OSCILLATION OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Some new sufficient conditions are obtained for the oscillation of the neutral differential equation

\[ r(t)(y(t) - cy(t - \tau))'' + p(t)y^\alpha(t - \sigma(t)) = 0 \]

where \( r(t) > 0, 0 < c < 1, p(t) \geq 0, \sigma(t) > \tau > 0 \) and \( \alpha = 1 \) or \( 0 < \alpha < 1 \).

I. INTRODUCTION

In the past several years the oscillation problem for second order neutral differential equations of the form

\[ (y(t) + cy(t - \tau))'' + py(t - \sigma) = 0 \text{ where } \tau > 0 \text{ and } \sigma > 0 \]

has been considered by a number of authors [1, 2, 4-8]. Most of these papers treat the case where \( c > 0 \). In [6, 7] the case \( c < 0 \) was also studied for Equation (1.1) with constant coefficients and constant delay.

In this paper we consider second order linear and sublinear neutral delay differential equations of the form

\[ [r(t)(y(t) - cy(t - \tau))']' + p(t)y^\alpha(t - \sigma(t)) = 0 \]

where \( r, p, \sigma \) are continuous, \( r(t) > 0, 0 < c < 1, 0 < \alpha \leq 1 \) is a quotient of odd integers \( \sigma(t) > \tau > 0, \sigma'(t) \leq 1, \lim_{t \to \infty} (t - \sigma(t)) = \infty \) and \( p(t) \geq 0 \).

As mentioned in [6] there are many important applications for neutral differential equations of the form (1.2).

As usual, a solution of Equation (1.2) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

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II. LINEAR EQUATIONS

The following Lemmas will be used to prove the main results.

LEMMA 2.1. Assume \(0 < g(t) < t\) for \(t > 0\), \(\lim_{t \to \infty} g(t) = \infty\), \(g \in C(\mathbb{R}^+, \mathbb{R}^+)\), \(r(t) \in C(\mathbb{R}^+, \mathbb{R}^+)\) and assume \(r(t)\) is either nonincreasing (in brief n.i.) or nondecreasing (in brief n.d.). Let \(y \in C(\mathbb{R}^+, \mathbb{R})\) be such that \(r(t)y'(t) \in C(\mathbb{R}^+, \mathbb{R})\) and \(y(t) > 0\), \(y'(t) > 0\) and \((r(t)y'(t))' \leq 0\) for \(t \geq T\).

Then for each \(0 < k < 1\) there is a \(T_k \geq T\) such that either

\[
\text{(2.1)} \quad y(g(t)) \geq \frac{kr(t)g(t)}{tr(T)} y(t), \quad \text{for } t \geq T_k \geq T \text{ and } r \in \text{n.i.}
\]

or

\[
\text{(2.2)} \quad y(g(t)) \geq \frac{k}{t} y(t), \quad \text{for } t \geq T_k \geq T \text{ and } r \in \text{n.d.}
\]

Lemma 2.1 is a generalisation of Erbe's Lemma [3]. The proof of this Lemma can be given by the same argument as used in [3] so we omit it here.

LEMMA 2.2. We consider the delay differential inequality

\[
\text{(2.3)} \quad (r(t)z'(t))' - \frac{p(t)}{c} z(t - \sigma(t) + \tau) \leq 0
\]

where \(r, p, \sigma, c\) and \(\tau\) satisfy the assumptions for (1.2) in Section 1. Further assume that either

\[
\text{(2.4)} \quad \limsup_{t \to \infty} \frac{1}{r(t - \sigma(t) + \tau)} \int_{t-\sigma(t)+r}^{t} [u - (t - \sigma(t) + \tau)]p(u)du > c \text{ for } r \in \text{n.i.}
\]

or

\[
\limsup_{t \to \infty} \frac{1}{r(t)} \int_{t-\sigma(t)+r}^{t} [u - (t - \sigma(t) + \tau)]p(u)du > c \text{ for } r \in \text{n.d.}
\]

Then (2.3) has no negative increasing solution.

PROOF: Suppose the contrary and let \(z(t)\) be a negative increasing solution of (2.3).

Integrating (2.3) we have, for \(t > s\)

\[
\text{(2.5)} \quad r(t)z'(t) - r(s)z'(s) - \frac{1}{c} \int_{s}^{t} p(u)z(u - \sigma(u) + \tau)du \leq 0.
\]
Integrating (2.5) in \( s \) from \( t - \sigma(t) + \tau \) to \( t \), we have

\[
r(t)z'(t)(\sigma(t) - \tau) - \int_{t-\sigma(t)+\tau}^{t} f(s)dz(s) \]
\[
- \frac{1}{c} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)z(u - \sigma(u) + \tau)du \leq 0.
\]

We note that \( z'(t) > 0 \) so integrating the first integral by parts we have

(2.6)

\[
-r(t)z(t) = r(t - \sigma(t) + \tau)z(t - \sigma(t) + \tau) + \int_{t-\sigma(t)+\tau}^{t} z(s)dr(s) \]
\[
- \frac{1}{c} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)z(u - \sigma(u) + \tau)du \leq 0.
\]

For \( r \in n.d. \), we have

(2.7)

\[
\int_{t-\sigma(t)+\tau}^{t} z(s)dr(s) \geq z(t - \sigma(t) + \tau)[r(t) - r(t - \sigma(t) + \tau)].
\]

Combining (2.6) and (2.7) we have

\[
r(t)[z(t - \sigma(t) + \tau) - z(t)] - \frac{1}{c} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)z(u - \sigma(u) + \tau)du \leq 0.
\]

Dividing the above inequality by \( r(t)z(t - \sigma(t) + \tau) \) and noting the negativity of this term, we have

\[
1 - \frac{z(t)}{z(t - \sigma(t) + \tau)} - \frac{1}{cz(t - \sigma(t) + \tau)r(t)} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)du \geq 0.
\]

Since \( z(t) < 0 \) and \( z'(t) > 0 \), we have

\[
1 - \frac{z(t)}{z(t - \sigma(t) + \tau)} - \frac{1}{cr(t)} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)du \geq 0.
\]

Hence

\[
\frac{1}{cr(t)} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)du \leq 1
\]

which contradicts (2.4).

The case that \( r \in n.i. \) may be proved in a similar way. We omit the details.
Remark 2.1. If \( p(t) \geq p_0 > 0 \) and \( p_0 \) is a constant, \( r(t) \equiv 1, \sigma(t) \equiv \sigma > \tau \), then (2.3) becomes

\[
(2.8) \quad z''(t) - \frac{1}{c}(p_0 + p(t) - p_0)z(t - \sigma + \tau) \leq 0.
\]

By a known result [9, Theorem 5.3.9], if

\[
(2.9) \quad \left( \frac{p_0}{c} \right)^{1/2} \frac{\sigma - \tau}{2} > \frac{1}{c}
\]

then (2.8) has no negative increasing solution.

**Lemma 2.3.** In addition to the assumptions for (1.2) in Section 1, further assume that \( \sigma(t) \) is nondecreasing and

\[
(2.10) \quad \liminf_{t \to \infty} \frac{1}{c} \int_{t-\sigma_1(t)}^t \frac{1}{\tau(s)} \int_s^{\sigma(t)-\tau} p(u) du ds > \frac{1}{c}
\]

where \( \sigma_1(t) = \sigma \left( t + \frac{\tau(t) - \tau}{2} \right) - \frac{\sigma(t) - \tau}{2} \).

Then

\[
(2.11) \quad z'(t) + \frac{1}{cr(t)} \int_t^\infty p(u)z(u - \sigma(u) + \tau) du \geq 0
\]

has no negative increasing solution.

**Proof:** If not, let \( z(t) \) be a negative increasing solution of (2.11), then

\[
(2.12) \quad z'(t) + \left( \frac{1}{cr(t)} \int_t^{t+\frac{\sigma - \tau}{2}} p(u) du \right) z \left( t + \frac{\sigma - \tau}{2} - \sigma \left( t + \frac{\sigma - \tau}{2} \right) + \tau \right) \geq 0.
\]

By a known result, [9, Theorem 2.1.1], (2.12) has no negative solution under the assumption (2.10). This contradiction proves the Lemma.

In this section we shall henceforth always assume that \( \alpha = 1 \) in (1.2).
THEOREM 2.1. In addition to the assumption for (1.2) in the first section we assume that $\int_T^\infty \frac{dt}{r(t)} = \infty$ and either the second order ODE

$$ (r(t)y'(t))' + \frac{\lambda p(t)r(t)(t - \sigma(t))}{t r(T)} y(t) = 0 $$

is oscillatory for some $0 < \lambda < 1$ and $r \in n.i.$ or

$$ (r(t)y'(t))' + \lambda \frac{p(t)}{t} \frac{t - \sigma(t)}{t} y(t) = 0 $$

is oscillatory for some $0 < \lambda < 1$ and $r \in n.d.$ Then every solution of (1.2) is either oscillatory or tends to zero as $t \to \infty$.

PROOF: Without loss of generality let $y(t)$ be an eventually positive solution of (1.2) and define

$$ z(t) = y(t) - cy(t - \tau). $$

From (1.2) we know that

$$ (r(t)z'(t))' \leq 0 \text{ for } t \geq T. $$

We shall show that

$$ r(t)z'(t) > 0 \text{ for } t \geq T. $$

In fact, if

$$ r(t)z'(t) < 0 \text{ for } t \geq T_1 > T. $$

Then

$$ p(t)z'(t) \leq -\ell < 0 \text{ for } t \geq T_1. $$

Hence

$$ z(t) \to -\infty \text{ as } t \to \infty, $$

since $\int_T^\infty \frac{dt}{r(t)} = \infty$.

On the other hand, if

$$ z(t) < 0 $$

we have

$$ 0 < y(t) < cy(t - \tau) < \ldots < c^ny(t - n\tau). $$

Hence $y(t) \to 0$ as $t \to \infty$, since $0 < c < 1$. Consequently $z(t) \to 0$ as $t \to \infty$ which contradicts the fact that $z(t) \to -\infty$. Therefore (2.17) is true.

There are two possible cases for $z(t)$:

(a) $z(t) > 0$ for $t \geq T_2 \geq T_1$,

(b) $z(t) < 0$ for $t \geq T_3$. 

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Let us consider the case (a). In this case the assumptions of Lemma 2.1 are satisfied. Therefore for each $0 < k < 1$ there is a $T_k \geq T_2$ such that

$$z(t - \sigma(t)) \geq \frac{kr(t)(t - \sigma(t))}{r(T)t} z(t), \quad t \geq T_k, \quad r \in n.i.$$ 

and

$$z(t - \sigma(t)) \geq \frac{k(t - \sigma(t))}{t} z(t), \quad t \geq T_k, \quad r \in n.d..$$

Since $0 < z(t) < y(t)$, from (1.2) we have

$$\lambda \leq k < 1, \quad r \in n.i,$$

which imply, respectively that (2.13) and (2.14) are nonoscillatory [3]. This contradicts the assumption.

The second possibility is that $z(t) < 0$ for $t \geq T$. As before, this time the corresponding solution $y(t)$ must tend to zero as $t \to \infty$.

**Theorem 2.2.** In addition to the assumptions of Theorem 2.1 assume further that (2.4) holds. Then every solution of (1.2) oscillates.

**Proof:** To prove this theorem it is sufficient to show that in the proof of Theorem 2.1 $z(t) < 0$, for $t \geq T$ is impossible under assumptions (2.4). Suppose that $(rz')' < 0, \quad rz' > 0$ and $z(t) < 0$ for $t \geq T$. By (2.15) we have

$$z(t - \sigma(t) + \tau) > -cy(t - \sigma(t)).$$

This together with (1.2) gives

$$z(t - \sigma(t) + \tau) \leq \frac{p(t)}{c} z(t - \sigma(t) + \tau) \leq 0.$$

By Lemma 2.2 (2.20) has no negative increasing solutions which proves the theorem.

**Theorem 2.3.** In addition to the assumptions of Theorem 2.1 assume further that $\sigma(t)$ is nondecreasing and (2.10) holds. Then every solution of (1.2) oscillates.

**Proof:** As mentioned earlier we continue the proof of Theorem 2.1 and consider the possible case that $(rz')' \leq 0, \quad rz' > 0$ and $z(t) < 0$ for $t \geq T$. From this we have

$$r(t)z'(t) \to d \geq 0,$$
exists. If $d > 0$ it follows that $x(t) \to \infty$ as $t \to \infty$ which contradicts the negativity of $x(t)$. Therefore $r(t)x'(t) \to 0$ as $t \to \infty$.

Integrating (1.2) from $t$ to infinity we have

$$r(t)x'(t) = \int_t^\infty p(s)y(s - \sigma(s))ds,$$

which, together with (2.19), yields

$$r(t)x'(t) \geq -\frac{1}{c} \int_t^\infty p(s)x(s - \sigma(s) + \tau)ds.$$

This is a contradiction, by Lemma 2.3. The proof is completed.

Remark 2.2. We consider a special case of (1.2) as follows:

$$y(t) - cy(t - \tau)'' + p(t)y(t - \sigma) = 0$$

where $0 < c < 1, \sigma > \tau > 0$ are constants and $p(t) \geq p_0 > 0$. It is obvious (2.13) holds for (2.21). By Remark 2.1 if (2.9) holds then every solution of (2.21) oscillates. Therefore Theorem 8 in [6] becomes a special case of Theorem 2.2.

Example. Consider

$$y(t) - cy(t - \pi)'' + (1 + c)y(t - 2\pi) = 0$$

where $0 < c < 1$. Every solution of (2.22) oscillates by Remark 2.2. In fact $y = \sin t$ is a solution of (2.22).

III. SUBLINEAR EQUATIONS

We now consider Equations (1.2) in the sublinear case, that is, $0 < \alpha < 1$.

Theorem 3.1. Assume that:

(i) the assumptions for (1.2) in Section 1 hold;
(ii) $R(t) = \int_0^t \frac{dt}{r(t)}$ and $R(t) \to \infty$ as $t \to \infty$;
(iii) every solution of the second order ordinary differential equation

$$\left(r(t)x'(t)\right)' + p(t)\left(\frac{\lambda(t)x(t) - \lambda(t)}{r(t)}\right)z^\alpha(t) = 0, \text{ if } r \in n.i.$$

or

$$\left(r(t)x'(t)\right)' + p(t)\left(\frac{\lambda(t)x(t) - \lambda(t)}{t}\right)z^\alpha(t) = 0, \text{ if } r \in n.d.$$
is oscillatory, where \(0 < \lambda < 1\) is a constant. Then every solution of (1.2) is either oscillatory or tends to zero as \(t \to \infty\).

**Proof:** Suppose the contrary and let \(y(t)\) be an eventually positive solution. As in the proof of Theorem 2.1 we have \((rz')' \leq 0\) and \(rz' > 0\) for \(t \geq T\). For the case that \(z(t) > 0\) for \(t \geq T\), by Lemma 2.1 and (1.2), we get differential inequalities: either

\[
(r(t)z'(t))' + p(t)\left(\frac{kr(t)(t - \sigma(t))^{\alpha}}{r(T)t}\right)z^{\alpha}(t) \leq 0
\]

for \(t \geq T_k\), \(r \in \text{n.i.}\) and \(0 < k < 1\) or

\[
(t(t)z'(t))' + p(t)\left(\frac{k(t - \sigma(t))}{t}\right)^{\alpha}z^{\alpha}(t) \leq 0.
\]

By the comparison method we know that (3.3) and (3.4) imply that (3.1) and (3.2) have a nonoscillatory solution [3, p.52], which contradicts assumption (iii). For the case that \(z(t) < 0\) for \(t \geq T\), as in the proof of Theorem 2.1 the corresponding solution \(y(t)\) tends to zero as \(t \to \infty\). The proof is completed.

**Remark 3.1.** There are many results for oscillation of second order sublinear ordinary differential equations (3.1) and (3.2). For example, if

\[
\int_{T}^{\infty} R^{\alpha}(t)p(t)\left(\frac{\lambda r(t)(t - \sigma(t))}{r(T)t}\right)^{\alpha} dt = \infty \text{ for } r \in \text{n.i.}
\]

or

\[
\int_{T}^{\infty} R^{\alpha}(t)p(t)\left(\frac{\lambda(t - \sigma(t))}{t}\right)^{\alpha} dt = \infty \text{ for } t \in \text{n.d.}
\]

then every solution of (3.1) or (3.2) respectively oscillates.

**Theorem 3.2.** In addition to the assumptions of Theorem 3.1 assume further that

\[
\lim_{t \to \infty} \frac{1}{r(t)(t - \sigma(t) + \tau)} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)du > 0 \text{ for } r \in \text{n.i.}
\]

or

\[
\lim_{t \to \infty} \frac{1}{r(t)} \int_{t-\sigma(t)+\tau}^{t} [u - (t - \sigma(t) + \tau)]p(u)du > 0 \text{ for } r \in \text{n.d.}
\]

then every solution of (1.2) oscillates.

**Proof:** Let \(y(t)\) be an eventually positive solution. As in the proof of Theorem 3.1, we have \((rz')' \leq 0\), \(rz' > 0\) and \(z(t) < 0\) for \(t \geq T\). From (2.19) and (1.2) we have

\[
(r(t)z'(t))' - \frac{p(t)}{c}z^{\alpha}(t - \sigma(t) + \tau) \leq 0.
\]

By the same arguments as used in the proof of Lemma 2.2 we can prove that (3.9) has no negative increasing solution under assumptions (3.7) and (3.8). Hence we get a contradiction, which proves the theorem.
THEOREM 3.3. In addition to the assumptions of Theorem 3.1 assume further that $\sigma(t)$ is nondecreasing and

\[
(3.10) \quad \int_T^\infty \frac{1}{r(s)} \int_s^{t+\frac{(\sigma-t)}{K}} P(u) \, du \, ds = \infty
\]

where $K > 1$ is some constant. Then every solution of (1.2) oscillates.

PROOF: If not, it is sufficient to consider the case that $(rz')' \leq 0$, $rz' > 0$ and $z(t) < 0$ for $t \geq T$. As in the proof of Theorem 2.2, we have

\[
(3.11) \quad r(t)z'(t) = \int_t^\infty p(s)\alpha(s - \sigma(s)) \, ds
\]

\[
\geq -\frac{1}{c\alpha} \int_t^\infty p(s)\alpha(s - \sigma(s) + \tau) \, ds
\]

\[
\geq -\frac{1}{c\alpha} \int_t^{t+\frac{(\sigma-t)}{K}} p(s)\alpha(s - \sigma(s) + \tau) \, ds
\]

\[
\geq -\left(\frac{1}{c\alpha} \int_t^{t+\frac{(\sigma-t)}{K}} p(s) \, ds\right)\alpha\left(t + \frac{(\sigma-t)}{K} - \sigma\left(t + \frac{\sigma-t}{K}\right) + \tau\right).
\]

This is a first order sublinear delay differential inequality. From a known result [9, Theorem 3.3.2] (3.11) has no negative solution under assumption (3.10). This contradiction proves the theorem.

Remark. It would be interesting to obtain results similar to those presented here for the superlinear case $\alpha > 1$ for equation (1.2).

REFERENCES


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