SUMS OF MULTIPLICATIVE FUNCTIONS OVER A BEATTY SEQUENCE

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Abstract

We study sums involving multiplicative functions that take values over a nonhomogenous Beatty sequence. We then apply our result in a few special cases to obtain asymptotic formulas for quantities such as the number of integers in a Beatty sequence that are representable as a sum of two squares up to a given magnitude.

Keywords and phrases: sums of multiplicative functions, Beatty sequences.

1. Introduction

Let $A \geq 1$ be an arbitrary constant, and let $\mathcal{F}_A$ be the set of multiplicative functions $f$ such that $|f(p)| \leq A$ for all primes $p$ and

$$
\sum_{n \leq N} |f(n)|^2 \leq A^2 N \quad (N \in \mathbb{N}).
$$

(1)

Exponential sums of the form

$$
S_{\alpha,f}(N) = \sum_{n \leq N} f(n)e(n\alpha) \quad (\alpha \in \mathbb{R}, \ f \in \mathcal{F}_A),
$$

(2)

where $e(z) = e^{2\pi iz}$ for $z \in \mathbb{R}$, occur frequently in analytic number theory. Montgomery and Vaughan have shown (see [8, Corollary 1]) that the upper bound

$$
S_{\alpha,f}(N) \ll_A \frac{N}{\log N} + \frac{N(\log R)^{3/2}}{R^{1/2}}
$$

(3)

holds uniformly for all $f \in \mathcal{F}_A$, provided that $|\alpha - a/q| \leq q^{-2}$ with some reduced fraction $a/q$ for which $2 \leq R \leq q \leq N/R$. In this paper, we use the Montgomery–Vaughan result to estimate sums of the form

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where $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, $f \in \mathcal{F}_A$, and $\mathcal{B}_{\alpha, \beta}$ is the nonhomogenous Beatty sequence defined by

$$\mathcal{B}_{\alpha, \beta} = \{ n \in \mathbb{N} : n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \in \mathbb{Z} \}.$$ 

Our results are uniform over the family $\mathcal{F}_A$ and nontrivial whenever

$$\lim_{N \to \infty} \frac{\log N}{N \log \log N} \left| \sum_{n \leq N} f(n) \right| = \infty,$$

a condition which guarantees that the error term in Theorem 1 is smaller than the main term. One can remove this condition, at the expense of losing uniformity with respect to $f$, and still obtain Theorem 1 for any bounded arithmetic function $f$ (not necessarily multiplicative) for which the exponential sums in (2) satisfy

$$S_{\alpha, f}(N) = o \left( \sum_{n \leq N} f(n) \right) \quad (N \to \infty).$$

The general problem of characterizing functions for which this relation holds appears to be rather difficult; see [1] for Bachman’s conjecture and his related work on this problem.

We shall also assume that $\alpha$ is irrational and of finite type $\tau$. For an irrational number $\gamma$, the type of $\gamma$ is defined by

$$\tau = \sup \left\{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \left\lfloor \gamma n \right\rfloor = 0 \right\},$$

where $\lfloor \cdot \rfloor$ denotes the distance to the nearest integer. Dirichlet’s approximation theorem implies that $\tau \geq 1$ for every irrational number $\gamma$. According to theorems of Khinchin [6] and Roth [10], $\tau = 1$ for almost all real numbers (in the sense of the Lebesgue measure) and all irrational algebraic numbers $\gamma$, respectively; also see [2, 11].

Our main result is the following theorem.

**Theorem 1.** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that $\alpha$ is irrational and of finite type. Then, for all $f \in \mathcal{F}_A$,

$$G_{\alpha, \beta, f}(N) = \alpha^{-1} \sum_{n \leq N} f(n) + O \left( \frac{N \log \log N}{\log N} \right),$$

where the implied constant depends only on $\alpha$ and $A$.

The following corollaries are immediate applications of Theorem 1.
Corollary 2. The number of integers not exceeding \( N \) that lie in the Beatty sequence \( B_{\alpha, \beta} \) and can be represented as a sum of two squares is

\[
\#\{n \leq N : n \in B_{\alpha, \beta}, n = \square + \square\} = \frac{C N}{\alpha \sqrt{\log N}} + O\left(\frac{N \log \log N}{\log N}\right),
\]

where

\[
C = 2^{-1/2} \prod_{p \equiv 3 \mod 4} (1 - p^{-2})^{-1/2} = 0.76422365 \ldots \tag{5}
\]

is the Landau–Ramanujan constant.

To state the next result, we recall that an integer \( n \) is said to be \( k \)-free if \( p^k \nmid n \) for every prime \( p \).

Corollary 3. For every \( k \geq 2 \), the number of \( k \)-free integers not exceeding \( N \) that lie in the Beatty sequence \( B_{\alpha, \beta} \) is

\[
\#\{n \leq N : n \in B_{\alpha, \beta}, n \text{ is } k\text{-free}\} = \alpha^{-1} \zeta^{-1}(k) N + O\left(\frac{N \log \log N}{\log N}\right),
\]

where \( \zeta(s) \) is the Riemann zeta function.

Finally, we consider the average value of the number of representations of an integer from a Beatty sequence as a sum of four squares.

Corollary 4. Let \( r_4(n) \) denote the number of representations of \( n \) as a sum of four squares. Then

\[
\sum_{\substack{n \leq N \\text{in } B_{\alpha, \beta}}} r_4(n) = \frac{\pi^2 N^2}{2\alpha} + O\left(\frac{N^2 \log \log N}{\log N}\right),
\]

where the implied constant depends only on \( \alpha \).

Any implied constants in the symbols \( O \) and \( \ll \) may depend on the parameters \( \alpha \) and \( A \) but are absolute otherwise. We recall that the notation \( X \ll Y \) is equivalent to \( X = O(Y) \).

2. Preliminaries

2.1. Discrepancy of fractional parts

We define the discrepancy \( D(M) \) of a sequence of real numbers \( b_1, b_2, \ldots, b_M \in [0, 1) \) by

\[
D(M) = \sup_{\mathcal{I} \subseteq [0, 1]} \left| \frac{\mathcal{V}(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|,
\]

where the supremum is taken over all possible subintervals \( \mathcal{I} = (a, c) \) of the interval \([0, 1)\), \( \mathcal{V}(\mathcal{I}, M) \) is the number of positive integers \( m \leq M \) such that \( b_m \in \mathcal{I} \), and \(|\mathcal{I}| = c - a \) is the length of \( \mathcal{I} \).
If an irrational number $\gamma$ is of finite type, we let $D_{\gamma,\delta}(M)$ denote the discrepancy of the sequence of fractional parts $\{\gamma m + \delta\}_{m=1}^{M}$. By [7, Theorem 3.2, Ch. 2], we have the following result.

**Lemma 5.** For a fixed irrational number $\gamma$ of finite type $\tau$ and for all $\delta \in \mathbb{R}$,

$$D_{\gamma,\delta}(M) \leq M^{-1/\tau + o(1)} \quad (M \to \infty),$$

where the function defined by $o(\cdot)$ depends only on $\gamma$.

### 2.2. Numbers in a Beatty sequence

The following result is standard in characterizing the elements of the Beatty sequence $\mathcal{B}_{\alpha,\beta}$.

**Lemma 6.** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and set $\gamma = \alpha^{-1}$ and $\delta = \alpha^{-1}(1 - \beta)$. Then $n = [\alpha m + \beta]$ for some $m \in \mathbb{Z}$ if and only if $0 < \{\gamma n + \delta\} \leq \gamma$.

From Lemma 6, an integer $n$ lies in $\mathcal{B}_{\alpha,\beta}$ if and only if $n \geq 1$ and $\psi(\gamma n + \delta) = 1$, where $\psi$ is the periodic function with period one whose values on the interval $(0, 1]$ are given by

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \leq \gamma, \\ 0 & \text{if } \gamma < x \leq 1. \end{cases}$$

We wish to approximate $\psi$ by a function whose Fourier series representation is well behaved. This will give rise to the aforementioned exponential sum $S_{\alpha,\beta}(N)$. To this end, we use the result of Vinogradov (see [15, Ch. I, Lemma 12]) which states that for any $\Delta$ such that

$$0 < \Delta < \frac{1}{8} \quad \text{and} \quad \Delta \leq \frac{1}{2} \min\{\gamma, 1 - \gamma\},$$

there exists a real-valued function $\Psi$ with the following properties:

(i) $\Psi$ is periodic with period one;
(ii) $0 \leq \Psi(x) \leq 1$ for all $x \in \mathbb{R}$;
(iii) $\Psi(x) = \psi(x)$ if $\Delta \leq \{x\} \leq \gamma - \Delta$ or if $\gamma + \Delta \leq \{x\} \leq 1 - \Delta$;
(iv) $\Psi$ can be represented by a Fourier series

$$\Psi(x) = \sum_{k \in \mathbb{Z}} g(k) e(kx),$$

where $g(0) = \gamma$ and the Fourier coefficients satisfy the uniform bound

$$g(k) \ll \min\{|k|^{-1}, |k|^{-2}\Delta^{-1}\} \quad (k \neq 0). \quad (7)$$

### 3. Proofs

#### 3.1. Proof of Theorem 1

Using Lemma 6, we rewrite the sum (4) in the form

$$G_{\alpha,\beta,f}(N) = \sum_{n \leq N} f(n) \psi(\gamma n + \delta).$$
Replacing $\psi$ by $\Psi$, we obtain

$$G_{\alpha, \beta, f}(N) = \sum_{n \leq N} f(n) \Psi(\gamma n + \delta) + O\left( \sum_{n \in V(\Delta, N)} f(n) \right),$$

(8)

where $V(\Delta, N)$ is the set of positive integers $n \leq N$ for which

$$\{\gamma n + \delta\} \in [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).$$

Since the length of each interval above is at most $2\Delta$, it follows from definition (6) and Lemma 5 that

$$|V(\Delta, N)| \ll \Delta N + N^{1-1/(2\tau)},$$

where we have used the fact that $\alpha$ and $\gamma$ have the same type $\tau$. Thus, taking (1) into account, Cauchy’s inequality gives

$$\left| \sum_{n \in V(\Delta, N)} f(n) \right| \leq \left| V(I, N) \right|^{1/2} \left( \sum_{n \leq N} |f(n)|^2 \right)^{1/2} \ll \left( (\Delta N)^{1/2} + N^{1/2 - 1/(4\tau)} \right)^{1/2} = \Delta^{1/2} N + N^{1-1/(4\tau)}.$$  

(9)

Next, let $K \geq \Delta^{-1}$ be a large real number (to be specified later), and let $\Psi_K$ be the trigonometric polynomial given by

$$\Psi_K(x) = \sum_{|k| \leq K} g(k)e(kx) = \gamma + \sum_{0 < |k| \leq K} g(k)e(kx) \quad (x \in \mathbb{R}).$$

(10)

Using (7), we see that the estimate

$$\Psi(x) = \Psi_K(x) + O(K^{-1}\Delta^{-1})$$

holds uniformly for all $x \in \mathbb{R}$; therefore,

$$\sum_{n \leq N} f(n) \Psi(\gamma n + \delta) = \sum_{n \leq N} f(n) \Psi_K(\gamma n + \delta) + O(K^{-1}\Delta^{-1} N),$$

(11)

where we have used the bound $\sum_{n \leq N} |f(n)| \ll N$ which follows from (1).

Combining (8), (9), (10) and (11), we derive that

$$G_{\alpha, \beta, f}(N) = \gamma \sum_{n \leq N} f(n) + H(N) + O(K^{-1}\Delta^{-1} N + \Delta^{1/2} N + N^{1-1/(4\tau)}),$$

where

$$H(N) = \sum_{0 < |k| \leq K} g(k)e(k\delta) S_{k\gamma, f}(N).$$

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Put $R = (\log N)^3$. We claim that if $N$ is sufficiently large, then for every $k$ in the above sum there is a reduced fraction $a/q$ such that $|k\gamma - a/q| \leq q^{-2}$ and $R \leq q \leq N/R$. Assuming this is true for the moment, (3) implies that

$$S_{k\gamma, f}(N) \ll \frac{N}{\log N} \quad (0 < |k| \leq K);$$

using (7), we then deduce that

$$H(N) \ll \frac{N \log K}{\log N}.$$

Therefore,

$$G_{\alpha, \beta, f}(N) - \gamma \sum_{n \leq N} f(n) \ll \frac{N \log K}{\log N} + K^{-1} \Delta^{-1} N + \Delta^{1/2} N + N^{1-1/4\tau}.$$

To balance the error terms, we choose

$$\Delta = (\log N)^{-2} \quad \text{and} \quad K = \Delta^{-3/2} = (\log N)^{3},$$

thus obtaining the bound stated in the theorem.

To prove the claim, let $k$ be an integer with $0 < |k| \leq K = (\log N)^3$, and let $r_i = a_i/q_i$ be the $i$th convergent in the continued fraction expansion of $k\gamma$. Since $\gamma$ is of finite type $\tau$, for every $\varepsilon > 0$ there is a constant $C = C(\gamma, \varepsilon)$ such that

$$C(|k|q_{i-1})^{-(\tau+\varepsilon)} \leq |\gamma||k|q_{i-1} - a_{i-1}| \leq q_i^{-1}.$$

Put $\varepsilon = \tau$, and let $j$ be the least positive integer for which $q_j \geq R$ (note that $j \geq 2$). Then

$$R \leq q_j \ll (|k|q_{i-1})^{2\tau} \leq (KR)^{2\tau} = (\log N)^{6\tau},$$

and it follows that $R \leq q_j \leq N/R$ if $N$ is sufficiently large, depending only on $\alpha$. This concludes the proof. $\square$

3.2. Proof of Corollary 2 Let $f(n)$ be the characteristic function of the set of integers that can be represented as a sum of two squares. It follows from [4, Theorem 366] that $f(n)$ is multiplicative. Hence Corollary 2 is an immediate consequence of Theorem 1 and the asymptotic formula

$$\sum_{n \leq N} f(n) = \frac{CN}{(\log N)^{1/2}} + O\left(\frac{N}{(\log N)^{3/2}}\right)$$

(see, for example, [12, 13]), where $C$ is given by (5). $\square$

3.3. Proof of Corollary 3 Fix $k \geq 2$ and let $f(n)$ be the characteristic function of the set of $k$-free integers. It is easily proved that $f(n)$ is multiplicative. Thus Corollary 3 follows from Theorem 1 and the following estimate of Gegenbauer [3] for the number of $k$-free integers not exceeding $N$:

$$\sum_{n \leq N} f(N) = \zeta^{-1}(k) N + O(N^{1/k}).$$

$\square$
3.4. Proof of Corollary 4  Put $f(n) = r_4(n)/(8n)$. From Jacobi’s formula for $r_4(n)$, namely

$$r_4(n) = 8(2 + (-1)^n) \sum_{d \mid n, \text{ odd}} d \quad (n \geq 1),$$

it follows that $f(n)$ is multiplicative and that $f(p) \leq 3/2$ for every prime $p$. Moreover, using the formula of Ramanujan [9] (see also [14]),

$$\sum_{n \leq N} \sigma^2(n) = \frac{5}{6} \xi(3)N^3 + O(N^2(\log N)^2)$$

where $\sigma$ is the sum of divisors function, we obtain

$$\sum_{n \leq N} |f(n)|^2 \leq \sum_{n \leq N} \frac{\sigma^2(n)}{n^2} = \frac{5}{2} \xi(3)N + O((\log N)^3)$$

by partial summation. Therefore, $f(n) \in \mathcal{F}_A$ for some constant $A \geq 1$. Applying Theorem 1, we deduce that

$$\sum_{n \leq N} \frac{r_4(n)}{n} = \alpha^{-1} \sum_{n \leq N} \frac{r_4(n)}{n} + O\left(\frac{N \log \log N}{\log N}\right),$$

where the implied constant depends only on $\alpha$.

From the asymptotic formula

$$\sum_{n \leq N} r_4(n) = \frac{\pi^2 N^2}{2} + O(N \log N)$$

(see for example [5, p. 22]), partial summation gives

$$\sum_{n \leq N} \frac{r_4(n)}{n} = \pi^2 N + O((\log N)^2).$$

Consequently,

$$\sum_{n \leq N} \frac{r_4(n)}{n, n \in \mathcal{B}_{\alpha, \beta}} = \alpha^{-1} \pi^2 N + O\left(\frac{N \log \log N}{\log N}\right).$$

Using partial summation once more, we obtain the statement of Corollary 4. \qed

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References


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