.

We conclude with a few examples.

(1) The first example shows that there exist integral domains R with right ACC_1 such that $V(a) = \{bR; bR \geq aR\}$ is a lattice for every $a \neq 0$ in R, but R does not have modular factor lattice. Let K be any commutative field, $K[t]_{(t)} = A$ the localization of the polynomial ring $K[t]$ on the prime ideal (t). There exists a monomorphism σ from A into

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A which maps t to t^2 and fixes the elements in K. Finally let $R_1 = A[[x, \sigma]]$ be the skew power series ring in one variable x with the elements $\sum_0^\infty a_i x^i$, a_i in A, and $xa = \sigma(a) . x$. We know from [3] that R_1 is an integral domain with right and left ACC_1, the intersection of any two principal right ideals is again a principal right ideal and the numbers of irreducible factors in a factorization of a is bounded by some constant dependent on a. This implies that $V(a)$ is a lattice, but this lattice is not modular in general as can be seen for $a = xt$ for example.

(2) One can choose the field K in the first example so that there exists a monomorphism τ from A into K. Let R_2 be the free power series ring $R_2 = A\{\{X, \tau\}\}$ in a set on non-commuting variables $X = \{x_i\}$, $i \in I$, such that A is not in the center, but that $a . x_i = x_i\tau(a)$ determines the multiplication. R_2 is a local Bezout domain with right ACC_1 and $R_2^* = S_2$.

(3) The last example shows that the uniqueness statement in Theorem 1 can be correct for elements in S_α in an integral domain R with right ACC_1, but wrong for elements in S_β with $\beta > \alpha$. We use the local ring A with its monomorphism τ from Example 2. Let T be the skew power series ring $A[[x, \tau]]$ in one variable over A with elements $\sum_{i=0}^\infty x^i a_i$, $a_i \in A$ and $ax = x\tau(a)$. Let R_3 be the subring of T consisting of all those elements for which the coefficient a_1 of x is zero. The 1-atoms are of the form $t\varepsilon$ for ε a unit in R_3 and every element in S_1 can be factored uniquely up to units. But x^2 and x^3 are both 2-atoms and $x^6 = (x^2)^3 = (x^3)^2$.

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