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DISJOINTLY ADDITIVE OPERATORS AND MODULAR SPACES

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0. Introduction. By now the literature concerning the representation of disjointly additive functionals and operators is quite extensive. A few entries on the subject are [**6, 7, 8, 11, 20, 21**]. In [**7, 8, 17**] further references can be found, in [**7**] the "prehistory" of the subject is also discussed.

To quote a typical result, we may take a 1967 theorem of Drewnowski and Orlicz ([**6**] Th. 3.2, [**17**] 12.4) which asserts that, under proper assumptions, an abstract modular (= disjointly countably additive functional) ρ on a "substantial" subspace D of L^0 can be realized by the formula

$$(1) \qquad \rho(x) = \int \phi(t, |x(t)|)d\nu.$$

Having the above formula for ρ, the theorem can also be interpreted as saying that for every such

$$(2) \qquad \rho : D \longrightarrow \mathbf{R}$$

there exists a modular space (notably, the familiar Musielak-Orlicz space L^φ) such that D is its subspace; furthermore (as it can be easily shown), the modular space in question is *canonical* in the following sense.

(i) L^φ with its F-norm topology τ is a Fatou Levi Riesz space.

(ii) $\rho : D \longrightarrow \mathbf{R}$ is τ-continuous.

(iii) L^φ is the largest modular solid subspace of L^0 such that (i) and (ii) hold.

The main result of this paper says, essentially, that replacing the real line $\mathbf{R}$ in (2) by an arbitrary Hausdorff topological vector space X, the latter version of the Drewnowski-Orlicz Theorem is still valid.

In other words, every disjointly countably additive operator $T : D \longrightarrow X$ 'generates' a canonical modular space L_T associated with T such that (i), (ii) and (iii) hold.

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1. Modular vector core. Let E be a set, and $\mathcal{F}$ a filter of subsets in E. Suppose that $\mathcal{F}$ has a (filter) base $\mathcal{V}$ consisting of sets having some properties,

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say $P, Q, \ldots$. As a rule, we will then qualify $\mathcal{V}$ as a $P, Q, \ldots$ filter base, and $\mathcal{F}$ as a locally $P, Q, \ldots$ filter. For instance, if E is a vector space and $\mathcal{F}$ is a filter of neighborhoods at 0 for some vector topology on E, then $\mathcal{F}$ is a locally balanced absorbent filter because there exists a base for the neighborhoods at 0 that is balanced and absorbent (i.e. consists of balanced absorbent sets). Sometimes the property under consideration is hereditary in the sense that if $\mathcal{V}$ has P then $\mathcal{F}$ has it (e.g. if $\mathcal{V}$ is absorbent then each $F \in \mathcal{F}$ is such). In such cases, we simply say that $\mathcal{F}$ is P (instead of locally P).

Let now E be a vector space and $\mathcal{F}$ a principal filter at 0. The filter $\mathcal{F}$, or its base $\mathcal{V}$, is said to be *summative* if

(s) $\quad \forall U \in \mathcal{V}\ \exists V \in \mathcal{V} : V + V \subset U.$

We will say that $\mathcal{V}$ (or $\mathcal{F}$) is *pseudo-summative* if it is p-summative for some $p \in (0, 1)$:

(p-s) $\quad \forall U \in \mathcal{V}\ \exists V \in \mathcal{V} : p(V + V) \subset U.$

Note that in this terminology summative = 1-summative.

If $\mathcal{F}$ is a locally balanced summative filter then $(E, \mathcal{F})$ is a topological group under addition whose filter of neighborhoods at 0 is precisely $\mathcal{F}$. The *vector core* [**12**] of $(E, \mathcal{F})$ is defined as follows:

$$X = \nu(E, \mathcal{F}) = \left\{ x \in E : \lim_n (1/n)x = 0\ (\mathcal{F}) \right\}$$

where $\lim_n (1/n)x = 0\ (\mathcal{F})$ means that

(1.1) $\quad \forall V \in \mathcal{F}\ \exists n \in \mathbf{N} : (1/n)x \in V.$

As is easily seen, X is a vector subspace of E and $\mathcal{F} \cap X = \{F \cap X : F \in \mathcal{F}\}$ is now absorbent on X and, consequently, defines a vector topology τ on X for which $\mathcal{F} \cap X$ is the filter of neighborhoods at 0. The vector core X is the largest vector space on which $\mathcal{F}$ defines the topological vector structure. (X, τ) is sometimes referred to as a *topological vector core* [**12**] of the pair $(E, \mathcal{F})$.

Suppose now that $\mathcal{F}$ is merely pseudo-summative. $X = \nu(E, \mathcal{F})$ is still a vector subspace of E and $\mathcal{F}' = \mathcal{F} \cap X$ is a locally balanced absorbent pseudo-summative filter on X. Such a filter or filter base will be called, following Leśniewicz and Orlicz [**16**], *modular*; the corresponding pair is called a *modular* (vector) *space*. Thus now $\nu(E, \mathcal{F})$ is the largest vector subspace of E on which $\mathcal{F}$ defines the modular space structure and we call $(X, \mathcal{F}')$ the *modular vector core* of $(E, \mathcal{F})$.

The *upper topology* τ on $(X, \mathcal{F}')$ is simply the weakest vector topology on X whose filter of neighborhoods at 0 is finer than $\mathcal{F}'$. It can be introduced, e.g., by the following base of neighborhoods at 0:

$$\mathcal{U} = \{\lambda V : \lambda > 0, V \in \mathcal{V}\} \quad \text{where } \mathcal{V} \text{ is a base of } \mathcal{F}.$$

A modular filter $\mathcal{F}$ is *Hausdorff* if its upper topology is Hausdorff.

2. Maximal modular enlargement. Let (S, Σ, ν) be a σ-finite positive measure space and $L^0 = L^0(S, \Sigma, \nu)$ the familiar (Riesz) space of a.e. finite measurable functions on (S, Σ, ν). In what follows our vector spaces will not be arbitrary but rather the Riesz subspaces of L^0. In this setting it will be natural to impose that filters (besides satisfying other conditions) be locally solid. Recall that a subset V in a Riesz subspace $L \subset L^0$ is *solid* (i.e., solid in L) if

$$(x \in V,\ y \in L, |y| \leqq |x|) \Rightarrow y \in V$$

and L will be said to be Σ-solid if for $x \in L$ and $E \in \Sigma$, $x1_E \in L$.

Our immediate aim is to apply the Abramovich process of extension (cf. [**1**][**12**]) to modular filters. Although the p-summative situation is more general than the linear topological (= 1-summative) case discussed in [**12**], no essentially new phenomena arise. Moreover, we will show that, due to the extremely simple connection between a modular base and its upper topology, the general case in some sense reduces to the 1-summative case. Thus we shall consider "the Fatou case" only: this is the one that will be needed in the sequel.

As is customary, we refer to the members of L^0 as functions. L^0_+ denotes the cone of positive functions in L^0. Our terminology concerning Riesz spaces follows [**2**] or [**12**]. We only recall that a solid set V in L is *order complete* if $0 \leqq x_\alpha \uparrow \subset V \Rightarrow \sup x_\alpha = x \in V$; V is *order closed* if $(x_\alpha) \subset V_+$, $x_\alpha \uparrow x \in L \Rightarrow x \in V$; (x_α) is a generic notation for a net. Following the Russian terminology, a solid order dense Riesz subspace of L^0 will be called, shortly, a *foundation*.

A pair $(L, \mathcal{F})$ is a *modular space* if L is a Riesz subspace in L^0 and $\mathcal{F}$ is a filter at 0 possessing a base which is *Hausdorff solid absorbent pseudo-summative*. In what follows we will fix $L^0 = L^0(S, \Sigma, \nu)$ and our modular spaces will always be *order dense* in this L^0.

Let $(L, \mathcal{F})$ be given and let $(M, \mathcal{G})$ be another modular space. We will say that $(M, \mathcal{G})$ *enlarges* $(L, \mathcal{F})$ or is an *enlargement* of $(L, \mathcal{F})$ if $M \supset L$ and $\mathcal{G} \mid L \simeq \mathcal{F}$. A set $B \subset L$ is *bounded in* $(L, \mathcal{F})$, or $\mathcal{F}$-bounded, if $\mathcal{F}$ absorbs B i.e., if for each $F \in \mathcal{F}$ there is $\lambda > 0$ such that $\lambda F \supset B$. Let $(L, \mathcal{F})$ be a modular space and let $\mathcal{V}$ be a (Hausdorff) absorbent solid p-summative base of $\mathcal{F}$. With the notation

$$L_x = L_{|x|} = \{u \in L : |u| \leqq |x|\}, \tag{2.1}$$

set

$$L^* = \{x \in L^0 : L_x \text{ is } \mathcal{F}\text{-bounded}\} \tag{2.2}$$

$$V^* = \{x \in L^* : L_u \subset V, V \in \mathcal{V}\}; \tag{2.3}$$

$$\mathcal{V}^* = \{V^* : V \in \mathcal{V}\} \text{ and } \mathcal{F}^* \text{ is the filter generated by } \mathcal{V}^*. \tag{2.4}$$

Let now $(L, \mathcal{F})$ be a *Fatou* modular space in L^0 i.e., we assume that $\mathcal{F}$ has a base $\mathcal{V}$ which, besides of being Hausdorff absorbent solid and p-summative has the property that the elements V in $\mathcal{V}$ are *order closed.*

We note that it is easy to see, from the very definition of L^* and $\mathcal{V}^*$, that

(2.5) L^* is a foundation of L^0.

(2.6) Given $V \in \mathcal{V}, V^*$ is solid (in L^* and so) in L^0

(2.7) $\mathcal{V}^*$ is Hausdorff absorbent

PROPOSITION 1. *$\mathcal{V}^*$ is p-summative.*

Proof. $L_{au} = aL_u$ for $a \geqq 0$ and, by decomposition property, $L_{u+v} \subset L_u + L_v$ for $u, v \geqq 0$. We now check that $p(V+V) \subset W \Rightarrow p(V^* + V^*) \subset W^*$.

If $0 \leqq f \in V^* + V^*$ then $f = u + v$, $u \in L^*$ and $v \in L^*$ with $L_u \subset V$, $L_v \subset V$. Thus $f \in L^*$, $pf = pu + pv$ and $pL_f = L_{pf} = L_{pu+pv} = pL_u + pL_v \subset p(V+V) \subset W$. □

Let $(L, \mathcal{F})$ be a Fatou modular space. The discussion above shows that $(L^*, \mathcal{F}^*)$ is a modular space enlarging $(L, \mathcal{F})$. Let τ be the upper topology of $(L, \mathcal{F})$ and τ^* the upper topology of $(L^*, \mathcal{F}^*)$. In **[12]** we have considered the Abramovich extension for a linear topological vector space denoting it by "#" (i.e., the extension of (L, τ) was $(L^\#, \tau^\#)$). As mentioned, nothing new happens since we have:

PROPOSITION 2. $L^* = L^\#$, $\tau^* = \tau^\#$.

Proof. We observe that ($\mathcal{V}$ is a modular base of $(L, \mathcal{F})$): a set $B \subset L$ is $\mathcal{V}$-bounded iff it is τ-bounded. This implies that $L^\# = L^*$.

Now, since $\tau^\#$ has a base of the form $\{\lambda V^\# : \lambda > 0, V \in \mathcal{V}\}$, τ^* has a base of the form $\{\lambda V^* : \lambda > 0, V \in \mathcal{V}\}$. But, since $L^* = L^\#$, $V^* = V^\#$ and it follows that $\tau^* = \tau^\#$.

PROPOSITION 3. *Let $(L, \mathcal{F})$ be a Fatou modular space. Then $(L^*, \mathcal{F}^*)$ is a locally boundedly order complete modular space.*

Proof. This is a consequence of Prop. 2 and Th. 4.2 in **[12]** taking into account the form of our base for the upper topology τ^*. □

Remark. According to our grammar, this means that $\mathcal{F}^*$ has a base consisting of (solid) boundedly order complete sets F (i.e., $(x_\alpha) \subset F$, (x_α) $\mathcal{F}$-bounded, (x_α) increasing implies that $x_\alpha \uparrow x \in F$). We note that, in particular, $\mathcal{F}^*$ is Fatou and $(L^*, \mathcal{F}^*)$ is *Levi* ($0 \leqq (x_\alpha)$ $\mathcal{F}^*$-bounded and (x_α) increasing implies that sup $x_\alpha = x \in L^*$).

Write $(L, \mathcal{F}) C_{\hookrightarrow}$ (resp. $C^{\hookrightarrow}$)$(M, \mathcal{G})$ if $L \subset M$ and $\mathcal{F}$ is finer (resp. coarser) than $\mathcal{G} \cap L$. $(L^*, \mathcal{F}^*)$ is the *maximal enlargement* of $(L, \mathcal{F})$ in the sense made precise by the following.

PROPOSITION 4. *If $(L, \mathcal{F}) C^{\hookrightarrow} (M, \mathcal{G})$, then $(M, \mathcal{G}) C_{\hookrightarrow} (L^*, \mathcal{F}^*)$. If $(M, \mathcal{G})$ is a Fatou enlargement of $(L, \mathcal{F})$ then $(L^*, \mathcal{F}^*)$ enlarges $(M, \mathcal{G})$.*

Proof. The first statement is easy (cf. [**12**] 3.3). Let now $(M, \mathcal{G})$ be Fatou and enlarging $(L, \mathcal{F})$. We first show that

$$(2.8) \qquad (L^*\mathcal{F}^*) C_{\hookrightarrow} (M^*, \mathcal{G}^*).$$

For each $G \in \mathcal{G}$ there exists $F \in \mathcal{F}$ with $F \subset G$. Working with a base rather than original filter, we may assume that G is solid order closed. Take x such that $L_x \in F$ (i.e., $x \in F^*$) and consider $y \in M_x$. Since L is order dense in M, we can find $0 \leqq x_\alpha \uparrow |y|$, $(x_\alpha) \subset L$. Clearly, $L_y \subset L_x \subset F \subset G$ and so $y \in G$ by order closedness and solidity of G. Hence $M_x \subset G$ which means also that $x \in G^*$. We have shown $F^* \subset G^*$ which implies (2.8).

On the other hand, since $(M^*, \mathcal{G}^*)$ enlarges $(L, \mathcal{F})$, by the first part we have $(M^*, \mathcal{G}^*) C_{\hookrightarrow} (L^*, \mathcal{F}^*)$. Consequently, we have $(M^*, \mathcal{G}^*) = (L^*, \mathcal{F}^*)$ which shows that $(L^*, \mathcal{F}^*)$ enlarges $(M, \mathcal{G})$. □

COROLLARY. *$(L^*, \mathcal{F}^*)$ is a unique locally bounded order complete modular space enlarging $(L, \mathcal{F})$.*

Proof. Suppose $(M, \mathcal{G})$ is another such enlargement. By Prop. 4, $(L^*, \mathcal{F}^*)$ enlarges $(M, \mathcal{F})$. Hence $(M^*, \mathcal{G}^*)$ enlarges $(L^*, \mathcal{F}^*)$ by Prop. 4 again. The result follows by observing that

$$(2.9) \qquad (M^*, \mathcal{G}^*) = (M, \mathcal{G}).$$

Indeed, $M^* = \{x \in L^0 : M_x \text{ is } \mathcal{V}\text{-bounded}\}$. If $x \notin M$, we can find $0 \leqq x_\alpha \uparrow |x|$, $(x_\alpha) \subset M$. Then $(x_\alpha) \subset (M)_x$ and is, therefore, $\mathcal{V}$-bounded. But then, by the completeness assumption, $|x|$ (whence x) belongs to M. Hence M cannot be further enlarged and (2.9) holds. □

Remark. One can apply the Abramovich extension procedure to the whole L^0 (setting, e.g., $V^0 = \{u \in L^0 : L_u \in V\}$ for $V \in \mathcal{V}$ in the formula 2.3). Then $\mathcal{V}^0 = \{V^0 : V \in \mathcal{V}\}$ is a Hausdorff solid p-summative filter base and it can be shown (cf. [**12**]) that (L^*, τ^*) is nothing else but the modular vector core of $(L^0, \mathcal{F}^0)$ (where $\mathcal{F}^0$ is the filter generated by $\mathcal{V}^0$).

3. Extension. In what follows $L^0 = L^0(S, \Sigma, \nu)$ is fixed; L^∞_{loc} is its foundation of essentially bounded functions with supports of finite measure; L^0 is considered with the topology of convergence in measure on sets of finite measure and L^∞_{loc} with the topology β of the inductive limit of topologies of uniform convergence on sets of finite measure; Y is a complete Hausdorff topogical vector space. We recall that, by the definition of $\beta, T : L^\infty \longrightarrow Y$ is β-continuous means that for every set of finite measure E and every sequence $(x_m) \subset L^\infty(E) = \{x 1_E : x \in L^\infty\}$, $\|x_m\|^E_\infty \longrightarrow 0$ implies $T(x_m) \longrightarrow 0$ (with $\|x\|^E_\infty = \text{supess}\,(|x 1_E|)$).

Let D be a Riesz subspace in L^0. By saying that an operator $T : D \cap L^\infty_{loc} \longrightarrow Y$ is uniformly β-continuous we mean that T is uniformly continuous as a map from the uniform space (associated with) $(D \cap L^\infty, \beta)$ into the uniform space Y. The term *Lebesgue topology* (on a Riesz space) means always a locally solid vector topology that is Lebesgue; *operator* means always a disjointly additive operator.

We recall few facts concerning Lebesgue topologies. Let D be an order dense Σ-solid Riesz subspace in L^0 equipped with a locally solid topology τ. The topology τ is often said to be **[19][22]** *absolutely continuous* if for each $x \in D$, $(E_n) \subset \Sigma$ and $E_n \downarrow \emptyset$, $x 1_{E_n} \longrightarrow 0(\tau)$ when $n \longrightarrow \infty$. Sometimes, especially in an abstract situation, the following terminology is used **[9]**. Let $(x_n) \subset D_+$; (x_n) is said to be *σ-laterally decreasing* to 0 ($x_n \urcorner 0$) if $x_n \downarrow 0$ $(x_n - x_{n+1}) \wedge x_{n+1} = 0$ ($\wedge = \inf$); an increasing (x_n) is *σ-laterally increasing* ($x_n \lrcorner$) if $x_{n+1} - x_n \wedge x_n = 0$; $x_n \lrcorner x$ means that $x_n \lrcorner$ and $x_n \uparrow x$. The topology τ is *laterally σ-order continuous* if $|x_n| \urcorner 0$ implies $x_n \longrightarrow 0(\tau)$; τ is said to be σ-Lebesgue **[2]** if: $(x_n) \subset D$, $|x_n| \downarrow 0$ implies $x_n \longrightarrow 0(\tau)$; τ is *Lebesgue* if it satisfies the previous condition for nets. The topology τ is *exhaustive* (or pre-Lebesgue **[2]**) if $(x_n) \subset D$, (x_n) disjoint and order bounded implies $x_n \longrightarrow 0(\tau)$.

For the following proposition see **[9]** Ch. X. §4. Th. 3 and **[19]** 2.3.5.

(3.1) *In (D, τ) absolute continuity, lateral σ-order continuity and σ-Lebesgue property coincide. Either one implies exhaustivity.*

We will say that an operator $T : D \longrightarrow Y$ is *countably additive* if for each $x \in D$ and $(E_n) \subset \Sigma$, $E_n \downarrow \emptyset$ we have $T(x 1_{E_n}) \longrightarrow 0$. It should be clear that such a T is also laterally σ-order continuous and exhaustive (with the obvious meaning of those terms). In order to be compatible with **[2]**, we adopt the "Lebesgue" terminology for topologies.

Given (D, τ), we denote by $(D, \tau)^a$ its subset of absolutely continuous elements: $x \in (D, \tau)^a$ iff $x_n \leqq |x|$, $x_n \downarrow 0 \Rightarrow x_n \longrightarrow 0(\tau)$. D^a can reduce to $\{0\}$ but is always a solid vector subspace in D and $\tau \mid D^a$ is σ-Lebesgue. Indeed D^a is the largest vector subspace of D on which τ is σ-Lebesgue.

From now on (D, τ) denotes an order dense Σ-solid Riesz subspace in L^∞_{loc} with a σ-Lebesgue topology τ. We note that since L^0 (ν is σ-finite) has the countable sup property τ is automatically Lebesgue. We have $(D, \tau) \hookrightarrow L^0$ (**[14]** §3, Cor. 12) and also τ is weaker than β.

Let (K, κ) be a Lebesgue enlargement of (D, τ).

PROPOSITION 1. *Let $T : D \longrightarrow Y$ be a τ-continuous disjointly additive operator which is also uniformly β-continuous. Then T extends uniquely to a (disjointly additive) continuous operator $T_K : (K, \kappa) \longrightarrow Y$.*

Proof. By the "principle of extension by continuity" (see [4] Ch. 1, §8, Theorem 1), we have to check that if $(x_n) \subset D$, $x \in K$ and $x_n \longrightarrow x(\kappa)$ then $(T(x_n))$ is a Cauchy sequence (and so converges to a limit in Y). We note that we can work with sequences rather than general nets since $L^\infty \cap K$ is sequentially dense in K. Thus we have to show the following lemma. Although its proof is routine, we give it here for the sake of completeness.

LEMMA 1. *Suppose that $x_n \longrightarrow x$ in (D,τ). Then $(T(x_n) : n \in \mathbf{N})$ is a Cauchy sequence in Y.*

Proof. We first show that given (x_n) and $(E_i) \subset \Sigma$, $E_i \downarrow \emptyset$, for each τ-neighborhood V of 0 there exists i_0 such that $x_n 1_{E_i} \in V$, $n \in \mathbf{N}$, $i \geqq i_0$. To this end note that the set functions $\Sigma \ni A \longmapsto x_n 1_A \in (D,\tau)$, $n \in \mathbf{N}$, are countably additive and converge pointwise on Σ to the set function $A \longmapsto x 1_A$. Hence by the Nikodym Theorem these set functions are uniformly countably additive. On the other hand, since τ-convergence implies convergence in measure on sets of finite measure, by the Egoroff Theorem we can find a sequence (F_i) of sets of finite measure such that $F_i \uparrow S$ and $x_n 1_{F_i} \longrightarrow x 1_{F_i}$ in the uniform topology for each $i \in \mathbf{N}$. Now setting $E_i = S \backslash F_i$, taking $\epsilon > 0$ and a continuous F-semi-norm on Y, we may write

$$\begin{aligned}\|T(x_n) - T(x_m)\| \leqq \|T(x_n 1_{F_{i_0}}) - T(x_m 1_{F_{i_0}})\| \\ + \|T(x_n 1_{E_{i_0}}) - T(x_m 1_{E_{i_0}})\| < 3\epsilon\end{aligned}$$

where the first term on the right hand side is less than ϵ by uniform β-continuity of T and the second term is less than 2ϵ, by the uniform countable additivity, for i_0 large enough. This shows that $(T(x_n))$ is Cauchy in Y.

The extension is disjointly additive since any function $x \in K$, by super order denseness of D, is a limit of a sequence in D such that the functions in the sequence have their supports contained in the support of x. □

Let us call a topology τ on D the *T-topology* if it is the weakest locally solid vector topology on D making the operator T continuous at 0.

Let V be a base of closed balanced neighborhoods of 0 in Y. Consider the coarsest locally solid filter $\mathcal{F}$ at 0 in D such that $T(\mathcal{F})$ is finer than $\mathcal{V}$. $\mathcal{F}$ is generated by the following base $\mathcal{U} = \{U = U(V) : V \in \mathcal{V}\}$ where $U(V) = \{x \in D : T(D_x) \subset V\}$.

(3.2) *The filter $\mathcal{F}$ is $(1/2)$-summative.* The proof of this fact is essentially the proof that if ϕ is an Orlicz function, then $\rho(x) = \int \phi(|x(t)|)d\nu$ is a modular (see e.g. [**18**] p. 10). Take $V \in \mathcal{V}$ and $V_1 \in \mathcal{V}$ such that $V_1 + V_1 \subset V$. We show that $(1/2)U(V_1) + (1/2)U(V_1) \subset U(V)$. To this end, take $x, y \in (1/2)U(V_1)$ and consider $T(x+y)$. Denote $E = \{s \in S : x(s) \geqq y(s)\}$ and $F = S \backslash E$. Then

$$T(x+y) = T[(x+y)1_E] + T[(x+y)1_F] \in T(D_{2x}) + T(D_{2y}).$$

Since $(1/2)U(V_1) = (1/2)\{u : T(D_u) \subset V_1\} = \{(1/2)u : D_u \subset V_1\} = \{x : T(D_{2x}) \subset V_1\} = \{y : T(D_{2y}) \subset V_1\}$, we have $T(x+y) \subset V_1 + V_1 \subset V$ which ends the proof. □

(3.3) *If $T : D \longrightarrow Y$ is β-continuous, then U is absorbent.* Since U is solid, it suffices to show that, given $x \in D_+$ and $U(V) \in \mathcal{U}$, $x \in nU(V)$ for some $n \in \mathbf{N}$. Equivalently, $T(D_{n^{-1}x}) \subset V$ for some $n \in \mathbf{N}$. Denying this condition, we can find $n_k \longrightarrow \infty$ and $x_k \in D_x$, $k \in \mathbf{N}$, such that $T(n_k^{-1}x_k) \notin V$. This is impossible since $n_k^{-1}x_k \longrightarrow 0$ in L^∞.

Now, assuming T to be uniformly β-continuous, by 3.5 and 3.6, $\mathcal{F}$ is a (1/2)-summative locally solid absorbent filter such that T is continuous at zero for the upper topology τ of $\mathcal{F}$ and by construction τ is the T-topology on D. □

LEMMA 2. *For each net $(x_\alpha) \subset D$, $x_\alpha \longrightarrow x(\tau)$, there exists a sequence $(x_{\alpha_n}) \subset (x_\alpha)$, $n \in \mathbf{N}$, such that $x_{\alpha_n} \longrightarrow x(\tau)$.*

Proof. Denote by μ the topology of convergence in measure on sets of finite measure in L^0 (and D). We have $\mu \subset \tau$ and μ is metrizable. Hence there exists $x_{\alpha_n} \longrightarrow x(\tau)$. Since $(x_{\alpha_n} : n \in \mathbf{N})$ are uniformly absolutely (τ-) continuous and by using the Egoroff Theorem in a similar fashion as in Lemma 1, given a τ-neighborhood V of 0 we will find E and n_0 so large that $(x_{\alpha_n} - x)1_E \in V$ for $n \geqq n_0$ and $(x_{\alpha_n} - x)1_{S\setminus E}$ for all $n \in \mathbf{N}$ which implies the Lemma. □

(3.4) *T is τ-continuous (for its T-topology τ).* In view of Lemma 2, we may work with sequences. Taking $x_n \longrightarrow x$ in (D, τ), it is easy to check, by using Egoroff Theorem and uniform absolute continuity of x_n's (as in the proof of the lemma above), that $T(x_n) \longrightarrow T(x)$.

The topology τ need not be Hausdorff. However, that it will be Hausdorff can be assured by a familiar procedure in which the "inessential" part of D is thrown out.

Let $\{\rho_\alpha : \alpha \in A\}$ be a family of monotone F-semi-norms defining τ on D. Call a set $E \in \Sigma$ a ρ_α-zero set if $\rho_\alpha(1_E) = 0$. Find, e.g. by Hausdorff Maximum Principle, a maximal disjoint family $\mathcal{E}_\alpha$ of ρ_α-zero sets of positive measure and observe that $\mathcal{E}_\alpha$ is at most countable since ν is σ-finite. Set

$$E_\alpha = \cup\{E : E \in \mathcal{E}_\alpha\} \quad \text{and} \quad E_0 = \cap\{E_\alpha : \alpha \in A\}.$$

Observe that since L^0 has the countable sup property, there exists a sequence $(A_n) \subset (A_\alpha)$ such that $\inf_\alpha A_\alpha = \inf A_n$ and so $A_0 = \cap A_n \in \Sigma$. By construction E_0 is a maximal τ-zero set in the sense that iff $E_0 \subset E \in \Sigma$ and $\nu(E \setminus E_0) > 0$ then $\rho_\alpha(1_E) > 0$ for some α. Thus τ is Hausdorff on $D(S\setminus E_0) = \{x1_{S\setminus E_0} : x \in D\}$. It can be checked that $D(E_0)$ is the subspace of all $x \in D$ having the property that, for each scalar a, $T(ax) = 0$. Thus, the band $D(E_0)$ of D is inessential and we can restrict our attention to the restriction of T on $D(S\setminus E_0)$ which is *T-saturated*: $x \in D(S\setminus E_0) \Rightarrow \exists a$ such that $T(ax) \neq 0$. □

In what follows we always assume that $D = D(S\setminus E_0)$, i.e., that D is T-saturated (and so our T-topology will automatically by Hausdorff).

Summing up, we have shown the following.

PROPOSITION 2. *Let $T : D \longrightarrow Y$ be uniformly β-continuous* (*and D be T-saturated*). *Then there exists a smallest Hausdorff locally solid vector topology*

τ on D such that T is τ-continuous. The topology τ is weaker than β and is the T-topology on D.

The question remains when τ is Lebesgue?

PROPOSITION 3. *Suppose that the operator $T : D \longrightarrow Y$ is countably additive. Then the T-topology τ on D is Lebesgue.*

Proof. It is sufficient to show that τ is σ-Lebesgue since D has the countable sup property. Let $u \in D_+$ and $(E_i) \subset \Sigma$, $E_i \downarrow \emptyset$ be given. Let us show that, given $a > 0$ and $U = U(V) \in \mathcal{U}$, there exists i so large that $u1_{Ei} \in aU$. Equivalently, we want to show that $T(D_{x_i}) \subset V$, where $a^{-1}u1_{E_i} = x_i$, $n \in \mathbf{N}$. We show first:

(3.5) *There exists $x \in \{x_i : i \in \mathbf{N}\}$ such that* $T([0,x]) \subset V$. We denoted $[0,x]$ the set $(D_+)_x$. Now let us deny (3.5) and set

$$A_n = \{x_i : i \geqq n\}$$
$$B_n = \{z \in D_+ : \exists k \in \mathbf{N} \text{ with } x_k \leqq z \leqq x_n\}.$$

For each $w \in [0, x_n]$, the sequence $(w \vee x_i : i \geqq n) \subset B_n$ and decreases to w. Further, for each $i \geqq n$, we can write $w \vee x_i$ as a disjoint sum: $w \vee x_i = w_i + x_i$ since x_i is a component of $w \vee x_i$. Moreover, $x_i \urcorner 0$ and $w_i \lrcorner w$ as $i \longrightarrow \infty$.

Consequently, $T(w \vee x_i) = T(w_i) + T(x_i) \longrightarrow T(w)$, which shows that $w \in \overline{T(B_n)}$. But V in $\mathcal{V}$ is closed and we are supposing that $T[0, x_n] \not\subset V$ so $T(B_n) \not\subset V$.

Set

$$C = \left\{ z \in D_+ : T(z) \notin V \;\&\; \underset{m>n}{\exists}\; x_m \leqq z \leqq x_n \right\}, m > n$$

and observe that

(i) $\forall x_n \in \{x_i : i \in \mathbf{N}\} \exists z_n \leqq x_n, z_n \in C$

(ii) $\forall z_n \in C \exists m > n$ with $x_m < z_n$.

Hence we can find a sequence (z_{i_k}) such that $z_{i_k} \leqq x_{i_k}$ and (x_{i_k}) is an infinite subsequence of (x_i). To simplify notation suppose that we have (x_i) and (z_i), $z_i \leqq x_i$, $T(z_i) \notin V$. Take $i_1 = 1$ and $z_{i_1} = z_1$. Since V is closed and T is countably additive, we can find i_2 so large that $T(z1_{E_{i_2}})$ is small enough to have $T(z_1') \notin V$ where $z_1' = z_1 - z_1 1_{E_{i_2}}$. Take $i_3 > i_2$. Since $x_i \urcorner 0$, z_{i_3} is disjoint with z_{i_1}. As above, find i_4 and $z_{i_3}' = z_{i_3} - 1_{E_{i_4}}$ such that $T(z_{i_3}') \notin V \ldots$. By continuing this process, we find a disjoint sequence (z_{i_k}'), contained in $[0, u]$, for which $T(z_{i_k}') \notin V$. This contradicts exhaustivity of T. Thus, (3.5) holds. The symmetric statement obtained by replacing $T([0,x])$ in (3.5) by $T([-x, 0])$ also holds (by duality: replace our order by antiorder and repeat the proof). Thus we can find i so large that for $x_i = a^{-1}u1_{E_i}$, $T([0, x_i]) \subset V$ and $T([-x_i, 0]) \subset V$. Take $x \in D_{x_i}$. Then $x_+ \in [0, x_i]$, $x_- \in [-x_i, 0]$ so that $T(x) = T(x_+) + T(x_-) \in V + V$. Hence $T(D_{x_i}) \subset V + V$ and, as V was arbitrary, this means that $T(u1_{E_i}) \longrightarrow 0$.

In the above $u \in D_+$ but, by symmetry again, the same will hold for $u \in D_-$. Now, let $u \in D$ be arbitrary, and $E_i \downarrow \emptyset$. Then

$$T(u1_{E_i}) = T(u_+1_{E_i}) + T(-u_-1_{E_i}) \longrightarrow 0$$

and the proof is finished. □

COROLLARY. *Let $T : D \longrightarrow Y$ be a countably additive uniformly β-continuous operator. Suppose that the T-topology is Hausdorff on D. Then T extends uniquely to a continuous disjointly additive operator $\tilde{T} : (D^\#, {}^\#)^a \longrightarrow Y$, where $D^\#$ denotes the (maximal) Abramovich extension of (D,τ).*

Let now (K,κ) denote a foundation K of L^0, with a Hausdorff solid vector topology κ, subject to the following conditions:

(i) $D \cap K$ is κ-dense in K

(ii) $T \mid K : K \cap D \longrightarrow Y$ is κ-continuous

(iii) $T \mid K$ entends by continuity to a disjointly additive operator $T_K : K \longrightarrow Y$.

Denote by $\mathcal{K}$ the family of all such (K,κ). We say that T is *$\mathcal{K}$-compatible* if for $K, M \in \mathcal{K}$ and $u \in K \cap M$ we have $T_K(u) = T_M(u)$.

PROPOSITION 4. *Let $T : D \longrightarrow Y$ be a uniformly β-continuous operator such that its T-topology τ is Hausdorff Lebesgue. Then for every $K \in \mathcal{K}$, $K \subset (D^\#, \tau^\#)^a$ and $T_K = \tilde{T} \mid K$; in particular, T is $\mathcal{K}$-compatible.*

Proof. We first show that κ is stronger than τ on $D \cap K$. Take a net $(u_\alpha) \subset D \cap K$, $u_\alpha \longrightarrow 0(\kappa)$. Since κ is locally solid, it is sufficient to show that $u_\alpha \longrightarrow 0(\mathcal{U})$. If not, then there exists a neighborhood $U(V)$ and a subnet (u_γ) of (u_α) such that $u_\gamma \notin U(V)$ for each γ. This means that we can find, for each γ, $x_\gamma \in D_{u_\gamma}$ such that $T(x_\gamma) \notin V$. This contradicts κ-continuity of T since, as K is solid, $(x_\gamma) \subset K$ and, as κ is locally solid, $x_\gamma \longrightarrow 0(\kappa)$.

Now by the unicity statement (§2, Corollary) we must have $(D \cap K, \tau)^\# = (D,\tau)^\#$. Also $(D \cap K, \tau) \hookrightarrow (K, \tau)$ so that (§2, Prop. 4) $(K,\kappa) \hookrightarrow (D \cap K, \tau)^\# = (D,\tau)^\#$. But it is implicit in (iii) that D is κ-dense in K and so D is $\tau^\#$-dense in K. Thus $\tau^\# \mid K$ is Lebesgue ([**2**] 10.6) and $K \subset D^\#$.

The results of this section give the following

MAIN THEOREM. *Let D be an order dense Σ-solid Riesz subspace of L^∞_{loc}, Y a complete Hausdorff topological vector space and $T : D \longrightarrow Y$ a disjointly additive operator such that D is T-saturated. Suppose that T is countably additive and uniformly β-continuous. Then there exists a Fatou Levi modular space $D^\#$ such that its upper topology $\tau^\#$ has the following properties*

(i) *$D \subset (D^\#, \tau^\#)^a$*

(ii) *T extends to a continuous disjointly additive operator $\tilde{T}$ on $(D^\#, \tau^\#)^a$.*

(iii) *$(D^\#, \tau^\#)^a$ is the largest Hausdorff locally solid foundation on which T can be extended by continuity.*

4. Comments. (1) We have assumed before Prop. 1 and throughout the paper that D, besides being order dense and Σ-solid, is contained in L^∞_{loc}. The latter assumption is technically convenient but otherwise is inessential in view of the following argument.

Suppose D is merely order dense and Σ-solid. Then $D_1 = D \cap L^{\infty}_{loc}$ can be considered instead of D (since it is still order dense Σ-solid). Suppose T-topology exists on D. Since $D_1 \, C^{\hookrightarrow} \, D$ we have $D^{\#} \, C_{\hookrightarrow} \, D_1^{\#}$ and so $D^{\#a} \subset D_1^{\#a}$. On the other hand, since $D^{\#a}$ is the largest Hausdorff locally solid foundation on which T can be extended, $D_1^{\#a} \subset D^{\#a}$ i.e., $D_1^{\#a} = D^{\#a}$.

(2) For what operators $T : D \longrightarrow Y$ does it happen that $(D^{\#})^a = D^{\#}$?

Recall that a sequence (x_n) or a series Σx_n, in a topological vector space, is said to be *perfectly bounded* if its set of all unordered partial sums $\{\sum_{n\in e} x_n : e$ finite subset of $\mathbf{N}\}$ is bounded. Consider the following condition

(*) For every disjoint sequence $(x_n) \subset D$ such that (x_n) is perfectly bounded in D, $(T(x_n))$ is perfectly bounded in Y.

PROPOSITION. *Let $T : D \longrightarrow Y$ be countably additive uniformly β-continuous operator. Suppose moreover that T satisfies condition* (*) *and $Y \not\supset c_0$ (i.e., Y has no copy of c_0). Then $(D, \tau^{\#})$ is Lebesgue* (*and T extends to the whole of $D^{\#}$*).

Proof. We recall that $Y \not\supset c_0$ means that there is no subspace $F \subset Y$ such that F is linearly homeomorphic to the Banach space c_0 of sequences convergent to 0. We show that $(D^{\#})^a = D^{\#}$. Since $(D^{\#}, \tau^{\#})$ is complete, it will be sufficient to show that $\tau^{\#}$ is exhaustive ([**2**] 10.3). Let (E_n) be a disjoint sequence in Σ. Suppose $x \in D^{\#}$ and $x1_{E_n} \not\rightarrow 0(\tau^{\#})$. Then we can find $a > 0$ and $U^{\#} = U(V)^{\#} \in \mathcal{U}^{\#}$ such that for some subsequence $(E_k) \subset (E_n)$, $x1_{E_k} \notin aU^{\#}$ which means also that for $\mathrm{v} = a^{-1}x1_{E_k}$, $T(D_{\mathrm{v}}) \not\subset V$ i.e. there exist x_k, $k = 1, 2, \ldots$, with $|x_k| \leqq |a^{-1}x1_{E_k}|$, $T(x_k) \notin V$. Consider the bounded linear operator $t : c_{00} \longrightarrow Y$ such that for $c = (c_n) \in c_{00}$ (sequences that are eventually zero) $t(c) = T(\sum_n c_n x_n)$. The operator t extends by continuity to c_0 and $t(e_n) = T(x_n) \not\rightarrow 0$. Hence Y contains c_0 [**10**][**5**].

(3) If Y is metrizable then $(D^{\#}, \tau^{\#})$ and its underlying modular space will be metrizable too.

(4) In the introduction we have emphasized the roots of the present paper in some classical work on the representation of modulars which certainly is historically true. The extension of the operator T to $\tilde{T}$ is a by-product. However this point of view can be reversed, and in fact in a recent paper [**11**] Kozlowski does similar things declaring the extension of a disjointly additive operator his main goal.

(5) The process of extension of T can also be viewed as a Daniell type extension of a not necessarily linear (vector) integral and is indeed a generalization of the classical extension of the integral (from the space of simple functions onto L^1). On the other hand, the use of the Abramovich extension "#" and some other techniques used above are in connection with the papers of Aronszajn-Szeptycki [**3**] and Labuda-Szeptycki [**15**] on extensions of integral operators.

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