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INTEGRAL MEANS ON RADIALLY WEIGHTED SPACES OF ANALYTIC FUNCTIONS

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Abstract

Hilbert spaces of analytic functions generated by rotationally symmetric measures on disks and annuli are studied. A domination relation between function norm and weighted sums of integral means on circles is developed. The function norm and the weighted sum take the same value for a specified class of polynomials. This class can be varied according to two parameters. Parts of the construction carry over to other Banach spaces of analytic or harmonic functions. Counterexamples illuminating properties of the complex method of interpolation appear as a byproduct.

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0. Introduction

Let Ω_1 be a generalized circular annulus $A(R_1, R_2) = \{z \in \mathbb{C}; R_1 < |z| < R_2\}$. We allow $R_1 < 0$ as well as $R_2 = +\infty$ in order to represent disks centered at the origin or at the point at infinity. Let Ω be the union of Ω_1 with any or none of $\{|z| = R_1\}$, $\{|z| = R_2\}$, except that $R_2 = +\infty$ may not be used. On Ω we take a fixed but initially arbitrary rotationally symmetric measure μ. The centre of rotation is the origin. Consider $\Gamma_\mu = \{k \in \mathbb{Z}; \int_\Omega |z^k|^2 d\mu(z) < \infty\}$ and let $L = \inf \Gamma_\mu$, $K = \sup \Gamma_\mu$. In case $L < 0$ we impose the condition that $0 \notin \Omega$. At times the interval $I_\Omega = \Omega \cap [0, \infty[$ will be used.

This paper is concerned with the generalized Bergman spaces $L^2_a = L^2_a(\Omega, \mu)$ consisting of the closure in $L^2(\Omega, \mu)$ of $\{\phi_k : \phi_k(z) = z^k, k \in \Gamma_\mu\}$. Due to rotational symmetry the ϕ_ks are orthogonal and L^2_a consists entirely of analytic functions.

The norm is $\|f\| = \left(\int_\Omega |f(z)|^2 d\mu(z)\right)^{1/2}$ and the natural inner product is used. Observe that the Fock-Bargmann space $L^2_a(\mathbb{C}, e^{-|z|^2} dA(z))$, the common Bergman spaces $L^2_a(\mathbb{U}, |z|^\alpha dA(z))$ as well as spaces containing functions with poles in the hole interior to Ω are contained in the setting above.

The rotational symmetry of μ allows a decomposition $d\mu = d\theta \otimes d\nu_1 = d\theta \otimes d\nu_2$ in the sense that

$$\int_\Omega |f(z)|^2 d\mu(z) = \iint_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}\, d\nu_1(r) = \iint_0^{2\pi} |f(\sqrt{x}e^{i\theta})|^2 \frac{d\theta}{2\pi}\, d\nu_2(x),$$

where ν_1 is supported in I_Ω and ν_2 in $J_\Omega = \{x \geq 0;\ \sqrt{x} \in I_\Omega\}$. Notice that there is a bijective correspondence $\mu \leftrightarrow \nu_2$ through the change of variables $z = \sqrt{x}\, e^{i\theta}$. We will make an assumption throughout this paper, that ν_2 *is not supported on finitely many points only.*

This suggests the integral mean mapping $M_f : L^2_a \to L^2(I_\Omega, \nu_1)$ given by

$$M_f(r) = \left\{ \int_{|z|=r} |f(z)|^2 \frac{d\theta}{2\pi} \right\}^{1/2}.$$

Each $f \in L^2_a$ can be expanded $f(z) = \sum_{n=L}^K a_n z^n$, whence $M_f(r)^2 = \sum |a_n|^2 r^{2n}$. It follows that

$$\|f\| = \left\{ \sum_{n=L}^{K} |a_n|^2 \int_{J_\Omega} x^n\, d\nu_2(x) \right\}^{1/2}.$$

In [A] the author used the above interpretation for the weighted Bergman spaces with $d\mu_{\alpha\beta}(z) = |z|^{2\beta}(1 - |z|^2)^\alpha dA(z)$, where dA is area measure on the unit disk. A central result of [A] was the following theorem.

THEOREM A. *Let $k \geq 0$, $l \geq 1$ be integers, $\alpha, \beta > -1$. Then there exist positive numbers A_j and r_j such that for every $f \in L^2_a(\mathbb{U}, \mu_{\alpha\beta})$ the following inequality holds.*

$$\sum_{j=1}^{l} A_j \int_{|z|=r_j} |f(z)|^2 \frac{d\theta}{2\pi} \leq \|f\|^2.$$

In addition, equality holds precisely for all $f(z) = a_k z^k + \cdots + a_{k+2l-1} z^{k+2l-1}$. The numbers r_j^2 are the zeros of the lth Jacobi polynomial $P_l^{(\alpha,\beta)}(2x - 1)$.

It is the purpose of this paper to expand this result to the general setting. The complete counterpart is stated in section 3. A key technique in [A] was to use Rodrigue's formula for the Jacobi polynomials. That formula is not available in general, so a different approach will be taken. As a notable side effect we will find that the integral means form a partial order of norms as the number of circles increases. For example, the mean along one circle does not exceed a particular weighted combination of integral means along two other circles. Finally, the last two sections deal with examples showing the shortcomings of the present technique if we on one hand study harmonic functions and on the other hand we want to look at the general $L^p(\mu)$-version of Bergman spaces.

The original motivation for this paper was to explain the observation made in [M]. As a result, with new methods we have also improved the results that so far have been an ingredient in studying the Korenblum conjecture for the standard Bergman space, see [Ko], [KORZ], [KR], and [S]. The techniques to date have made use of the totally monotone functions. The direct connection of the present construction to the conjecture was discussed in [A]. As an added benefit the method happens to produce a new example related to the complex method of interpolation. This is explained in the last section.

1. Domination and restriction

DEFINITION 1.1. Say that a measure κ on Ω is a *contractive Carleson measure* for L^2_a if for every $f \in L^2_a$ its norm is controlled by $\|f\|_{L^2(\Omega,\kappa)} \leq \|f\|$, that is, $\int |f(z)|^2\, d\kappa(z) \leq \int |f(z)|^2\, d\mu(z)$; κ is said to be *saturated* in the case that equality can occur.

With Theorem A in view, the case of radially finitely supported measures is useful.

DEFINITION 1.2. We say that $\lambda = \sum_{j=1}^{l} A_j \delta_{r_j}$, where $A_j \geq 0$ are real and δ_{r_j} is the point mass at $r_j > 0$, is a *contractive restriction measure* if $\|M_f\|_{L^2(\lambda)}$ is at most $\|f\|$, that is, if for all $f \in L^2_a$

$$\sum_{j=1}^{l} A_j \int_{|z|=r_j} |f(z)|^2 \frac{d\theta}{2\pi} \leq \int_\Omega |f(z)|^2\, d\mu(z).$$

PROPOSITION 1.3. *A rotationally symmetric measure κ is a contractive Carleson measure if and only if $\int_\Omega |z^n|^2\, d\kappa(z) \leq \int_{J_\Omega} x^n\, d\nu_2(x)$ for all $n \geq 0$. It is saturated precisely in the case that equality occurs for some n.*

In particular $\lambda = \sum_{j=1}^{l} A_j \delta_{r_j}$ is a contractive restriction measure if and only if for all $n \in \mathbb{Z}$

$$\sum_{j=1}^{l} A_j r_j^{2n} \leq \int_{J_\Omega} x^n\, d\nu_2(x).$$

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The proof is easily obtained.

It is more convenient to construct the restriction measure using notation inherent in ν_2. We will hence be mainly concerned with the relation

$$\sum_{j=1}^{l} A_j x_j^n \leq \int_{J_\Omega} x^n \, d\nu_2(x). \tag{1.1}$$

OBSERVATION 1.4. *Suppose that given the measure ν_2 and an integer $l \geq 1$, it is possible to determine $A_j > 0$, $j = 1, \dots, l$, and x_j in the interior of J_Ω such that* (1.1) *holds for all $n \in \mathbb{Z}$. Then $r_j = \sqrt{x_j}$ and A_j produce a contractive restriction measure for $d\mu = d\theta \otimes d\nu_2$ as defined in the introduction.*

This is a straightforward consequence of the material so far.

The strategy to follow from now on is this. We fix a positive measure ν_2 on $[0, \infty[$ and strive to make (1.1) valid for every n. However, we will add more demands on $\{x_j\}$, $\{A_j\}$. They must also be such that (1.1) turns into *equality* for all n in $\{k, k+1 \dots, k+2l-1\}$, where we specify k in advance. Provided the construction is successful, Theorem A carries over to the present setting.

In fact, this will be achieved using the polynomials $\{p_l\}_{l=0}^{\infty}$ orthogonal with respect to $d\alpha_k(x) = x^k \, d\nu_2(x)$. The nodes $\{x_j\}_1^l$ will be the zeros of $p_l(x)$ and the weights $\{A_j\}_1^l$ will be the slightly modified Christoffel numbers of order l for α_k.

DEFINITION 1.5. Let m_n denote the *nth moment*: $m_n = \int x^n \, d\nu_2(x)$. It is easily realized that $m_n = \|z^n\|^2 = \int_\Omega |z^n|^2 \, d\mu(z)$.

The single radius case is simple enough to be demonstrated at once.

PROPOSITION 1.6. *Fix $k \in \Gamma_\mu$, with $k+1 \in \Gamma_\mu$, and put $l = 1$, $x_1 = m_{k+1}/m_k$, as well as $A_1 = m_k^{k+1}/m_{k+1}^k$. Then* (1.1) *holds for all $n \in \mathbb{Z}$. This means that for all $f \in L_a^2$ we have*

$$\frac{m_k^{k+1}}{m_{k+1}^k} \int_{|z|=\sqrt{m_{k+1}/m_k}} |f(z)|^2 \, \frac{d\theta}{2\pi} \leq \|f\|^2_{L_a^2(\Omega,\mu)}.$$

In addition, equality holds only for $f(z) = a_k z^k + a_{k+1} z^{k+1}$, all $a_k, a_{k+1} \in \mathbb{C}$.

PROOF. We need to prove that (1.1) holds and that equality occurs only for $n = k$ and $n = k+1$. It suffices to assume $n \in \Gamma_\mu$.

Starting with equality for $n = k, k+1$, (1.1) demands

$$\left\{ \begin{array}{c} A_1 x_1^k = m_k \\ A_1 x_1^{k+1} = m_{k+1} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} x_1 = m_{k+1}/m_k \\ A_1 = m_k^{k+1}/m_{k+1}^k \end{array} \right\}.$$

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With this choice, (1.1) for arbitrary n requires

$$A_1 x_1^n \leq m_n \quad \Leftrightarrow \quad m_{k+1}^{n-k} \leq m_n m_k^{n-k-1}. \tag{1.2}$$

For each $n > k+1$ Hölder's inequality with $p = n-k$, $p' = (n-k)/(n-k-1)$ yields

$$m_{k+1} \leq \left(\int x^{np/(n-k)}\, d\nu_2(x) \right)^{1/p} \left(\int x^{k(n-k-1)p'/(n-k)}\, d\nu_2(x) \right)^{1/p'} = m_n^{1/(n-k)} m_k^{(n-k-1)/(n-k)}$$

which is (1.2). For $n < k$ the condition (1.2) takes the form $m_k^{k+1-n} \leq m_n m_{k+1}^{k-n}$ and Hölder's inequality is applied analogously with $p = k+1-n$ and $p' = (k+1-n)/(k-n)$. The claim for equality follows from the well known conditions on equality in Hölder's inequality. This finishes the proof. □

REMARK. This proposition is the one closest to [M]. The entities x_1 and A_1 will reappear below for the polynomial of first degree connected to $d\alpha_k = x^k d\nu_2$.

2. Quadrature formulas with poles

This section begins with the facts on orthogonal polynomials to be used in the construction.

Again we fix $k \in \Gamma_\mu$ and consider $d\alpha_k(x) = x^k\, d\nu_2(x)$ on $[0, \infty[$. Let $\{p_n\}_{n=0}^\infty$ be a sequence of polynomials obtained by successive orthogonalization of $1, x, x^2, \ldots$ with respect to α_k. The polynomial p_n of degree n is unique up to a constant factor, which we will choose presently.

PROPOSITION 2.1. *The polynomials $\{p_n\}_{n=0}^\infty$ given by*

$$p_n(x) = \begin{vmatrix} 1 & x & \ldots & x^n \\ m_k & m_{k+1} & \ldots & m_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k+n-1} & m_{k+n} & \ldots & m_{k+2n-1} \end{vmatrix}$$

are orthogonal with respect to α_k.

PROOF. It suffices to prove $\int x^l p_n(x)\, d\alpha_k(x) = 0$ for $l = 0, \ldots, n-1$. The integral is readily rewritten as

$$\begin{vmatrix} m_{k+l} & m_{k+l+1} & \ldots & m_{k+l+n} \\ m_k & m_{k+1} & \ldots & m_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k+n-1} & m_{k+n} & \ldots & m_{k+2n-1} \end{vmatrix}$$

which obviously is zero for all $l = 0, \ldots, n-1$. □

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From now on we fix the orthogonal system $\{p_n\}_{n=0}^{\infty}$ according to Proposition 2.1. We will for each k deal with only this choice.

PROPOSITION 2.2 ([F, Satz I.2.2]). *Every p_n has exactly n real and simple zeros. If ν_2 has support in $[0, R[$, $R = \infty$ allowed, then all zeros lie in $]0, R[$.*

The proof hinges only on the orthogonality and not on the representation as a determinant. The standard demonstration is omitted here.

Our way to establish (1.1) is now set.

PROPOSITION 2.3. *Let ν_2 be a measure on $[0, \infty[$ and put $\Gamma = \{n \in \mathbb{Z}; \int x^n d\nu_2(x) < \infty\}$. Furthermore, take integers $k \in \Gamma$ and $l \geq 1$ such that $k + 2l - 1 \in \Gamma$. Then there are explicit points $\{x_j\}_{j=1}^{l}$ in $]0, \infty[$, and positive numbers $\{A_j\}_{j=1}^{l}$ such that the following formula holds. For every $n \in \Gamma$*

$$\int x^n \, d\nu_2(x) \geq \sum_{j=1}^{l} A_j x_j^n. \tag{2.1}$$

Equality holds precisely in the case that n belongs to $\{k, k+1, \ldots, k+2l-1\}$.

The proof will be conducted in three steps.

(i) Existence of real A_j.

Recall the technique of Hermitian polynomial interpolation:

For every set $\{y_j\}_1^m$ of nodes with multiplicities $\{d_j\}_1^m$, where $\sum_i^m d_j = n$, and for every smooth function f there exists a polynomial $P_n(x; f)$ of degree at most $n - 1$ such that

$$P_n^{(i)}(y_j; f) = f^{(i)}(y_j), \quad 1 \leq j \leq m, \; 0 \leq i < d_j.$$

For polynomials f of degree at most $n - 1$ the relation $P_n(\cdot\,; f) = f$ holds.

This is proved in [Kr, pages 45-49] and [F, Hilfssatz I.1.3].

We first take the measure $d\alpha_k(x) = x^k \, d\nu_2(x)$, whose set of finite moments is $-k + \Gamma$. In particular, the moments of order $0, \ldots, 2l - 1$ are finite. Let p_l be the lth degree polynomial orthogonal with respect to α_k and take $\{x_j\}_{j=1}^{l}$ to be its zeros according to Proposition 2.2.

We now consider the interpolation nodes $\{x_j\}_{j=1}^{l}$ and multiplicities $d_j = 1$. The corresponding Lagrange interpolation polynomial is of degree $l - 1$:

$$P_{l-1}(x; f) = \sum_{j=1}^{l} \beta_j(x) f(x_j). \tag{2.2}$$

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Integrating this we find a quadrature formula exact for polynomials of degree at most $l-1$:

$$\int f(x)x^k \, d\nu_2(x) = \sum_{j=1}^{l} A_j x_j^k f(x_j). \tag{2.3}$$

The constants are for notational convenience chosen with factors x_j^k. Next, (2.3) must hold for degrees not greater than $2l-1$. Every such polynomial r is given by $r(x) = r_1(x) + p_l(x)r_2(x)$, where r_1 and r_2 have degrees not exceeding $l-1$. From Proposition 2.1 it follows that $\int p_l(x)r_2(x)x^k \, d\nu_2(x) = 0$, whence $\int r(x)x^k \, d\nu_2(x) = \int r_1(x)x^k \, d\nu_2(x)$. On the other hand, the value of the right-hand side in (2.3) is the same for $f = r$ and $f = r_1$.

(ii) Positivity of A_j.

Put $r_j(x) = [p(x)/(x-x_j)]^2$, which is of degree $2l-2$. Using (2.3)

$$0 < \int r_j(x)x^k \, d\nu_2(x) = A_j x_j^k p'(x_j)^2,$$

and consequently A_j is strictly positive.

(iii) The inequality for $n < k$ and $n \geq k + 2l$.

Take $n \in \Gamma$ with $n < k$. Let $Q_n(x)$ be the polynomial specified by Hermitian interpolation:

$$\begin{cases} Q_n(x_j) = x_j^{n-k}, & j = 1, \ldots, l, \\ Q_n'(x_j) = (n-k)x_j^{n-k-1}, & j = 1, \ldots, l. \end{cases} \tag{2.4}$$

Consider $R_n(x) = x^{n-k} - Q_n(x)$, which has zeros of multiplicity two at each x_j, and is smooth in $]0, \infty[$. In addition, $R_n^{(2l)}(x) > 0$ on the right half-line, whence $R_n^{(2l-1)}$ has at most one zero there. Then $R_n^{(2l-2)}$ can have at most two zeros in $]0, \infty[$. Repetition of this argument proves that R_n itself has at most $2l$ zeros counting multiplicity. The construction of Q_n now forces $2l$ to be the exact number. The zero at every x_j has exact order two, and since $\lim_{x \to 0^+} R_n(x) = +\infty$, we find $R_n \geq 0$ in $]0, \infty[$, that is, $x^{n-k} \geq Q_n(x)$ on the right half-line. Next, $Q_n(x)$ has degree $2l-1$, whence application of (2.3) and (2.4) shows that

$$\int x^n \, d\nu_2(x) > \int x^k Q_n(x) \, d\nu_2(x) = \sum_{j=1}^{l} A_j x_j^k Q_n(x_j) = \sum_{j=1}^{l} A_j x_j^n.$$

The strict inequality is due to the assumption that the support is not finite.

For the remaining case $n \geq k + 2l$ the argument is the same, except that now $\lim_{x \to +\infty} R_n(x) = +\infty$ is used in order to gain $R_n(x) \geq 0$ on the right half-line. This completes the proof of Proposition 2.3. □

REMARK. The proof of the above inequalities has been influenced by [F, Hilfssatz III.1.5]. Such results were first found by Shohat.

3. Restriction operators

The careful formulation of the result in the preceding section is as follows.

THEOREM 3.1. *Let μ be a rotationally symmetric measure on $\mathbb{C}$ with radial component measure ν_2. Fix integers $l \geq 1$ and $k \in \Gamma_\mu$ such that $k + 2l - 1 \in \Gamma_\mu$. Denote by $x_1, \ldots, x_l$ the zeros of the lth polynomial orthogonal with respect to $x^k\, d\nu_2(x)$. Determine the (positive) numbers $A_1, \ldots, A_l$ through the equations $\sum_{j=1}^{l} A_j x_j^n = \int x^n\, d\nu_2(x)$, $n = k, \ldots, k + l - 1$. Then the norm on $L_a^2(\mu)$ has the property that, for $r_j = \sqrt{x_j}$ and all $f \in L_a^2(\mu)$,*

$$\sum_{j=1}^{l} A_j \int_{|z|=r_j} |f(z)|^2 \frac{d\theta}{2\pi} \leq \|f\|^2.$$

Moreover, equality holds precisely in case that f is of the form $f(z) = a_k z^k + \cdots + a_{k+2l-1} z^{k+2l-1}$.

DEFINITION 3.2. The *Jacobi restriction measure* for μ with parameters $k \in \Gamma_\mu$, $l \geq 1$, is the measure $\lambda = \lambda_{kl}(\mu)$ on $\mathbb{C}$ specified by: λ has support $\bigcup_{j=1}^{l} \{|z| = r_j\}$ and is uniform of mass A_j on each circle $\{|z| = r_j\}$, where r_j and A_j are taken from Theorem 3.1.

This concept of Jacobi restriction measure conforms to Definition 1.2, as we see from Theorem 3.1. Using a natural extension of Definition 1.1 we formulate the following theorem.

THEOREM 3.3. *The Jacobi restriction measure $\lambda = \lambda_{kl}$ for μ is a contractive Carleson measure for $L_a^{2m}(\Omega, \mu)$, where m is a positive integer, that is, for all f in $L_a^{2m}(\Omega, \mu)$ the inequality $\|f\|_{L^{2m}(\lambda)} \leq \|f\|_{L_a^{2m}(\Omega, \mu)}$ obtains. In addition, norm equality holds exactly when $f(z)^m = a_k z^k + \cdots + a_{k+2l-1} z^{k+2l-1}$.*

The proof is simply an application of Theorem 3.1 to the analytic function $f(z)^m$ in L_a^2, which appears for every $f \in L_a^{2m}(\Omega, \mu)$.

By variation of the parameters k and l we can even get a partial ordering of the corresponding Jacobi restriction measures.

THEOREM 3.4. *Suppose that $k \leq p < p + 2q - 1 \leq k + 2l - 1$. Then*

$$\|f\|_{L^{2m}(\lambda_{pq})} \leq \|f\|_{L^{2m}(\lambda_{kl})} \leq \|f\|_{L_a^{2m}(\mu)} \tag{3.1}$$

holds for all f analytic on Ω. Furthermore, the first equality holds precisely when the two outer members are equal.

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The technique used so far to compute moments reduces the proof to the demonstration of:

PROPOSITION 3.5. *Assume the relation* $k \le p < p+2q-1 \le k+2l-1$. *Denote the data for the restriction measure* $\lambda_{kl} = \lambda_{kl}(\mu)$ *by* $\{A_j\}_{j=1}^{l}$, $\{r_j = \sqrt{x_j}\}_{j=1}^{l}$, *and those for* $\lambda_{pq} = \lambda_{pq}(\mu)$ *by* $\{B_j\}_{j=1}^{q}$, $\{s_j = \sqrt{y_j}\}_{j=1}^{q}$. *Then for every* $n \in \mathbb{Z}$, *the nth moments satisfy*

$$\sum\nolimits_{j=1}^{q} B_j\, y_j^n \le \sum\nolimits_{j=1}^{l} A_j\, x_j^n. \tag{3.2}$$

Equality holds only for $n = p, p+1, \ldots, p+2q-1$.

PROOF. It suffices to prove the case $q = l-1$ and the subcases $p = k$ and $p = k+1$. The general case is proved by several applications of these reduced cases; in fact, as long as $k < p$ or $q < l-1$. Both of the subcases will be handled simultaneously. Hence we assume $q = l-1$ as well as $p = k$ or $p = k-1$. Write

$$K_1(t) = \sum\nolimits_{j=1}^{l} A_j\, x_j^t, \qquad K_2(t) = \sum\nolimits_{j=1}^{l-1} B_j\, y_j^t.$$

The original construction yields $K_1(n) = \int x^n\, d\nu_2(x)$, for $n = k, \ldots, k+2l-1$ as well as $K_2(n) = \int x^n\, d\nu_2(x)$, for $n = p, \ldots, p+2l-3$. In particular, $K_1(t) - K_2(t)$ has zeros at $t = p, \ldots, p+2l-3$, that is, at least $2l-2$ zeros. We need the following lemma which is a special case of [PS, section V, problem 75].

LEMMA 3.6. *Any real equation* $b_1 e^{a_1 x} + \cdots + b_m e^{a_m x} = 0$ *has at most* $m-1$ *real zeros.*

From the lemma we conclude that $K_1(t) - K_2(t)$ has exactly $2l-2$ zeros, namely the points $t = p, \ldots, p+2l-3$. From Section 2 we have $K_2(p-1) < \int x^{p-1}\, d\nu_2(x)$ and $K_2(p+2l-2) < \int x^{p+2l-2}\, d\nu_2(x)$. In case $p = k+1$ this means that $K_1(k) - K_2(k) > 0$ and in case $p = k$ we have $K_1(k+2l-2) - K_2(k+2l-2) > 0$. Since the zeros are distinct and even in number, $K_1(t) - K_2(t) > 0$ outside $[p, p+2l-3]$ in both subcases. Hence $K_1(n) \ge K_2(n)$ for all $n \in \mathbb{Z}$ and equality holds only for $n = p, \ldots, p+2l-3$. The proof is complete. □

REMARK. The partial ordering above is, in fact, cofinal: given two restriction measures we just choose parameters of a third measure so as to make it exact for at least all the moments for which either of the two given measures are exact. Obviously this net of measures converges to the original measure μ throughout Ω.

4. Harmonic functions

Not all of the preceding results are true if one relaxes the assumption on analyticity to only harmonicity. This will be demonstrated by counterexamples as well as by new proofs where something can be rescued. We will write $L_h^2(\Omega, \mu)$ for the space of complex valued *harmonic* functions in $L^2(\Omega, \mu)$.

PROPOSITION 4.1. *Suppose Ω is a disk and that g is harmonic in Ω. Then for all parameters k and l the inequality $\|g\|_{L^2(\lambda_{kl})} \leq \|g\|_{L_h^2(\mu)}$ holds. Hence the Jacobi restriction measure is a contractive Carleson measure for $L_h^2(\Omega, \mu)$.*

PROOF. Any function harmonic in a zero-centered disk can be represented $g(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$. Then we find $\int_{|z|=\sqrt{x}} |g(z)|^2 \, d\theta/2\pi = \sum_{n=-\infty}^{\infty} |a_n|^2 x^{|n|}$ due to orthogonality. This means we may apply the reasoning from the analytic case to deduce

$$\|g\|^2_{L_h^2(\Omega,\mu)} = \int \left(\sum_{n=-\infty}^{\infty} |a_n|^2 x^{|n|} \right) d\nu_2(x) \geq \sum_{j=1}^{l} A_j \int_{|z|=\sqrt{x_j}} |g(z)|^2 \frac{d\theta}{2\pi} = \|g\|^2_{L^2(\lambda_{kl})}.$$

□

PROPOSITION 4.2. *Let g be harmonic in Ω. Assume that $\int_{|z|=r} g(z) \, d\theta/2\pi$ is independent of r and that $0 \in \Gamma_\mu$, that is, $\mu(\Omega) < \infty$. For all parameters k and l such that $k \leq 0 \leq 2l + k - 1$, the norm inequality $\|g\|_{L^2(\lambda_{kl})} \leq \|g\|_{L_h^2(\mu)}$ holds.*

PROOF. The assumption on $g \in L^2(\mu)$ implies a representation

$$g(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \left(a_n r^n + b_n r^{-n}\right) e^{in\theta}.$$

It follows that

$$\int_{|z|=\sqrt{x}} |g(z)|^2 \frac{d\theta}{2\pi} = \sum_{n=-\infty}^{\infty} \left(|a_n|^2 x^n + |b_n|^2 x^{-n} + 2 \operatorname{Re} a_n \overline{b_n} \right). \tag{4.1}$$

Since $k \leq 0 \leq 2l + k - 1$, we know that the general relation $\int x^n \, d\nu_2(x) \geq \sum_{j=1}^{l} A_j x_j^n$ is an equality for $n = 0$. On the grounds that every coefficient in (4.1) with $n \neq 0$ is non-negative, a termwise integration yields

$$\int_\Omega |g(z)|^2 \, d\mu(z) \geq \int |g(z)|^2 \, d\lambda_{kl}(z),$$

which is the claimed property. □

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EXAMPLE 4.3. We construct two instances where relaxation of the conditions in Proposition 4.2 produces the reversed inequality.

Consider any ring domain and any accompanying ν_2 with $\int x^n \nu_2(x) < \infty$ for $n = -2, -1, 0, 1, 2$. Choose $a > 0$ such that $J_\Omega \subseteq]3/2a, \infty[$. It is claimed that for the parameters $l = 1$ and $k = 1$ there is a harmonic function $g \in L_h^2(\Omega, \mu)$ such that $\|g\|_{L_h^2(\mu)} < \|g\|_{L^2(\lambda_{11})}$. In fact, $g(z) = az - 1/\bar{z}$ is a good choice.

Put $F(x) = x^{-1} \int_{|z|=\sqrt{x}} |g(z)|^2 d\theta/2\pi = x^{-1}(a^2x - 2a + 1/x)$. The choice of a yields $F'(x) > 0$ and $F''(x) < 0$ throughout $]3/2a, \infty[$. We now use the technique from Section 2. Consider the one point quadrature formula for $x\, d\nu_2(x)$ specified by A_1 and x_1. Let $Q(x) = F'(x_1)(x - x_1) + F(x_1)$ be the Hermite interpolant at $x = x_1$ of order two. Then $F(x) - Q(x)$ has only one double zero in $]3/2a, \infty[$ and no other zeros. In fact, $F(x) \leq Q(x)$ throughout $]3/2a, \infty[$ with strict inequality for $x \neq x_1$. Integrating, we find (with the usual non-finiteness assumption on the support)

$$\int F(x)\, x d\nu_2(x) < \int Q(x)\, x d\nu_2(x) = A_1 x_1 Q(x_1) = A_1 x_1 F(x_1).$$

The definition of $F(x)$ finally yields

$$\int_\Omega |g(z)|^2\, d\mu(z) < A_1 x_1 F(x_1) = A_1 \int_{|z|=\sqrt{x_1}} |g(z)|^2 \frac{d\theta}{2\pi}.$$

Using a more intricate calculation we can get also $\|g\|_{L^2(\mu)} < \|g\|_{L^2(\lambda_{12})}$, which is an example with two circles. Let A_1, A_2, x_1, x_2, be the data for λ_{12}, and take $Q_2(x)$ to be the Hermite interpolant for the previous $F(x)$ with nodes x_1, x_2, and orders $d_j = 2$. One finds that $Q_2(x) = 2(x_2 - x_1)^{-2}(x_1^{-1} - x_2^{-1})^2(a - x_1^{-1} - x_2^{-1})x^3 +$ lower terms. Choosing a so large that $J_\Omega \subseteq]5/2a, \infty[$, one finds that $F^{(4)}(x) < 0$ and $F'''(x) > 0$ on $]5/2a, \infty[$. As before, the inequality $F(x) \leq Q_2(x)$ holds on J_Ω with equality only for $x = x_1, x_2$. Integrating this, $\int |g|^2\, d\mu < \int |g|^2\, d\lambda_{12}$ follows.

EXAMPLE 4.4. In the statement of Proposition 4.2 one contribution was excluded, namely the harmonic function $\log|z|$. Its inclusion will produce a further counterexample.

Consider $g(z) = \log a|z|$. Then $F(x) = \int_{|z|=\sqrt{x}} |g|^2\, d\theta/2\pi$ can, by a proper choice of a, give $F''(x) < 0$, $F'(x) > 0$ on any fixed $[\epsilon, \infty[$. Calculations similar to Example 4.3 will prove that $\|g\|_{L_h^2(\mu)} < \|g\|_{L^2(\lambda_{01})}$.

5. The intermediary L_a^p-spaces

It is natural to ask about the validity of the previous results for the spaces $L_a^p(\Omega, \mu)$. Some negative examples and positive results have been collected in this section.

Consider the subspace $L_l^p(\Omega, \mu)$ being the closure of the analytic polynomials in $L_a^p(\Omega, \mu)$. Each such function has a Laurent expansion in Ω containing only powers with non-negative exponents. In particular, they can be analytically continued into the hole around the origin, which Ω encloses. Using the method of [A] we have a differentiability result:

PROPOSITION 5.1. *For each $L_l^p(\Omega, \mu)$, $p \geq 2n-2$, the pth integral mean $M_p f(r) = \int_{|z|=\sqrt{r}} |f|^p \, d\theta/2\pi$ has n positive and continuous derivatives for r less than the outer radius of Ω.*

LEMMA 5.2. *Let $l \geq 1$ and $k \geq 0$. For any polynomial Q with non-negative coefficients of each order not contained in $\{k, k+1, \ldots, 2l+k-1\}$, a quadrature estimate holds: $\int Q \, d\nu_2 \geq \sum A_j Q(x_j)$, where x_j are the points constructed for λ_{kl}.*

PROOF. Write $Q(x) = q_1(x) + x^k q_2(x) + x^{2l+k} q_3(x)$, with $\deg q_1 \leq k-1$ and $\deg q_2 \leq 2l-1$. We know that $\sum A_j q_2(x_j) x_j^k$ is equal to $\int q_2(x) x^k \, d\nu_2$ by construction. Furthermore, $\sum A_j x_j^n \leq \int x^n \, d\nu_2$ for all integers n, so using the non-negativity of the coefficients in q_1 and q_3, the addition of all contributions including q_2 establishes the claim. $\square$

A result for general Bergman spaces is given by the following proposition.

PROPOSITION 5.3. *Suppose the ring domain Ω is bounded and $l \geq 1$, $k \in \Gamma_\mu$. Put $\kappa = \max(0, k)$ and suppose that $p \geq 4l + 2\kappa - 2$. Then every $f \in L_l^p(\Omega, \mu)$ satisfies for $\lambda = \lambda_{kl}$*

$$\|f\|_{L^p(\lambda)} \leq \|f\|_{L_a^p(\mu)}.$$

REMARK. Our method of proof is not suitable for functions with a proper Laurent expansion so they will be ignored for this result.

PROOF. Let $M(r) = M_p f(r)$ be the pth integral mean of f. From Proposition 5.1 we know that $M \in C^{(\kappa+2l)}([0, R])$ and all derivatives are non-negative. Consider the Hermite interpolant $P(r)$ for $M(r)$ with nodes $x_1, \ldots, x_l$ of degree 2 and $x_0 = 0$ of degree κ. Following [A] or [Kr, pages 45–49] we have

$$M(r) = P(r) + \frac{\omega(r)^2 r^\kappa}{(\kappa + 2l)!} M^{(\kappa+2l)}(\xi),$$

where ξ is in $[0, R]$ and $\omega(r)$ is the polynomial with zeros $x_1, \ldots, x_n$ and suitably normalized. Since no derivative of M is negative, we know that the terms of orders $0, 1, \ldots, \kappa-1$ in $P(r)$ are non-negative. In the case $k \geq 1$ we may invoke Lemma 5.2.

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The non-negative contribution from the remainder term is unproblematic and we conclude that

$$\|f\|^p_{L^p(\lambda)} = \int M(|\cdot|)\,d\lambda \leq \int M\,d\nu_2 = \|f\|^p_{L^p_a(\mu)}.$$

For the possibility $\kappa = 0$, that is, $k \leq 0$, a result similar to Lemma 5.2 but with exactness for the terms of order $\{0, \dots, 2l+k-1\}$ (or for none at all when $k < -2l+1$) is incorporated to yield the same conclusion. $\square$

Presently we will produce examples showing that the range of the exponent p is optimal to some extent, but first we will reinterpret the method in the way it depends on convexity. This will produce an ample supply of counterexamples.

After a change of measure the partial ordering at the end of Section 3 says that if $0 \leq n < 2l-1$, then the inequality $\int |f|^2\,d\lambda_{n1} \leq \int |f|^2\,d\lambda_{0l}$ holds. To see the meaning of this relation we let $g(x)$ be an arbitrary function for which the inequality $\int g\,d\lambda_{n1} \leq \int g\,d\lambda_{0l}$ holds. Denote the data for λ_{0l} by x_j, A_j and those for λ_{n1} by y_1, B_1. Our inequality claims that

$$B_1 g(y_1) \leq \sum\nolimits_{j=1}^{l} A_j g(x_j). \tag{5.1}$$

However, we also know that

$$B_1 y_1^n = \int x^n\,d\nu_2 = \sum A_j x_j^n,$$

$$B_1 y_1^{n+1} = \int x^{n+1}\,d\nu_2 = \sum A_j x_j^{n+1}.$$

Observe first the representation

$$y_1 = \sum_j \frac{A_j x_j^n}{\sum_k A_k x_k^n} x_j,$$

which expresses y_1 as a convex combination of all x_j. Next, (5.1) becomes

$$\frac{g(y_1)}{y_1^n} \leq \sum_j \frac{A_j x_j^n}{B_1 y_1^n}\frac{g(x_j)}{x_j^n} = \sum_j \frac{A_j x_j^n}{\sum_k A_k x_k^n}\frac{g(x_j)}{x_j^n},$$

where the right-hand side has the same convex combination as y_1 has.

OBSERVATION 5.4. (1) *For functions g such that $x^{-n}g(x)$ is convex, the inequality $\int g\,d\lambda_{n1} \leq \int g\,d\lambda_{0l}$ always holds.*

(2) *In the case that $x^{-n}g(x)$ is non-convex, the inequality in* (1) *is false as soon as the interval $[x_1, x_l]$ is an interval of concavity for this function.*

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Interpreting this for our integral means, we have established a geometrically improved result, related to Theorem 3.4:

PROPOSITION 5.5. *Suppose that* $k \le n < k + 2l - 1$, $p \ge 2$. *Then*

$$\|f\|_{L^p(\lambda_{n1})} \le \|f\|_{L^p(\lambda_{kl})}, \qquad \|f\|_{L^p(\lambda_{n1})} \le \|f\|_{L^p_a(\mu)}$$

hold for all measures μ *and all* f *analytic on* Ω *such that* $x^{-n} M_p f(x)$ *is convex. For each* f *giving non-convexity of* $x^{-n} M_p f(x)$ *there are measures* μ *simultaneously negating both inequalities for this particular function* f.

The proof is entirely contained in Observation 5.4 except the result for λ_{n1} and μ. This involves one radius and a quadrature formula exact for linear polynomials. The relation of the tangent to (respectively) a convex and non-convex curve is all that is needed for the integration to produce the claimed result. Observe also that for p an even integer convexity always results and this explains the previous success.

Now we can produce illuminating counterexamples.

EXAMPLE 5.6 (Sharpness of Proposition 5.3 with offset restriction). Take a non-negative integer n and consider $L^p_a(\Omega, \mu)$ with the only condition $2n < p < 2n + 2$. The Jacobi restriction measure $\lambda = \lambda_{n1}(\mu)$ will produce the required example.

The polynomial $f(z) = z$ has normalized integral mean $x^{-n} M_p f(x) = x^{p/2-n}$, which is strictly concave on the positive axis. By Proposition 5.5 and Observation 5.4 (2) the inequality $\|z\|_{L^p(\lambda)} > \|z\|_{L^p_a(\Omega,\mu)}$ holds for any measure μ on Ω. At the same time we find that for any other restriction measure we have $\|z\|_{L^p(\lambda_{n1})} > \|z\|_{L^p(\lambda_{kl})}$ as soon as $k \le n < k + 2l - 1$.

REMARK. Probably it is not worth the effort to try to construct counterexamples negating only one of the inequalities in Proposition 5.5.

EXAMPLE 5.7 (Sharpness of Proposition 5.3 for l radii). Let $l \ge 1$ and suppose that $4(l-1) < p < 4l - 2$. Our claim is that $\|z\|_{L^p(\mu)} < \|z\|_{L^p(\lambda)}$ for the Jacobi restriction measure $\lambda = \lambda_{0l}$ independently of μ. In fact, for $f(z) = z$ we have $d^{2l} M_p f(r)/dr^{2l}$ negative on $]0, \infty[$, since $M_p f(r) = r^{p/2}$ and $2l - 2 < p/2 < 2l - 1$. Let $Q(r)$ be the Hermite interpolant with l nodes of multiplicity two at each x_j corresponding to a radius in the construction of λ_{0l}. Since Q has degree $2l - 1$, which is greater than the power $p/2$ occurring in $M_p f(r)$, the observation $d^{2l}\{M_p f(r) - Q(r)\}/dr^{2l} < 0$ everywhere suffices to prove $M_p f \le Q$ with equality only at l points. The radially infinite support of μ then forces the claimed norm inequality.

Finally, let us cast one aspect of the material in a slightly different form. Theorem 3.3 says that the restriction operator $\mathscr{R}_{kl}$ has operator norm 1 from $L_a^{2m}(\mu, \Omega)$ to $L^{2m}(\lambda_{kl}, \cup_1^l\{|z| = r_j\})$, for each integer m. Were each $L_a^p(\mu)$ the correct interpolating space with respect to the Riesz–Thorin interpolation, then the operator norm $\mathscr{R}_{kl} : L_a^p(\mu) \mapsto L^p(\lambda_{kl})$ necessarily would have to be at most one for all $p \geq 2$. However, Examples 5.6 and 5.7 demonstrate that $\|\mathscr{R}_{k1}\|_{p\to p} > 1$ and $\|\mathscr{R}_{0l}\|_{p\to p} > 1$ for suitable p in the relevant range. Thus the interpolation between a couple of Bergman spaces yields spaces not isometric to the correct space in the same scale of Bergman spaces. Likewise by the same examples, not even the complex method of interpolation with respect to the full family $\{L_a^{2m}(\mu)\}_{m=1}^{\infty}$, in the sense of St. Louis spaces, isometrically preserves the Bergman spaces. This is so since the operator norm for St. Louis spaces obeys the same log-convexity property as expressed in the Riesz–Thorin theorem. Therefore we have found a new example consisting of analytic spaces illustrating the shortcomings of a complex method interpolation. For this particular operator $\mathscr{R}_{kl}$ the convexity of the integral means is governed by the expression $\sqrt{x}$ as seen above, whereas the log-convexity in general is incompatible with this notion of convexity. The logarithmic convexity is, however, the correct one in the three circles theorem of Hadamard and hence appears in the interpolation method.

Addendum: higher dimensions

The basic construction of the restriction measures λ_{kl} may be used also for harmonic functions in higher dimensional euclidean spaces. Only one result is stated here in order to present the mechanisms.

We consider measures μ on $\mathbb{R}^d$ invariant with respect to the action of $\mathrm{SO}(\mathbb{R}^d)$. These measures can be decomposed into a radial component and a normalized surface measure on the unit sphere, in a manner imitating the decomposition in the complex plane. The definition of the Jacobi restriction measure $\lambda_{kl}(\mu)$ is reinterpreted as producing a measure supported on the union $\bigcup_{j=1}^{l}\{x \in \mathbb{R}^d : |x| = r_j\}$. On each component $\{x \in \mathbb{R}^d : |x| = r_j\}$ the restriction of λ_{kl} is a multiple A_j of the normalized surface measure. This measure λ_{kl} exists precisely when $\int |x|^{2k}\, d\mu(x)$ and $\int |x|^{2(k+2l-1)}\, d\mu(x)$ are finite.

PROPOSITION. *Let $\Omega \subseteq \mathbb{R}^d$ be a ball centered at the origin. Suppose further that $k \leq p < p + 2q - 1 \leq k + 2l - 1$ holds and that λ_{kl} exists. Then every function f harmonic in Ω satisfies*

$$\|f\|_{L^2(\lambda_{pq})} \leq \|f\|_{L^2(\lambda_{kl})} \leq \|f\|_{L^2(\mu)}.$$

PROOF. Since λ_{kl} exists, so does λ_{pq} according to the condition on its parameters.

Using the Laplace's series for f, we have $f \sim \sum_{m=0}^{\infty} r^m Y_m$, where $Y_m \in \mathscr{H}_m$ is homogeneous of degree m. It follows that for normalized surface measure σ

$$\int_{|x|=1} |f(rx)|^2 \, d\sigma(x) = \sum_{m=0}^{\infty} \|Y_m\|_{L^2(\sigma)}^2 r^{2m}.$$

Using Proposition 2.3 as well as Proposition 3.5 we get the claimed ordering of the three norms. □

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