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A Simple Proof and Strengthening of a Uniqueness Theorem for L-functions

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Abstract. We give a simple proof and strengthening of a uniqueness theorem for functions in the extended Selberg class.

This paper concerns the question of when two L-functions are identically equal in terms of the zeros of $L - h$ for a value or more generally, a so-called moving target h (see below). L-functions are Dirichlet series with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as the prototype. The Selberg class of L-functions is the set of all Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of a complex variable $s = \sigma + it$ with $a(1) = 1$, satisfying the following axioms (see [5, 6]):

(i) (Ramanujan hypothesis) $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.

(ii) (Analytic continuation) There is a non-negative integer k such that $(s-1)^k L(s)$ is an entire function of finite order.

(iii) (Functional equation) L satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\bar{s})},$$

where $\Lambda_L(s) = L(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j)$ with positive real numbers Q, λ_j, and complex numbers μ_j, ω with $\mathrm{Re}\mu_j \geq 0$ and $|\omega| = 1$.

(iv) (Euler product) $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

The Selberg class, which has been extensively studied, includes the Riemann zeta-function ζ and essentially those Dirichlet series where one might expect the analogue of the Riemann hypothesis. At the same time, there are a whole host of interesting Dirichlet series not possessing Euler product. Throughout the paper, all L-functions are assumed to be functions from the extended Selberg class of those only satisfying axioms (i)–(iii) (see [6]). Thus, the results obtained in this paper apply particularly to L-functions in the Selberg class.

The uniqueness question of when two L-functions are identically equal has been studied in various settings (see [1–3, 6], etc., for related results and references therein). In particular, the following result was given in [6, p. 152].

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Theorem 1 *Two L-functions L_1 and L_2 are identically equal if $L_1 - a$ and $L_2 - a$ have the same zeros (counting multiplicities) for a complex number a.*

In [1], the above uniqueness problem was considered in terms of the zeros of $L - R(s)$ by changing the so-called fixed target a in Theorem 1 to a rational moving target $R(s)$.

Theorem 2 *Let R be a rational function with $\lim_{s\to\infty} R(s) \neq 1$. Two L-functions L_1 and L_2 are identically equal if $L_1 - R$ and $L_2 - R$ have the same zeros (counting multiplicities).*

In this short article, we first show that Theorem 1 is actually false when $a = 1$ (see Remark 4). We then give a simpler proof and also a strengthening of Theorems 2 and 1 (with $a \neq 1$) by improving a and R in the above theorems to a meromorphic function $h(s)$ of finite order in the complex plane (note that a rational function is of order 0), which will also be shown to be best possible (see Remark 5). More specifically, we have the following theorem.

Theorem 3 *Let h be a meromorphic function of finite order with*

$$\lim_{\Re(s)\to+\infty} h(s) = \alpha \neq 1$$

(α may be ∞). Two L-functions L_1 and L_2 are identically equal if $L_1 - h$ and $L_2 - h$ have the same zeros (counting multiplicities). The conclusion need not hold if h is of infinite order.

Remark 4 Consider $L_1(s) = 1 + \frac{2}{4^s}$ and $L_2 = 1 + \frac{3}{9^s}$. Then L_1 and L_2 trivially satisfy axioms (i) and (ii). Also, one can check that L_1 satisfies the functional equation

$$2^s L(s) = 2^{1-s}\overline{L(1-\bar{s})},$$

and L_2 satisfies the functional equation

$$3^s L(s) = 3^{1-s}\overline{L(1-\bar{s})}.$$

Thus, L_1 and L_2 also satisfy axiom (iii). It is clear that $L_1 - 1$ and $L_2 - 1$ do not have any zeros and thus satisfy the conditions of Theorem 1 with $a = 1$, but $L_1 \neq L_2$. This example shows that Theorem 3 is false when $\alpha = 1$. We note that the proof of [6, Theorem A] uses a result based on a Riemann–von Mangoldt formula for the zeros of $L - c$, which holds for both $c \neq 1$ and $c = 1$ with a modification for the case of $c = 1$ (see [6, Theorem 7.7, p. 147]). However, the assertion that $\lim_{s\to\infty} \ell(s) = 1$ in the proof of [6, Theorem A, p. 153] does not hold when $c = 1$.

Remark 5 We show that Theorem 3 is best possible in the following senses.

(i) The same example in Remark 4 shows that the condition $\lim_{\Re(s)\to+\infty} h(s) = \alpha \neq 1$ in Theorem 3 cannot be dropped. We include here an example with a non-constant h. Let $h = L$ be an arbitrary L-function, which is of finite order. Let $l_1 = L_1 L$ and $l_2 = L_2 L$, where L_1 and L_2 are the same L-functions in Remark 4. Then l_1 and l_2, as a product of two L-functions, are still L-functions. Clearly,

$l_1 - h = L_1L - L = (L_1 - 1)L$ and $l_2 - L = (L_2 - 1)L$ have the same zeros with counting multiplicities. But, $l_1 \neq l_2$.

(ii) The condition "counting multiplicities" cannot be dropped. In fact, for any nonconstant L-function $L(s)$, $L(s)$, and $L^2(s)$ have the same zeros (without counting multiplicities), but they are not equal.

(iii) A moving target of a function is usually assumed to grow more slowly than the function. The function h in Theorem 3, however, does not have this restriction and can be of any finite order, which may grow more quickly than L_1 and L_2, which are of finite order by axiom (ii) (actually order ≤ 1, see *e.g.*, [6, p. 150]). Furthermore, as stated in the theorem, the order of h cannot be further improved to infinite order.

We next give the proof of Theorem 3, which is short and elementary using only the basic fact that the orders of $\frac{f}{g}$ and $f + g$ cannot exceed the maximum of the orders of f, g (see *e.g.*, [4, p. 216]). In particular, it does not require the Riemann–von Mangoldt formula for L-functions or the result that L-functions are of order ≤ 1 (*cf.* [1, 6]). Thus, the proof enables one to extend the theorem to more general Dirichlet series and meromorphic functions, which is omitted here.

Proof of Theorem 3 Consider the function $G(s) = \frac{L_1(s)-h(s)}{L_2(s)-h(s)}$. Under the given conditions, it is clear that $G(s)$ cannot have any zeros or poles except possibly at $s = 1$. Since L_1, L_2, h are all of finite order, the function G is also of finite order, as noted above. Thus, we have that

$$G(s) = \frac{L_1(s) - h(s)}{L_2(s) - h(s)} = (s-1)^m e^{p(s)},$$

where m is an integer and $p(s)$ is a polynomial, and then that

$$|G(\sigma + it)| = |\sigma + it - 1|^m e^{\Re\{p(\sigma+it)\}}. \tag{1}$$

We may write

$$\Re p(\sigma + it) = a_n(t)\sigma^n + a_{n-1}(t)\sigma^{n-1} + \cdots + a_0(t),$$

a polynomial in σ with $a_n(t), \ldots, a_0(t)$ being polynomials in t. If $a_n(t) \not\equiv 0$ and $n \geq 1$, then we can take a t_0 such that $a_n(t_0) \neq 0$. Noting that $\lim_{\sigma\to+\infty} G(s) = 1$ (even when $\alpha = \infty$), by letting $\sigma \to +\infty$ in (1) with $t = t_0$ we deduce that $1 = 0$ (when $a_n(t_0) < 0$) or $1 = \infty$ (when $a_n(t_0) > 0$), which is absurd. This shows that $\Re p(\sigma + it) = a_0(t)$ and then $|G(\sigma + it)| = |\sigma + it - 1|^m e^{a_0(t)}$, which clearly implies that $m = 0$, since otherwise $G(s) \to 0$ when $m < 0$ and $G(s) \to \infty$ when $m > 0$ as $\sigma \to +\infty$ for a fixed t, a contradiction to the fact that $\lim_{\sigma\to+\infty} G(s) = 1$. We thus obtain that $|G(s)| = e^{a_0(t)}$. By letting $\sigma \to +\infty$ again, we deduce that $e^{a_0(t)} \equiv 1$ for each t. Hence, $|G(s)| \equiv 1$, which implies that $G(s) \equiv c$, a constant. Letting $\sigma \to +\infty$ again, we obtain that $G(s) \equiv 1$, which implies that $L_1 \equiv L_2$.

Next, we show that the theorem need not hold if the function h is of infinite order. Let $L_1 = 1 + \frac{2}{4^s}, L_2 = 1 + \frac{3}{9^s}$ (*cf.* Remark 4), and let $h = \frac{L_1 e^{L_1-1} - L_2}{e^{L_1-1} - 1}$. Then it is easy to check that $\frac{L_2-h}{L_1-h} = e^{L_1-1}$, which implies that $L_1 - h$ and $L_2 - h$ have the same zeros, counting multiplicities. It is elementary to check that

$$\lim_{\Re(s)\to+\infty} h(s) = \lim_{\Re(s)\to+\infty} \left(L_1 + \frac{L_1 - L_2}{e^{L_1-1} - 1} \right) = 2.$$

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Thus, L_1, L_2, h satisfy all the conditions of Theorem 3 except the "finite order" assumption on h, but $L_1 \neq L_2$. ■

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