NORMAL OPERATORS ON BANACH SPACES

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1. The main result and its consequences. A (bounded, linear) operator $H$ on a Banach space $X$ is said to be hermitian if $\|\exp(itH)\| = 1$ for all real $t$. An operator $N$ on $X$ is said to be normal if $N = H + iK$, where $H$ and $K$ are commuting hermitian operators. These definitions generalize those familiar concepts of operators on Hilbert spaces. Also, the normal derivations defined in [1] are normal operators. For more details about hermitian operators and normal operators on general Banach spaces, see [4]. The main result concerning normal operators in the present paper is the following theorem.

**Theorem A.** Suppose that $N$ in $\mathcal{L}(X)$ is normal. Then $\|Nx + w\| \geq \|w\|$ for each $x$ in $X$ and each $w$ in $\ker N$. (In other words, the kernel of $N$ is orthogonal to the range of $N$.)

The proof of Theorem A will be presented in the next section. Granting this theorem for the moment, we can deduce the following corollaries.

**Corollary 1.** If $N$ is a normal operator on $X$ and $\lambda, \mu$ are distinct eigenvalues of $N$, then $\|x + y\| \geq \|x\|$ for $x \in \ker (N - \lambda)$ and $y \in \ker (N - \mu)$. In other words, eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.

**Proof.** Since $N$ is normal, so is $N - \lambda$. Hence, by Theorem A, $\|x + y\| = \|(N - \lambda)((\lambda - \mu)^{-1}y) + x\| \geq \|x\|.$

**Corollary 2.** If $N$ is a normal operator on a separable space, then there are at most countably many eigenvalues.

**Proof.** This follows from Corollary 1 and a topological consideration.

The next corollary is a special case of Proposition 1 in [7].

**Corollary 3.** If $N$ is a normal operator and $N^2x = 0$, then $Nx = 0$.

**Proof.** Let $w = Nx$. Then $Nw = 0$ and hence, by Theorem A, $0 = \|N(-x) + w\| \geq \|w\|.$

**Corollary 4.** Let $N$ be a normal operator on $X$. If $0$ is in the spectrum $\sigma(N)$ of $N$ and the range $N\mathcal{X}$ is closed, then $0$ is an isolated point in $\sigma(N)$, $\mathcal{X} = \ker N \oplus N\mathcal{X}$ and $\|P\| = 1$ where $P$ is the projection from $X$ onto $\ker N$ along $N\mathcal{X}$.

The proof of this corollary is similar to that of Proposition 3 in [9] and hence omitted.

**Corollary 5.** Let $N$ be a normal operator on $X$. If $\{w_n\}$ is a sequence of unit vectors

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in $X$ such that $\lim_n \|N w_n\| = 0$, then, for each bounded sequence $\{x_n\}$ in $X$, 
\[ \lim \sup_n \|N x_n - w_n\| \geq 1. \]

**Proof.** First let us recall the "Berberian–Quigley extension". (See [6].) Let $\ell^\infty(X)$ be the Banach space of all bounded sequences in $X$ with sup-norm and let $c_0(X)$ be the subspace of $\ell^\infty(X)$ consisting of those $\{x_n\}$ with $\lim_n x_n = 0$. Let $\ell^\infty/c_0(X)$ be the quotient space $\ell^\infty(X)/c_0(X)$. For every $T$ in $\mathcal{L}(X)$, the mapping $\{x_n\} \mapsto \{Tx_n\}$ sends $\ell^\infty(X)$ into itself and hence it induces an operator $T^\circ$ on $\ell^\infty/c_0(X)$ with $\|T^\circ\| = \|T\|$. It is easy to see that if $T$ is hermitian or normal, then so is $T^\circ$. Now the corollary follows from an application of Theorem A to $N^\circ$.

**Corollary 6** (See [1] and [2].) Let $a_1$ and $a_2$ be normal operators on a Hilbert space $X$. Define $N \in \mathcal{L}(X)$ by $N x = a_1 x - a_2 x$. Then we have: (1) $\|N x + w\| \geq \|w\|$ for all $x$ in $\mathcal{L}(X)$ and $w$ in $\ker N$, (2) $\|x + y\| \geq \|x\|$ if $N x = \lambda x$, $Ny = \mu y$ and $\lambda \neq \mu$, (3) $N^2 x = 0$ implies $N x = 0$, and (4) if, furthermore, the range of $N$ is closed, then the spectra of both $a_1$ and $a_2$ are finite.

**Proof.** Suppose $a_j = h_j + i k_j$ ($j = 1,2$), where $h_j$ and $k_j$ are commuting hermitian operators on $X$. Then $N x = H x + i K x$, where $H x = h_1 x - x h_2$ and $K x = k_1 x - x k_2$. Note that both $H$ and $K$ are hermitian and $H K x = h_1 k_1 x + x h_2 k_2 - h_1 x h_2 - k_1 x k_2 = KH x$. Now (1), (2) and (3) follows from Theorem A, Corollary 1 and Corollary 3 respectively and (4) follows from Corollary 4 and Rosenblum’s theorem [8].

**Remark.** Corollary 6 still holds if $L(X)$ is replaced by a Banach algebra and $a_1$ and $a_2$ are assumed to be normal elements in it.

**2. Proof of the main result.** Now we proceed to prove Theorem A. It depends on the construction of certain projections in $L(X^*)$ (where $X^*$ is the dual space of $X$) which resemble conditional expectations in the theory of $C^*$-algebra.

To begin with, let $V$ be a power bounded operator on $X$, say $\|V^n\| \leq M$ for every positive integer $n$. Let $\text{glim}$ be a generalized (Banach) limit. For each $\phi \in X^*$, the map $E_\phi : x \mapsto \text{glim} (\phi, V^n x)$ is a bounded linear functional on $X$ with $\|E_\phi\| \leq M\|\phi\|$. Thus we obtain an operator $E$ in $L(X^*)$. Note that, in case $\|V\| \leq 1$, we have $\|E_\phi\| \leq \|\phi\|$ for all $\phi$ in $X^*$. We list some properties of $E$ in the following lemma.

**Lemma 1.** Suppose that $V \in L(X)$ is power bounded and $E$ is an operator in $L(X^*)$ defined as above. Then we have:

1. If $A \in L(X)$ and $AV = VA$, then $A^* E = EA^*$.
2. $V^* E = E V^* = E$.
3. For $w \in X$, $V w = w$ if and only if $\langle E_\phi, w \rangle = \langle \phi, w \rangle$ for all $\phi \in X^*$.
4. For $\phi \in X^*$, $V^* \phi = \phi$ if and only if $E_\phi = \phi$.
5. $E^2 = E$. 

In particular, \( E \) is a projection from \( \mathcal{X}^* \) onto \( \{ \phi \in \mathcal{X}^* : V^* \phi = \phi \} \).

**Proof.**

(1) For \( x \in \mathcal{X}, \phi \in \mathcal{X}^* \), we have \( \langle EA^* \phi, x \rangle = \lim_{n \to \infty} \langle A^* \phi, V^n x \rangle = \langle E \phi, Ax \rangle = \langle A^* E \phi, x \rangle \) and hence \( EA^* = A^* E \).

(2) For \( x \in \mathcal{X}, \phi \in \mathcal{X}^* \), we have \( \langle V^* E \phi, x \rangle = \langle E \phi, Vx \rangle = \lim_{n \to \infty} \langle \phi, V^{n+1} x \rangle = \langle \phi, V^n x \rangle = \langle E \phi, x \rangle \). Hence \( V^* E = E \).

(3) If \( Vw = w \), then \( \langle E \phi, w \rangle = \lim_{n \to \infty} \langle \phi, V^n w \rangle = \langle \phi, w \rangle \). Conversely, suppose that \( \langle E \phi, w \rangle = \langle \phi, w \rangle \) for all \( \phi \) in \( \mathcal{X}^* \). Then, by (2), we have \( \langle \phi, Vw \rangle = \langle V^* \phi, w \rangle = \langle E \phi, w \rangle = \langle \phi, w \rangle \) for all \( \phi \) in \( \mathcal{X}^* \). Hence \( Vw = w \).

(4) If \( V^* \phi = \phi \), then \( \langle E \phi, x \rangle = \lim_{n \to \infty} \langle \phi, V^n x \rangle = \lim_{n \to \infty} \langle V^n \phi, x \rangle = \langle \phi, x \rangle \) for all \( x \) and hence \( E \phi = \phi \). Conversely, if \( E \phi = \phi \), then, by (2), we have \( V^* \phi = E \phi = \phi \).

(5) follows from (2) and (4).

As an aside, we give a different proof of Sinclair’s result [9; Proposition 1] by applying the above lemma (and without using Kakutani’s fixed point theorem.) First we need a technical lemma.

**Lemma 2.** If \( T \in \mathcal{L}(\mathcal{X}), x \in \mathcal{X}, (\exp T)x = x \) and \( |\exp \lambda - 1| < 1 \) for all \( \lambda \) in \( \sigma(T) \), then \( Tx = 0 \).

**Proof.** The lemma follows by applying the expansion \( T = - \sum_{n=1}^{\infty} n^{-1}(I - \exp T)^n \) to the vector \( x \).

**Corollary 7 (Sinclair [9]).** Let \( T \in \mathcal{L}(\mathcal{X}) \). If \( 0 \) is in the boundary of the closed convex hull of the (spatial) numerical range of \( T \), then \( \|Tx + w\| \geq \|w\| \) for \( x \in \mathcal{X} \) and \( w \in \ker T \).

**Proof.** By multiplying \( T \) by a suitable constant which is small enough in modulus, we may assume that \( \text{Re} \lambda \leq 0 \) and \( |\exp \lambda - 1| < 1 \) for \( \lambda \) in the closed convex hull of the numerical range of \( T \). Let \( V = \exp T \). Then, by [4, Theorem 3.6], \( \|V\| \leq 1 \). Let \( w \in \ker T \). Then \( Vw = w \). By the spectral mapping theorem, \( \sigma(V^*) = \sigma(V) = \exp \sigma(T) \subseteq \{ \lambda : |\lambda - 1| < 1 \} \). By Lemma 1, there exists a projection \( E \) in \( \mathcal{L}(\mathcal{X}^*) \) with \( \{ \phi \in \mathcal{X}^* : V^* \phi = \phi \} \) as its range such that \( \|E\| \leq 1 \), \( EV^* = V^* E = E \) and \( \langle E \phi, w \rangle = \langle \phi, w \rangle \) for all \( \phi \in \mathcal{X}^* \). If \( E \phi = \phi \), then \( \exp (T^*) \phi = V^* E \phi = E \phi = \phi \) and hence, by Lemma 2, \( T^* \phi = 0 \). Therefore \( T^* E = 0 \). If \( \|E\| \leq 1 \), then \( \|E \phi\| \leq 1 \) and hence

\[
\|Tx + w\| \geq \|\langle E \phi, Tx + w \rangle\| = |\langle T^* E \phi, x \rangle + \langle E \phi, w \rangle| = |\langle \phi, w \rangle|.
\]

Therefore \( \|Tx + w\| \geq \sup \{ |\langle \phi, w \rangle| : \|\phi\| \leq 1 \} = \|w\| \).

The next lemma is already known. (See [7].) However, for the convenience of the reader, a proof of it is included here.

**Lemma 3.** If \( N = H + iK \) is a normal operator, where \( H \) and \( K \) are commuting hermitian operators and \( Nx = 0 \), then \( Hx = Kx = 0 \).
Proof. From the assumption we have \( Kx = iHx \). Since \( HK = KH \), by induction, we have \( K^n x = (iH)^n x \) for \( n = 1, 2, 3, \ldots \). Hence \( \exp(\lambda K)x = \exp(i\lambda H)x \) for every complex number \( \lambda \). Now, suppose \( \lambda = \alpha + i\beta \), where \( \alpha \) and \( \beta \) are real. Then we have

\[
\exp(\lambda K)x = \exp(i\beta K)\exp(\alpha K)x = \exp(i\beta K)\exp(i\alpha H)x.
\]

Hence \( \|\exp(\lambda K)x\| \leq \|x\| \). By Liouville’s theorem, \( \exp(\lambda K)x \) is a constant function. Differentiate this function at \( \lambda = 0 \). We obtain \( \exp(\lambda K)Kx = 0 \) and hence \( Kx = 0 \).

Remark. From the proof of the above lemma we see that, besides the commutativity of \( H \) and \( K \), all we need is the boundedness of \( \alpha \mapsto \exp(i\alpha H) \) and \( \beta \mapsto \exp(i\beta K) \) (\( \alpha, \beta \in \mathbb{R} \)). Hence this lemma can be generalized for those \( H + iK \), where \( H \) and \( K \) are commuting pre-hermitian operators. By an argument similar to that of Corollary 6, we can deduce Berkson, Dowson and Elliott’s extension of Fuglede’s theorem [3; Theorem 1] from this fact.

Proof of Theorem A. By Lemma 3, we have \( Hw = Kw = 0 \). Let \( \text{glim} \) be a generalized limit. Define \( E, F \in \mathcal{L}(\mathcal{H}^*) \) by the identities \( \langle E\phi, x \rangle = \text{glim} \langle \phi, \exp(\alpha H)x \rangle \) and \( \langle F\phi, x \rangle = \text{glim} \langle \phi, \exp(i\beta K)x \rangle \). Then, by Lemma 1, \( E \) and \( F \) are projections satisfying \( \|E\| \leq 1, \|F\| \leq 1 \) and \( \langle E\phi, w \rangle = \langle F\phi, w \rangle \) for all \( \phi \) in \( \mathcal{H}^* \). Furthermore, \( (\exp iH)^*E = E(\exp iH)^* = E \) and \( (\exp iK)^*F = F(\exp iK)^* = F \). By Lemma 2, it is easy to see that \( H^*E = 0 \) and \( K^*F = 0 \). Note that, since \( KH = HK \), we have \( K^*E = EK^* \) and hence \( K^*EF = EK^*F = 0 \). For \( \phi \in \mathcal{H}^* \) with \( \|\phi\| \leq 1 \), we have

\[
\|Nx + w\| \geq |\langle EF\phi, Nx + w \rangle| = |\langle H^*EF\phi, x \rangle + i\langle K^*EF\phi, x \rangle + \langle EF\phi, w \rangle| = |\langle \phi, w \rangle|.
\]

Hence \( \|Nx + w\| \geq \|w\| \). The proof is complete.

Remark. We have mentioned at the beginning of this section that the projection \( E \) in Lemma 1 resembles conditional expectations in the theory of \( C^* \)-algebra. To make this statement more clear, we consider the following special case. Let \( \mathcal{H} \) be a Hilbert space \( \mathcal{H} = \mathcal{L}(\mathcal{H}) \) and let \( h \) be a hermitian operator on \( \mathcal{H} \). Define \( H \) and \( V \) in \( \mathcal{L}(\mathcal{H}) \) by \( Hx = hx - xh \) and \( Vx = \exp(iH)x = \exp(i\alpha H)x \). Let \( \text{glim} \) be a generalized limit and \( P \), in \( \mathcal{L}(\mathcal{H}) \), be the projection defined in such a way that \( \langle Px, \xi, \eta \rangle = \text{glim} \langle (V^*x)\xi, \eta \rangle \), where \( \xi, \eta \) are in \( \mathcal{H} \). It is easy to check that \( P \) is a conditional expectation from \( \mathcal{L}(\mathcal{H}) \) onto the von Neumann algebra \( \{x \in \mathcal{L}(\mathcal{H}) : xh = hx\} \), the commutant of \( h \). Define \( E \in \mathcal{L}(\mathcal{H}^*) \) in the same way as that in the beginning of this section, i.e., \( \langle E\phi, x \rangle = \text{glim} \langle \phi, V^*x \rangle \). Then we have \( P^*\phi = E\phi \) if \( \phi \) is of the form \( \phi(x) = \langle x\xi, \eta \rangle \) for some vectors \( \xi, \eta \) in \( \mathcal{H} \). Thus \( E \) is “almost” the dual of \( P \). One can check that Lemma 1 still holds if \( E \) is replaced by \( P^* \).

3. Compact normal operators. Many results concerning compact hermitian operators (see [5, §28]) can be generalized to compact normal operators.
Proposition 1. Let \( T \in \mathcal{L}(\mathcal{H}) \) be compact and normal, let \( \lambda_n \) be the non-zero (distinct) eigenvalues of \( T \) arranged such that \( |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \) and let \( P_n \) be the spectral projection corresponding to \( \lambda_n \). Then the following statements hold. (1) Each eigenvalue of \( T \) has ascent 1. (2) \( \|P_n\| = 1 \). (3) If each \( P_n \) is hermitian, then \( T = \sum \lambda_n P_n \), a norm-convergent series. (4) If \( \lim n\lambda_n = 0 \), then \( T = \sum \lambda_n P_n \).

Proof. (1) and (2) follows from Corollary 4 in §1. One can modify the proof of Theorem 28.1 in [5, p. 82] to obtain (3) and (4).

Next we show that if the underlying space \( \mathcal{H} \) is weakly complete, then the linear span of eigenspaces of \( T \) is dense in \( \mathcal{H} \). First we need a technical lemma.

Lemma 3. Let \( \mathcal{A} \) be a Banach algebra, \( \mathfrak{I} \) a closed two-sided ideal in \( \mathcal{A} \) and \( \nu: \mathcal{A} \to \mathcal{A}/\mathfrak{I} \) the quotient map. Then the following two statements hold.

1. If \( h \in \mathcal{A} \) is hermitian, then so is \( \nu(h) \).
2. If \( h, k \in \mathcal{A} \) are hermitian, \( n = h + ik \) is normal and \( n \in \mathfrak{I} \), then \( h, k \in \mathfrak{I} \).

Proof. (1) for each real \( t \), we have \( \|\exp(\pm it\nu(h))\| = \|\nu(\exp(\pm ith))\| \leq 1 \). On the other hand, \( \|\exp(-it\nu(h))\|\|\exp(it\nu(h))\| \geq \|\exp(-it\nu(h))\exp(it\nu(h))\| = 1 \). Hence \( \|\exp(it\nu(h))\| = 1 \) for all real \( t \). Therefore \( \nu(h) \) is hermitian.

(2) By (1), \( \nu(n) = \nu(h) + iv(k) \) is normal. By the assumption, \( \nu(n) = 0 \). Hence \( \nu(h) = \nu(k) = 0 \); i.e., \( h, k \in \mathfrak{I} \).

Proposition 2. Let \( \mathcal{H} \) be a weakly complete Banach space, \( T \) a compact normal operator \( \mathcal{H} \) and let \( \lambda_n, \ P_n \) be as in Proposition 1. Then \( \mathcal{H} \) is the closed linear span of eigenvectors of \( T \).

Proof. Let \( T = H + iK \), where \( H \) and \( K \) are commuting hermitian operators. Since \( T \) is compact, by Lemma 3, both \( H \) and \( K \) are compact. Let \( \alpha_n = \text{Re} \lambda_n \) and \( \beta_n = \text{Im} \lambda_n \). Suppose \( x \in P_n \mathcal{H} \). Then \( Kx = \lambda_n x \); that is, \( ((H - \alpha_n) + i(K - \beta_n))x = 0 \) and hence, by Lemma 2 in §2, \( (H - \alpha_n)x = 0 \) and \( (K - \beta_n)x = 0 \). Thus, if \( \alpha \) is a non-zero eigenvalue of \( H \), \( M_{\alpha} \) is the eigenspace \( \{x: Hx = \alpha x\} \) and \( \text{Re} \lambda_n = \alpha \), then \( P_n \mathcal{H} \subseteq M_{\alpha} \). Since \( T \) commutes with \( H \), \( M_{\alpha} \) is invariant under \( T \). It is not hard to see that \( T \) restricted to \( M_{\alpha} \) is a normal operator on a finite dimensional space \( M_{\alpha} \). If \( x \in M_{\alpha} \) is an eigenvector of \( T \) with \( Tx = \lambda x \), then, by the same argument as before we obtain \( Hx = (\text{Re} \lambda)x \) and hence \( \text{Re} \lambda = \alpha \). We conclude that \( M_{\alpha} \) is the direct sum of all those \( P_n \mathcal{H} \) with \( \text{Re} \lambda_n = \alpha \). Similarly, if \( \beta \) is a non-zero eigenvalue of \( K \), we write \( N_{\beta} \) for \( \{x \in \mathcal{H}: Kx = \beta x\} \), then \( N_{\beta} \) is the direct sum \( \sum \{P_n \mathcal{H}: \text{Im} \lambda_n = \beta \} \). Observe that \( P_n \mathcal{H} = M_{\alpha} \cap N_{\beta} \).

By Theorem 28.5 of [5, p. 87], \( \mathcal{H} \) is the closed linear span of the eigenspaces of \( H \) (or \( K \)). Now it is not difficult to see that \( \mathcal{H} \) is the closed linear span of eigenspaces of \( T \).

References


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