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The $C^{1,1}$ conclusions in Gromov's theory

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Abstract. According to M. Gromov, any sequence of Riemann manifolds with uniformly bounded geometry has a subsequence that converges to a limit. It is shown here that this limit Riemann structure is Lipschitz, generates a Lipschitz geodesic flow, and consequently, as Gromov asserted, the limit distance function is of class $C^{1,1}$. Sharpness of the results is discussed. A simple, extrinsic proof of Gromov's Theorem is included.

1. *Introduction*

As presented in [**8**], M. Gromov's theory of Riemann manifolds with pinched curvature contains the following:

COMPACTNESS THEOREM. *Any sequence of Riemann manifolds with bounded geometry, as described below, has a subsequence of mutually diffeomorphic manifolds converging* (*in a certain isometric sense*) *to a limit Riemann manifold.*

The class $\mathscr{C} = \mathscr{C}(n, \Lambda, \delta_0, v_0)$ consists of all n-dimensional compact C^∞ Riemann manifolds whose absolute sectional curvature is $\leq \Lambda^2$, whose diameter is $\leq \delta_0$, and whose volume is $\geq v_0$. These manifolds are said to have *bounded geometry.* The parameters n, Λ, δ_0, v_0 are arbitrary but fixed.

The Riemann structure g on the limit manifold M is asserted to be continuous and its distance function d is asserted to be $C^{1,1}$ - continuously differentiable with Lipschitz first derivatives. Since the proof of this finite differentiability has not met with complete acceptance (see [**15**] for instance) I offer an alternative one here. It was inspired by lectures of H. H. Wu in which he presented similar results of his and R. Green's [**6**].

Here, existence of the limit Riemann manifold is established extrinsically via the cheap Whitney embedding method (see § 3) whereas Greene and Wu proceed intrinsically. In the Greene–Wu regularity analysis (and also in a similar proof given by S. Peters [**17**]) harmonic coordinates and PDE estimates play an important role. They conclude that the limit Riemann structure is of class $C^{1,\nu}$ (it has ν-Hölder continuous first derivatives, $\nu < 1$) in their special coordinate systems and the distance function is $C^{1,\alpha}$, $\alpha < 1$. The regularity analysis below uses less delicate coordinates and ODE estimates to prove that the limit Riemann structure is $C^{0,1}$ (i.e. Lipschitz) and that its distance function is $C^{1,1}$. In § 4, sharpness of these differentiability results is discussed.

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2. *Lipschitz geodesic flows*

Let us fix some terminology. Consider a single smooth compact manifold M and a continuous Riemann structure g on M. The *g-length* of a piecewise C^1 curve $\beta:[a, b]\to M$ is

$$l_g(\beta)=\int_a^b |\beta'(t)|_g\, dt.$$

The *g-distance* from p to q in M is the infimum of g-lengths of such β's that join p and q. It is denoted $d_g(p, q)$ and d_g is a metric on M. A C^1 curve $\gamma:(a, b)\to M$ is a *g-geodesic of speed c* if the g-norm of its tangent vector is identically c and, for any t, $t^*\in(a, b)$ with $|t-t^*|$ small,

$$c|t-t^*|=d_g(\gamma(t), \gamma(t^*)).$$

A flow φ on TM is a *g-geodesic flow* if the projection $\pi: TM\to M$ sends φ-trajectories onto g-geodesics and $\varphi_t(v)$ is the tangent to $\pi(\varphi_t(v))$.

The following definition is implicit in the work of Gromov. It specifies the type of convergence and Riemann structure to be studied.

Definition. A sequence of smooth Riemann structures g_k on M *$\mathscr{G}$-converges* to a continuous Riemann structure g provided:

($\mathscr{G}$a) $g_k \rightrightarrows g$ as $k\to\infty$ in the sense of C^0 Riemann structures;

($\mathscr{G}$b) the (Levi-Civita) connection of g_k stays bounded as $k\to\infty$;

($\mathscr{G}$c) the absolute sectional curvatures of g_k are uniformly bounded.

We say that these g_k *$\mathscr{G}$-approximate* g, that g is a *Gromov limit*, and that g is of *class $\mathscr{G}$.*

To understand the meaning of $\mathscr{G}$-convergence, fix a finite atlas of smooth compact M-charts $\mathscr{A}=\{\psi\}$ and express each g_k in ψ-coordinates. Do the same for g. Over each ψ this gives real $n\times n$ matrix-valued functions $g_{kij}(x)$ and $g_{ij}(x)$. The condition ($\mathscr{G}$a) requires uniform convergence of the former to the latter as $k\to\infty$ and x varies in the ψ-chart. In the same vein, ($\mathscr{G}$b) requires that the Christoffel symbols $\Gamma^l_{kij}(x)$ of g_k in the ψ-coordinates stay uniformly bounded as $k\to\infty$. Since all the first partial derivatives of the g_{kij} respecting the ψ-coordinates are determined as recombinations of the Christoffel symbols and the g_{kij} themselves [**18**, p. 55, formulas (3), (4)], ($\mathscr{G}$b) can be replaced by

($\mathscr{G}$b') Respecting the atlas $\mathscr{A}$ on M, the C^1 size of the ψ-chart expression of g_k is uniformly bounded as $k\to\infty$.

Note: ($\mathscr{G}$b') *implies g is Lipschitz.* By compactness of M, ($\mathscr{G}$b') is independent of $\mathscr{A}$.

THEOREM 1. *A Riemann structure g of class $\mathscr{G}$ generates a Lipschitz geodesic flow.* It is not immediate, but existence of g-geodesics follows from continuity of g alone. The existence proof becomes easier when Lipschitzness of g is used. For uniqueness and Lipschitz dependence on initial data, we need g to be of class $\mathscr{G}$. As a first step we make a local estimate in $\mathbb{R}^n$.

LEMMA 1. *Let g be a Lipschitz Riemann structure on $\mathbb{R}^n$. Then, for small r,*

$$1 - C^2 Lr \le \frac{d_g(x, y)}{|x-y|_0} \le 1 + C^2 Lr,$$

where d_g denotes g-distance, $|\ |_z$ denotes the $g(z)$-norm, L is the Lipschitz constant of g, C is a constant comparing $|\ |_z$ against the Euclidean norm, and $|x|+|y|+|z| \le r$.

Proof. Let $|\ |$ denote the Euclidean norm. By continuity of g there exists a constant C such that

$$C^{-1} \le \frac{|v|_z}{|v|} \le C \tag{1}$$

for all non-zero v in $\mathbb{R}^n$ and all z near the origin. Since g is Lipschitz,

$$|\langle v, w\rangle_{g(x)} - \langle v, w\rangle_{g(y)}| \le L|x-y||v||w|,$$

for some constant L, all x, y near the origin and all vectors v, w in $\mathbb{R}^n$. Setting $v = w$ gives $||v|_x^2 - |v|_y^2| \le L|x-y||v|^2$. Thus,

$$||v|_x - |v|_y| \le \frac{L|x-y||v|^2}{|v|_x + |v|_y} \le CL|x-y||v|. \tag{2}$$

For x, y near the origin, there is a piecewise C^1 curve γ such that $d_g(x, y) \doteq \int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt$. (Exact equality is attained but we need not use this.) Clearly γ is near the origin too. Then

$$\begin{aligned} d_g(x, y) &= \int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt = \int_0^1 |\gamma'(t)|_0 + \{|\gamma'(t)|_{\gamma(t)} - |\gamma'(t)|_0\}\, dt \\ &\ge \int_0^1 |\gamma'(t)|_0 - CL|\gamma(t)||\gamma'(t)|\, dt \\ &\ge \int_0^1 |\gamma'(t)|_0 - CLrC|\gamma'(t)|_0\, dt, \end{aligned}$$

by (2) and (1). Since $\int_0^1 |\gamma'(t)|_0\, dt \gtrsim |x-y|_0$ we get the first inequality claimed in the lemma. The second is similar. Let $\sigma(t)$ be the segment from x to y, $\sigma(t) = ty + (1-t)x$. Then

$$\begin{aligned} d_g(x, y) \le l_g(\sigma) &= \int_0^1 |y-x|_{\sigma(t)}\, dt \\ &= \int_0^1 |y-x|_0 + \{|y-x|_{\sigma(t)} - |y-x|_0\}\, dt \\ &\le \int_0^1 |y-x|_0 + CLr|y-x|\, dt \le \{1 + C^2 Lr\}|y-x|_0. \end{aligned}$$

QED

LEMMA 2. *Suppose that a sequence g_k of smooth Riemann structures $\mathcal{G}$-converges to g as $k \to \infty$. Then, for some $r > 0$, the g_k-exponential maps restricted to the g-discs of radius r in TM converge uniformly,*

$$\lim_k \exp_{k,p} \stackrel{\text{def}}{=} e_p : T_pM(r) \to M.$$

These e_p are uniform Lipeomorphisms to neighbourhoods of $p \in M$. Their radial segments, $t \mapsto e_p(tv)$, are g-geodesics and g has no other local geodesics.

Proof. By the Rauch Comparison theorem [**1**, p. 251] and Klingenberg's lemma [**2**] there is a uniform radius $r > 0$ such that $\exp_{k,p}$ smoothly embeds each $T_pM(2r)$ and, on this disc, its derivative has uniformly bounded norm and conorm. It follows that these $\exp_{k,p}$-radii minimize g_k-length and that $\{\exp_{k,p}\}$ is equicontinuous. For a subsequence g_m,

$$e_p = \lim_m \exp_{m,p}$$

is a Lipeomorphic embedding. Suppose that for some second subsequence, g_m^*,

$$e_p^* = \lim_m \exp_{m,p}^*.$$

We must show $e_p = e_p^*$.

Fix a smooth coordinate system at p. Assumption ($\mathscr{G}$b) says that the Christoffel matrices for g_k in these coordinates are uniformly bounded. Since the g_k-geodesics γ_k solve

$$\ddot{\gamma}_k + \Gamma(\gamma_k)(\dot{\gamma}_k, \dot{\gamma}_k) = 0,$$

it follows that $\ddot{\gamma}_k$ is uniformly bounded, i.e. that the geodesics are uniformly $C^{1,1}$-equicontinuous. Thus, the radii of e_p and e_p^* are of class $C^{1,1}$.

Let γ_m and γ be the radii $\gamma_m : t \mapsto \exp_{m,p}(tv_m)$ and $\gamma : t \mapsto e_p(tv)$ where $0 \le t \le 2r$ and $v, v_m \in T_pM$ satisfy

$$|v_m|_{g_m} = 1 = |v|_g, \qquad \lim_m v_m = v,$$
$$\gamma_m(r_m) = \gamma(r), \qquad \lim_m r_m = r.$$

By construction, $\lim_m \gamma_m = \gamma$ and $|\dot{\gamma}_m|_{g_m} = 1 = |\dot{\gamma}|_g$. Hence,

$$d_{g_m}(p, \gamma_m(t)) = l_{g_m}(\gamma_m|_{[0,t]}) = t = l_g(\gamma|_{[0,t]}), \tag{3}$$

for $0 \le t \le 2r$. We claim that

$$d_g(p, \gamma(t)) = t, \qquad 0 \le t \le 2r. \tag{4}$$

First suppose that $d_g(p, \gamma(\tau)) < \tau$, for some $\tau \in [0, 2r]$. Then there exists a piecewise C^1 curve β from p to $q \overset{\text{def}}{=} \gamma(\tau)$ whose g-length is less than $\tau = l_g(\gamma|_{[0,\tau]})$. But clearly $l_{g_m}(\beta)$ converges to $l_g(\beta)$ as $m \to \infty$. Thus, for large m and some $\delta > 0$,

$$l_{g_m}(\beta) \le \tau - \delta.$$

Since $\gamma_m \rightrightarrows \gamma$ as $m \to \infty$ and since β almost joins p to $\gamma_m(\tau)$, it follows that $d_{g_m}(p, \gamma_m(\tau)) \le \tau - \frac{1}{2}\delta$ for large m, contradicting (3). Thus,

$$d_g(p, \gamma(t)) \ge t, \qquad 0 \le t \le 2r.$$

The reverse inequality is always true because, being C^1, $\gamma|_{[0,t]}$ is one of the paths over which $d_g(p, q)$ is defined by infimization. This verifies (4).

From (4) we see that both the e_p-radii and the e_p^*-radii are arc-length minimizing, unit speed, g-geodesics. We next claim

(5) Any piecewise C^1 curve β from p to $q \overset{\text{def}}{=} \gamma(\tau)$ with $\tau \in [0, r]$ and $l_g(\beta) = d_g(p, q)$ is a reparameterization of $\gamma|_{[0,\tau]}$.

Again, $\gamma(t) \overset{\text{def}}{=} e_p(tv)$, $0 \leq t \leq 2r$. Fix a β as in (5) and observe that it minimizes arc-length everywhere along itself:

$$d_g(\beta(t), \beta(s)) = l_g(\beta|_{[t,s]}), \qquad 0 \leq t \leq s \leq \tau. \tag{6}$$

For otherwise there is a path from p to q, shorter than β. To prove (5) we show

$$\beta \text{ is tangent to } \gamma \text{ at } q. \tag{7}$$

Choose smooth coordinates which are g-orthonormal at q. Let ϕ be the g-angle between β and γ at q, $0 \leq \phi \leq \pi$. Draw a triangle BqC in the q-coordinates so that B lies on β, C lies on the extension of γ past q, and, letting $| \ |$ denote the $g(q)$-norm, $|Bq| = |qC|$. Then let B and C approach q. See figure 1.

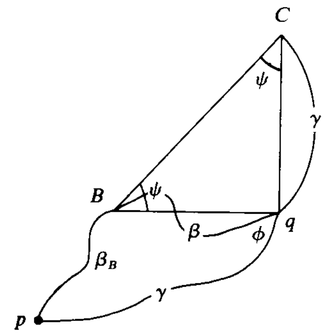

FIGURE 1. The isosceles triangle BqC.

The vertex angle of BqC converges to $\pi - \phi$ as B and C converge to q, so the equal base angles ψ converge to $\frac{1}{2}\phi$. Hence

$$\frac{|BC|}{|Bq| + |qC|} \to \cos\left(\tfrac{1}{2}\phi\right) \qquad \text{as } B, C \to q. \tag{8}$$

Consider the piecewise C^1 path $\lambda = \beta_B \cup BC$ where β_B is the part of β from p to B, and BC is the segment from B to C in the q-coordinates. By (6),

$$\begin{aligned} d_g(p, C) &\leq l_g(\lambda) = l_g(\beta_B) + l_g(BC) \\ &= \{d_g(p, q) - d_g(B, q)\} + |BC| \frac{l_g(BC)}{|BC|}. \end{aligned}$$

By lemma 1, $|BC| = (1+\varepsilon) d_g(B, C)$ with $\varepsilon \to 0$ as B and C converge to q. By continuity of g, the ratio $l_g(BC)/|BC|$ tends to 1 as $|BC| \to 0$. Thus, by (8),

$$\begin{aligned} d_g(p, C) &\leq \{d_g(p, q) - d_g(B, q)\} + (1+\varepsilon)(\cos\left(\tfrac{1}{2}\phi\right) + \varepsilon') 2 d_g(q, C) \\ &= d_g(p, q) + [(2\cos(\tfrac{1}{2}\phi) + \varepsilon'') - 1] d_g(q, C), \end{aligned}$$

where ε, ε', and ε'' tend to 0 as B and C converge to q. If $\phi \neq 0$ then the factor inside brackets is eventually <1, so $d_g(p, C) < d_g(p, q) + d_g(q, C)$ which contradicts (4). Hence $\phi = 0$ and (7) is verified. (Note that in the case $\phi = \pi$, the triangle may be degenerate: B may equal C and ψ may equal $\frac{1}{2}\pi$.)

Returning to (5), we think of e_p as a fixed Lipeomorphic chart and draw β in it. At every point, β is tangent to the e_p-radii. Since their foliation is Lipschitz, it follows from van Kampen's Uniqueness theorem [**9**, p. 35] that β coincides with one of the e_p-radii; i.e. $\beta = \gamma|_{[0,\tau]}$ and (5) is proved. In particular, the e_p-radii and the e_p^*-radii coincide, so the subsequence chosen to make $\exp_{k,p}$ converge was superfluous: the whole sequence $\exp_{k,p}$ converges to e_p as $k \to \infty$, which is what we claimed at the outset. QED

To prove theorem 1, we must measure Lipschitz constants in TM and to do so it is convenient to use the Sasaki–Riemann structure there. We recall its construction [**13**, p. 137–139]. Let a smooth Riemann structure g on M be given. Its Christoffel symbols Γ^k_{ij} define the *horizontal sub-bundle* $\mathcal{H}$ of $T(TM)$ as follows. For each $v \in TM$, $\mathcal{H}_v$ is the linear subspace of $T_v(TM)$ given by

$$\mathcal{H}_v = \{w \in T_v(TM)\colon w = \sum_i a^i \frac{\partial}{\partial x^i} + \sum_k b^k \frac{\partial}{\partial \xi^k} \quad \text{and} \quad \sum_{i,j} \Gamma^k_{ij} a^i v^j + b^k = 0, \quad k = 1, \dots, n\}$$

where the tangent vector v is expressed in a smooth set of coordinates $x^1, x^2, \dots, x^n$ at p as

$$v = \sum_j v^j \left(\frac{\partial}{\partial x^j}\right)_p.$$

(The $\xi^1, \xi^2, \dots, \xi^n$ are the corresponding tangent coordinates in TM. Thus, $x^1, x^2, \dots, x^n, \xi^1, \xi^2, \dots, \xi^n$ coordinatizes the part of TM over the x-chart.) A second set of coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ at p leads to the same subspace $\mathcal{H}_v$ because of the way that Christoffel symbols transform. See [**18**, p. 58 formula (7)]. As v varies over TM, the subspaces $\mathcal{H}_v$ fill out the sub-bundle $\mathcal{H}$.

The *vertical sub-bundle* $\mathcal{V}$ of TM is simpler to describe. Its fibre at $v \in TM$ is $\mathcal{V}_v = \ker T_v\pi$ where π is the projection of TM onto M sending each T_pM onto p. Thus, from g we obtain a canonical splitting

$$T(TM) = \mathcal{H} \oplus \mathcal{V}.$$

The *Sasaki Riemann structure* G on TM is uniquely defined by requiring, for all $v \in TM$,

(Sa) $\mathcal{H}_v \perp \mathcal{V}_v$ respecting $G(v)$;

(Sb) $T\pi$ is an isometry from $(\mathcal{H}_v, G(v))$ to $(T_pM, g(p))$, $p = \pi(v)$;

(Sc) The canonical identification

$$(\mathcal{V}_v, G(v)) = (T_v(T_pM), G(v)) \leftrightarrow (T_pM, g(p))$$

is an isometry.

From this local description of G the following uniformity result is straightforward.

LEMMA 3. *Let G_0 be a fixed Riemann structure on TM and let $\mathcal{S}$ be a set of smooth Riemann structures on M such that, respecting some fixed atlas $\mathcal{A}$ on M, the g_{ij} and g^{ij} matrices of the $g \in \mathcal{S}$ are uniformly C^1 bounded, say by B. Let $S \subset TM$ be compact. Then, for some constant C depending only on B, S, and G_0,*

$$C^{-1} \le \frac{|w|_G}{|w|_{G_0}} \le C,$$

where G is the Sasaki Riemann structure of $g \in \mathcal{S}$, $w \in T_v(TM)$ is non-zero, and $v \in S$.

Proof. The Christoffel symbols of $g \in \mathcal{S}$ respecting the charts in $\mathcal{A}$ are uniformly bounded because they are universal combinations of the g^{ij} and the first partials of the g_{kl}. Thus, measured respecting the fixed Riemann structure G_0, the angle between $\mathcal{H}_v$ and $\mathcal{V}_v$ is uniformly bounded away from zero as v ranges over S. Expressing $w \in T_v(TM)$ as $w_1 + w_2$ with $w_1 \in \mathcal{H}_v$ and $w_2 \in \mathcal{V}_v$, we then see that

$$|w| \le |w_1| + |w_2| \le c|w|,$$

for some uniform constant c. The asserted comparison follows from (Sb), (Sc), and the hypothesized bound B on the $g \in \mathcal{S}$. QED

Proof of theorem 1. By lemma 2, g has unique geodesics and they are of class $C^{1,1}$. Thus, the g-geodesic flow

$$\varphi_t(v) = \dot\gamma(t)$$

is well defined where γ is the g-geodesic with $\dot\gamma(0) = v \in TM$. It is not yet clear that φ is continuous, let alone Lipschitz.

Since g is of class $\mathcal{G}$, it can be $\mathcal{G}$-approximated by smooth Riemann structures $\tilde g$. Their geodesic flows $\tilde\varphi = \tilde\varphi_t(v)$ are C^∞ and by lemma 2 they uniformly approximate φ on compact (t, v)-subsets of $\mathbb{R} \times TM$. Thus, to prove that φ is Lipschitz, it suffices to show that on compact subsets of $\mathbb{R} \times TM$, $\tilde\varphi$ is uniformly C^1 bounded with respect to some fixed Riemann structure G_0 on TM; i.e. it suffices to show

(9) $|\tilde\varphi'|_{G_0}$ and $\|T\tilde\varphi_t\|_{G_0}$ are uniformly bounded on compact subsets of $\mathbb{R} \times TM$ as $\tilde g$ $\mathcal{G}$-approximates g.

According to lemma 3 and ($\mathcal{G}$b′) it is equivalent to show

(10) $|\tilde\varphi'|_{\tilde G}$ and $\|T\tilde\varphi_t\|_{\tilde G}$ are uniformly bounded on compact subsets of $\mathbb{R} \times TM$ where $\tilde G$ is the Sasaki Riemann structure of $\tilde g$ and $\tilde g$ $\mathcal{G}$-approximates g.

The trick for (10) is a calculation in a special chart.

Let $\tilde\gamma$ be a unit speed $\tilde g$-geodesic at p; choose a $\tilde g$-orthonormal basis $e_1, e_2, \ldots, e_n$ of T_pM with $e_1 = \tilde\gamma'(0)$; then $\tilde g$-parallel-translate the $e_2, \ldots, e_n$ down $\tilde\gamma$; then $\tilde g$-exponentiate them into M. This gives the *Fermi chart along* γ, call it f. See [**7**, p. 113–114] and [**11**]. Call F the tangent-chart over f. It is a TM-chart. The geodesic flow $\tilde\varphi$, represented in F, solves the ODE

$$(11)\qquad \begin{pmatrix} x \\ \xi \end{pmatrix}' = \begin{pmatrix} \xi \\ -\tilde\Gamma_f(x)(\xi, \xi) \end{pmatrix} \qquad \begin{array}{l} x(0) = x_0, \\ \xi(0) = \xi_0, \end{array}$$

where $\tilde\Gamma_f$ is the Christoffel matrix of $\tilde g$ respecting the f-chart. (The same is true for any tangent chart, not just the Fermi chart.) The tangent flow $T\tilde\varphi$ lives on $T(TM)$, and, represented in the TF-chart, solves the First Variation Equation of (11), namely

$$(12)\qquad W' = AW, \qquad W(0) = I_{2n},$$

where I_{2n} is the $2n \times 2n$ identity matrix and

$$A = \begin{pmatrix} 0 & I_n \\ -\dfrac{\partial\tilde\Gamma_f}{\partial x}(\cdot)(\xi, \xi) & -2\tilde\Gamma_f(x)(\xi, \cdot) \end{pmatrix} \qquad \begin{array}{l} x = x(t, x_0, \xi_0), \\ \xi = \xi(t, x_0, \xi_0). \end{array}$$

Now, in the f-chart at all points of $\tilde{\gamma}$, $\tilde{\Gamma}_f = 0$ and $\partial\tilde{\Gamma}_f/\partial x$ is the sectional curvature K. Hence,

$$A = \begin{pmatrix} 0 & I_n \\ -K & 0 \end{pmatrix} \quad \text{along } \tilde{\gamma}'.$$

It follows that the solution of (12) satisfies the inequality

$$\| W(t) \| \le e^{|t|\Lambda}$$

where $|K| \le \Lambda^2$. Also, the vector field $\tilde{\varphi}'$ represented in the F-chart, is just the ODE (11); along $\tilde{\gamma}'$ this is

$$\begin{pmatrix} x \\ \xi \end{pmatrix}' = \begin{pmatrix} \xi \\ 0 \end{pmatrix}.$$

Thus, (10) is verified along $\tilde{\gamma}'$ in the F-chart. But the Sasaki Riemann structure $\tilde{G}$ agrees exactly with the F-chart Riemann structure along $\tilde{\gamma}'$ because $\tilde{\Gamma}_f = 0$ along $\tilde{\gamma}$. Thus (10) is verified and φ is Lipschitz. QED

COROLLARY. *The g-exponential map $e: TM(r) \to M$ is Lipschitz.*

Proof. $e_p(v) = \pi \circ \varphi_1(v)$ where $v \in T_pM(r)$, φ_1 is the time one map of the geodesic flow, and $\pi: TM \to M$ is the projection. QED

THEOREM 2. *The distance function $d_g(x, y)$ of a Riemann structure g of class $\mathcal{G}$ is $C^{1,1}$ on a punctured neighbourhood of the diagonal Δ in $M \times M$.*

Proof. Let $\tilde{g}$ $\mathcal{G}$-approximate g and let $\tilde{\varphi}$ be its geodesic flow. For any $(x, y) \in M \times M$ near Δ, define

$$\tilde{R}_x(y) = \tilde{\varphi}_1(\tilde{e}_x^{-1}(y)), \quad \tilde{R}_y(x) = \tilde{\varphi}_1(\tilde{e}_y^{-1}(x)), \quad \tilde{R}(x, y) = (\tilde{R}_y(x), \tilde{R}_x(y)).$$

Then $\tilde{R}$ is a C^∞ tangent vector field on a neighbourhood of Δ in $M \times M$, vanishing on Δ only. We claim that the derivative of the distance-function $d_{\tilde{g}}$ at $(x, y) \notin \Delta$ is given by

(13) $$T_{x,y}d_{\tilde{g}}: T_{x,y}(M \times M) \to \mathbb{R}$$
$$w \mapsto \left\langle w, \frac{\tilde{R}(x, y)}{d_{\tilde{g}}(x, y)} \right\rangle_{\tilde{g}\oplus\tilde{g}}$$

where $\tilde{g} \oplus \tilde{g}$ is the direct sum Riemann structure on $M \times M$.

Fix x and vary y. By the Gauss lemma, the $d_{\tilde{g}}$-level surface passing through y is $\tilde{g}$-orthogonal to $\tilde{R}_x(y)$ and $d_{\tilde{g}}$ increases along the $\tilde{e}_x$-radius through y as $\tilde{g}$-arc-length. Thus,

$$\frac{\partial d_{\tilde{g}}}{\partial y} = \left\langle \cdot, \frac{\tilde{R}_x(y)}{d_{\tilde{g}}(x, y)} \right\rangle_{\tilde{g}\oplus\tilde{g}},$$

since $|\tilde{R}_x(y)|_{\tilde{g}} = d_{\tilde{g}}(x, y)$. Similarly for $\partial d_{\tilde{g}}/\partial x$. The sum of the partials is the total derivative, so (13) is proved.

Write

$$R(x, y) = (\varphi_1(e_y^{-1}(x)), \varphi_1(e_x^{-1}(y))),$$

where φ is the g-geodesic flow and e is the g-exponential map. According to theorem 1 and the preceding corollary,

(14) R and d_g are Lipschitz.

(15) $d_{\tilde{g}} \rightrightarrows d_g$, $\tilde{\varphi} \rightrightarrows \varphi$, and $\tilde{R} \rightrightarrows R$ as $\tilde{g}$ $\mathcal{G}$-converges to g.

In particular, $d_{\tilde{g}}$ C^1-converges. Under C^1-convergence, limits and differentiation interchange, so d_g is C^1 and

$$T_{x,y}d_g(w) = \left\langle w, \frac{R(x,y)}{d_g(x,y)} \right\rangle_{\tilde{g}\oplus\tilde{g}}.$$

By (14), Td_g is Lipschitz, so d_g is of class $C^{1,1}$. QED

Remark. Using the above formula for Td_g it is not hard to give uniform estimates concerning the rate at which Td_g blows up at $x = y$. Another way to handle R, suggested by R. Greene, involves the Greene–Wu Hessian Comparison theorem [**5**, p. 19–20].

3. *Gromov Compactness*

In [**6**] Greene and Wu show how to prove the Gromov Compactness theorem (with $C^{1+\alpha}$ distance function, $\alpha < 1$) based on the intrinsic techniques of S. Peters [**16**]. In contrast, here is a direct *extrinsic* proof. It relies on two things:

(i) The almost linear charts of Jost and Karcher [**14**];

(ii) The cheap Whitney embedding method [**19**, p. 113–114].

The use of (ii) parallels part of the Cheeger–Gromov Approximation theorem [**3**, theorem 2.5] but is considerably simpler. Along the way, the Cheeger Finiteness theorem drops out. Moreover, it would not be hard to give a quick proof of the latter result using exponential charts instead of almost linear ones.

Note that lattices and intrinsic centre of mass constructions are avoided. They are all replaced by (ii).

First we discuss the almost linear charts. For each M in a class $\mathscr{C}$ of manifolds with bounded geometry, Jost and Karcher produce a finite C^∞ atlas $\mathscr{A}$ of charts $\psi_i : U_i \to \mathbb{R}^n$, $i = 1, \ldots, N$, such that

(16) $\psi_i(U_i)$ contains in its interior the closed ball $B(r)$ at the origin.

(17) The ψ_i-coordinate expressions for the Riemann structure g and its inverse g^{-1} have C^1 size $\leq G$.

(18) N, r, and G are the same for all $M \in \mathscr{C}$.

From [**18**, p. 56] we see that (17), (18) are equivalent to

(19) The C^2-size of $\psi_i \circ \psi_j^{-1}$ is uniformly bounded.

It is easy to refine and shrink the above construction so that for some constants (uniform over all $M \in \mathscr{C}$) satisfying

(20) $$0 < a < b < r,$$

we have, in addition to (16)–(19),

(21) $\{U_i(a)\}$ covers M.

(22) If $U_i(a)$ meets $U_j(b)$ then $U_i(a) \subset U_j(r)$.

By $U_i(t)$ we denote the ψ_i-ball of radius t, $\psi_i^{-1}(B(t))$, $0 < t \leq r$.

Next we discuss the cheap way to embed M in $\mathbb{R}^{nN+N}$. Let $\sigma_1, \ldots, \sigma_N$ be C^∞ bump functions on M such that σ_i has support in U_i and is identically 1 on $U_i(r)$.

Then [19, p. 113–114]

$$f: M \to \mathbb{R}_1^n \times \cdots \times \mathbb{R}_N^n \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N$$
$$u \mapsto (\sigma_1\psi_1(u), \ldots, \sigma_N\psi_N(u), \sigma_1(u), \ldots, \sigma_N(u))$$

smoothly embeds M. We understand $\sigma_i\psi_i(u) = 0$ when $u \notin U_i$. A useful property of this embedding is its *uniform injectivity*. That is,

(23) If $p \in U_i(a)$ and $|f(p) - f(q)| \leq (b-a)/(b+1)$ then $q \in U_i(b)$.

To check this, observe that $\sigma_i(p) = 1$ and $|\sigma_i(p) - \sigma_i(q)| \leq (b-a)/(b+1)$ imply

$$\sigma_i(q) > 1 - \frac{b-a}{b+1} = \frac{a+1}{b+1}.$$

Then $|\psi_i(p) - \sigma_i\psi_i(q)| = |\sigma_i\psi_i(p) - \sigma_i\psi_i(q)| \leq (b-a)/(b+1)$ and $|\psi_i(p)| \leq a$ imply

$$|\psi_i(q)| < \left(\frac{1}{\sigma_i(q)}\right)\left(a + \frac{b-a}{b+1}\right) < b,$$

which verifies (23).

Finally, we observe that $f(M)$ is covered by a natural family of *plaques*, $\{\theta_i\}$. These are graph-embedded n-discs – see [12] for some dynamic plaquation discussion. The plaque θ_i is by definition

$$\theta_i = f(U_i(r)),$$

and $\Theta_i \overset{\text{def}}{=} f \circ \psi_1^{-1}|_{B(r)}$ is its *plaque chart*. Since $\sigma_i \equiv 1$ on $U_i(r)$ we can write

$$f(u) = (f_1(u), \ldots, f_{i-1}(u), x, f_{i+1}(u), \ldots, f_N(u), \sigma_1(u), \ldots, \sigma_N(u)),$$

where $u = \psi_i^{-1}(x) \in U_i(r)$ and $x \in B(r)$. This exhibits Θ_i as the graph of a smooth map. By (19) and the Higher Order Chain Rule we deduce

(24) The C^2-size of Θ_i is uniformly bounded as (M, g) varies in any class $\mathscr{C}$ of manifolds with bounded geometry.

Proof of Gromov's Compactness theorem. Let (M_k, g_k) be an arbitrary sequence in some class $\mathscr{C}$ of manifolds with bounded geometry. Let $f_k: M_k \to \mathbb{R}^{nN+N}$ be the cheap Whitney embedding constructed above, let $\{\theta_{ik}\}$ be its plaquation, and let Θ_{ik} be the plaque charts. By (24) and the Arzela–Ascoli theorem, we may (choose N sub-sequences and) assume

$$\Theta_{ik} \rightrightarrows \bar{\Theta}_i \qquad 1 \leq i \leq N \qquad \text{as } k \to \infty$$

where $\bar{\Theta}_i$ is of class $C^{1,1}$. Call $\bar{\theta}_i$ the image of $\bar{\Theta}_i$ and $\bar{\theta}_i(t)$ the image of its restriction to $B(t)$, $0 < t \leq r$. Since $\{U_{ik}(a)\}$ covers M_k, it is clear that

$$\bar{M} \overset{\text{def}}{=} \lim_k f_k(M_k) = \bigcup_i \bar{\theta}_i(a) = \bigcup_i \bar{\theta}_i(r),$$

where convergence takes place in the space of compact subsets of $\mathbb{R}^{nN+N}$.

We claim

(25) If $\bar{\theta}_i(a)$ meets $\bar{\theta}_j(a)$ then $\bar{\theta}_i(a) \subset \bar{\theta}_j(r)$.

Suppose that $\bar{\theta}_i(a) \cap \bar{\theta}_j(a) \neq \varnothing$. Since θ_{ik} and θ_{jk} converge to $\bar{\theta}_i$ and $\bar{\theta}_j$ as $k \to \infty$, there are points $u_{ik} \in U_{ik}(a)$ and $u_{jk} \in U_{jk}(a)$ such that

$$|f_k(u_{ik}) - f_k(u_{jk})| \to 0 \qquad \text{as } k \to \infty.$$

By (23), $u_{jk} \in U_{ik}(b)$ for all large k; i.e. $\theta_{ik}(a) \cap \theta_{jk}(b) \neq \varnothing$. By (22), $\theta_{ik}(a) \subset \theta_{jk}(r)$ and the inclusion persists in the limit, verifying (25). The same analysis shows that for large k, the only plaque of $f_k(M_k)$ near $\bar{\theta}_i(r)$ is $\theta_{ik}(r)$. Hence, as $k \to \infty$, plaques converge to plaques and no new intersections occur. It follows that:

$\bar{M}$ is a $C^{1,1}$ sub-manifold of $\mathbb{R}^{nN+N}$ to which the sequence of embedded sub-manifolds $f_k(M_k)$ $C^{1,1}$-converges.

To apply theorem 2 directly, we need a sequence of Riemann structures all defined on the same manifold. This leads us to smooth the tubular neighbourhood of $\bar{M}$ as follows.

Let $\bar{N}$ be the Euclidean-normal plane field to $\bar{M}$ in $\mathbb{R}^{nN+N}$. Fix a continuous extension $\hat{N}$ of $\bar{N}$, defined on a neighbourhood of $\bar{M}$. Then fix a smooth approximation $\tilde{N}$ of $\hat{N}$, so that $\tilde{N}$ is still transverse to $\bar{M}$. Since $f_k(M_k)$ converges to $\bar{M}$ as sub-manifolds, and since $\tilde{N}$ is fixed, it follows that the injectivity radius of the $\tilde{N}$-tubular neighbourhood of $f_k(M_k)$ stays bounded away from 0 as $k \to \infty$. See [**19**, p. 121]. Thus, we can fix a large $k = \kappa$, call $M = f_\kappa(M_\kappa)$, define a C^∞ tubular neighbourhood retraction

$$\rho : V \to M$$

where V contains all the $f_k(M_k)$ with $k \geq \kappa$, and observe that

$$h_k \overset{\text{def}}{=} \rho \circ f_k : M_k \to M$$

is a sequence of C^∞ diffeomorphisms. (In particular, $M_k \approx M$, proving Cheeger's Finiteness theorem.)

Since $M = f_\kappa(M_\kappa)$, it has a preferred atlas $\mathscr{A}$ of the plaque-charts, $\Theta_{i\kappa} : B(r) \to \theta_{i\kappa}$, but it also has atlasses $\mathscr{A}_k = \{\rho \circ \Theta_{ik}\}$, $k \geq \kappa$. Since ρ is fixed, C^∞, and the Θ_{ik} are uniformly C^2 bounded according to (24), it follows that the charts $\rho \circ \Theta_{ik}$ are all C^2 uniformly related to each other. In fact, $\rho \circ \Theta_{ik} C^{1,1}$-converges to $\rho \circ \bar{\Theta}_i$ as $k \to \infty$. Write $\mathscr{A}^*$ for the union of all the atlasses $\mathscr{A}_k$ with $k \geq \kappa$.

Now, let $h_k = \rho \circ f_k$ push ahead the Riemann structure g_k on M_k to a Riemann structure $\tilde{g}_k \overset{\text{def}}{=} h_{k*} g_k$ on M. The coordinate expressions for g_k and g_k^{-1} respecting the almost linear atlasses $\{\psi_{ik}\}$ are uniformly C^1 bounded, and the same must be true for any atlasses which are C^2 uniformly related to $\{\psi_{ik}\}$. See [**19**, p. 58]. Thus, the $\mathscr{A}^*$-coordinate expressions for $\tilde{g}_k$ and $\tilde{g}_k^{-1}$ are uniformly C^1 bounded. Choosing a subsequence, we may assume that $\tilde{g}_k$ converges to a Lipschitz Riemann structure g on M. Curvature, diameter, and volume are intrinsic, so the sequence $(M, \tilde{g}_k)$ has bounded geometry; that is,

$$\tilde{g}_k \ \mathscr{G}\text{-converges to } g \text{ on } M.$$

By theorem 2, its distance function d is $C^{1,1}$ and this is exactly what Gromov asserted: *A subsequence of (M_k, g_k) becomes more and more isometric to (M, g) where g (is Lipschitz and) has $C^{1,1}$ distance function.*

4. *The capped cylinder*

Let M be an embedded 2-sphere in $\mathbb{R}^3$ consisting of two hemi-spheres glued onto the ends of a cylinder [**6**, p. 4]; M carries the Riemann structure g inherited from $\mathbb{R}^3$. See figure 2.

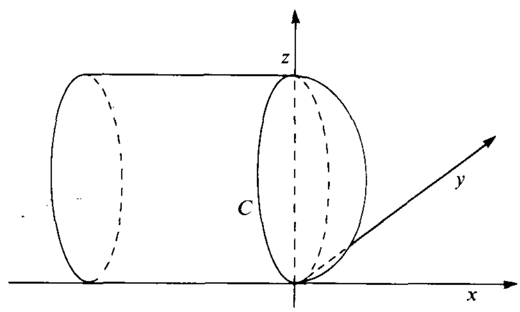

FIGURE 2. Half the capped cylinder.

As a surface in $\mathbb{R}^3$, M is of class $C^{1,1}$. It is easy to $\mathscr{G}$-approximate (M, g) by C^∞ spheres with bounded geometry; (M, g) is of class $\mathscr{G}$.

Observation 1. *Respecting the natural $C^{1,1}$ differentiable structure on M, g is Lipschitz but not C^1.*

Reasons. The equations for half of the capped cylinder are

$$\begin{array}{cc} x \le 0 & x \ge 0 \\ (z-1)^2 + y^2 = 1 & (z-1)^2 + x^2 + y^2 = 1. \\ \text{cylinder} & \text{hemi-sphere} \end{array}$$

The cylinder meets the hemisphere in the circle $C = M \cap \{x = 0\}$. The lower half of M is the image of $h = h(x, y)$ where

$$\begin{array}{cc} x \le 0 & x \ge 0 \\ h(x, y) = (x, y, 1 - \sqrt{(1-y^2)}) & h(x, y) = (x, y, 1 - \sqrt{(1-(x^2+y^2)})). \end{array}$$

Let e_1, e_2 be the standard unit vectors in $\mathbb{R}^2$. Their images under h_* are the tangent vectors to M at $h(x, y)$:

$$\begin{array}{cc} x \le 0 & x \ge 0 \\ h_*(e_1) = (1, 0, 0) & h_*(e_1) = \left(1, 0, \dfrac{x}{\sqrt{(1-(x^2+y^2))}}\right) \\ h_*(e_2) = \left(0, 1, \dfrac{y}{\sqrt{(1-y^2)}}\right) & h_*(e_2) = \left(0, 1, \dfrac{y}{\sqrt{(1-(x^2+y^2))}}\right). \end{array}$$

We then compute $g_{ij}(x) = \langle h_*(e_i), h_*(e_j) \rangle$ where $\langle\ ,\ \rangle$ is the Euclidean dot product in $\mathbb{R}^3$. For g_{12} this gives

$$\begin{array}{cc} x \le 0 & x \ge 0 \\ g_{12}(x, y) = 0 & g_{12}(x, y) = \dfrac{xy}{1-(x^2+y^2)} \\ \dfrac{\partial g_{12}}{\partial x} = 0 & \dfrac{\partial g_{12}}{\partial x} = \dfrac{y(1-(x^2+y^2)) + 2x^2 y}{(1-(x^2+y^2))^2}. \end{array}$$

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Thus, for $y \neq 0$, $\partial g_{12}/\partial x$ does not converge to the same limit when x tends to 0 from the left as from the right. This means that g is not C^1 at the circle C although it *is* Lipschitz vector field? g' is the corresponding Riemann structure $f_* g$ on M'. Such

Observation 2. *The g-exponentials of the capped cylinder are not of class C^1. Consequently, the g-geodesic flow is not C^1.*

Reasons. Consider the geodesics spraying out of the point $x = y = 0$ on M, and project them into $\mathbb{R}^2$ by the vertical projection. On the $x > 0$ side we see straight radii emerging from the origin. On the $x < 0$ side we see projected helices, $t \mapsto (-at, \sin t)$ with $a > 0$. Fix $0 < y_1 < y_2 < 1$ and draw short segments τ_1, τ_2 at $(0, y_1)$, $(0, y_2)$ parallel to the x-axis. See figure 3.

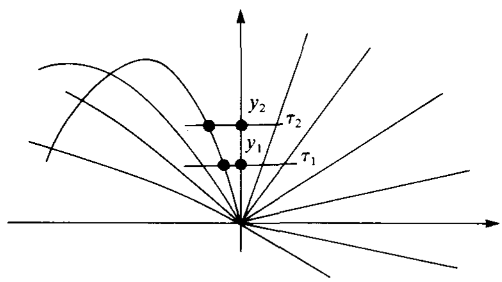

FIGURE 3. Top view of geodesics spraying out of the origin.

The projected helices are the graphs of

$$x = -a \arcsin(y) \qquad 0 \le y \le 1.$$

Such a curve crosses τ_1 at $x_1 = -a \arcsin(y_1)$ and crosses τ_2 at $x_2 = -a \arcsin(y_2)$. Thus, the holonomy map along the radial foliation (seen in the h-chart) from τ_1 to τ_2 is

$$\begin{array}{cc} x<0 & x>0 \\ \tau_1 \ni x_1 \mapsto x_1 \dfrac{\arcsin(y_2)}{\arcsin(y_1)} = x_2 \in \tau_2 & \qquad \tau_1 \ni x_1 \mapsto x_1 \dfrac{y_2}{y_1} = x_2 \in \tau_2 \end{array}$$

Since $0 < y_1 < y_2 < 1$ implies

$$\frac{\arcsin(y_2)}{\arcsin(y_1)} > \frac{y_2}{y_1},$$

we see that the two holonomy maps do not have a common derivative at $x = 0$. Hence, the vertical projection of the radial foliation at 0 is not C^1. Since this projection, restricted to M, is a $C^{1,1}$ diffeomorphism, the radial foliation is not C^1 in M. Clearly, then, the g-exponential map and the g-geodesic flow are not C^1, *nor* (*in contrast with the Riemann structure* g) *can they be made* C^1 *by looking at nicer charts,* C^1 *related to h.*

Remark. It is quite interesting that the sphere foliation, $d_g = \text{constant}$, is $C^{1,1}$, while the radial foliation is only Lipschitz. It's another instance of codimension 1 being more regular than dimension 1, cf. [**11**], [**12**].

Observation 3. *The vector field X tangent to the g-geodesic flow is undefined at tangent vectors $v \in T_pM$ when $p \in C$ and $v \pitchfork C$.*

Reasons. Similar. The Christoffel matrix determines X and it is discontinuous at C.

Question. Let (M, g) be a Gromov limit. Is there a C^1 diffeomorphism f of M onto a smooth manifold M' such that the g'-geodesic flow on $T(M')$ is generated by a Lipschitz vector field? g' is the corresponding Riemann structure f_*g on M'. Such an f is a *smoothing* of (M, g). It is an isometry from (M, g) to (M', g').

Remark 1. In [**4**] O. Durumeric asserts that the Gromov limit Riemann structure is of class C^1 'in a smooth coordinate chart'. Thus, he claims that there exists a smoothing of (M, g) such that the g'-geodesic flow is generated by a continuous vector field. See also remark 3, below.

Remark 2. D. Hart's smoothing theorem [**9**] asserts that any C^r flow is C^r equivalent to a flow generated by a C^r vector field provided that r is an integer ≥ 1. It is unknown if his result is valid for r = Lipschitz, but if it is then the preceding question has an affirmative answer and we get a more elementary explanation of theorem 2. On the other hand, note that there exist C^0 *flows* on $\mathbb{R}^4$ that are not topologically equivalent to flows generated by C^0 vector fields, so Hart's result is not valid for $r = 0$. (For $\mathbb{R}^4$ is homeomorphic to $\mathbb{R} \times W$ where W is a Whitehead set that is nowhere Euclidean. The corresponding flow has no local transversal and is therefore inequivalent to a vector field generated flow.)

Remark 3. There exists no smoothing of the capped cylinder which makes the geodesic flow generated by a C^1 vector field because the radial exponential foliation is not C^1 (observation 3) and this is a property invariant under C^1 isometry. *Lipschitz is the maximum expectable regularity.* For the same reason, there might exist a smoothing in which g' is of class $C^{1+\mathrm{Lip}}$ but there can be none in which it is of class C^2. In fact, in [**6**] Greene and Wu get g' of class $C^{1+\nu}$ for all $\nu < 1$, which just barely falls short.

Remark 4. A more general $\mathcal{G}$-limit manifold than the capped cylinder would be composed of countably many smooth pieces glued $C^{1,1}$ along their edges. For example, one could take countably many latitudinal bands of two-spheres and glue them to appropriate latitudinal bands of cones, cylinders, or the bugle surface. A characterization of the *generic* $\mathcal{G}$-limit manifold might concern the degree to which its geodesic flow fails to be C^1.

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