1. Introduction

Let $\Omega$ be a bounded domain of $n$-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. On $\Omega$ we consider the biharmonic equation

$$\Delta^2 u = \left( \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \right)^2 u = 0.\tag{1}$$

A function $u$ in $C^4(\Omega)$ is called biharmonic in $\Omega$ if it satisfies the equation (1). In this note we shall deal with the following boundary value problems. Find a biharmonic function $u$ in $\Omega$ such that the following couples of functions have boundary values given on the boundary of $\Omega$:

(a) $\frac{\partial u}{\partial n}$, $\frac{\partial (\Delta u)}{\partial n}$;
(b) $\Delta u$, $\frac{\partial u}{\partial n}$;
(c) $u$, $\frac{\partial (\Delta u)}{\partial n}$.

J. L. Lions [4] treated these problems for the operator $\Delta^2 + I$ and gave solutions in case that $\Omega$ is a Nikodym domain. But in his method, the boundary of $\Omega$ or boundary functions are not referred to.

In this note we take as the boundary the Martin boundary $M$ of $\Omega$ and define notations $\gamma_0(u)$ and $\gamma_1(u)$ for a function $u$ on $\Omega$ as follows. If $u$ has a fine boundary function $f$ on $M$ we denote $f$ by $\gamma_0(u)$ and if $u$ has $\varphi$, as generalized normal derivative of Doob [3] (in a slightly modified sense), we denote $\varphi$ by $\gamma_1(u)$ (c.f. Definitions 1 and 2).

Now our boundary value problems are described as follows. Find a biharmonic function $u$ in $\Omega$ such that the following couples of functions are equal to boundary functions given on the Martin boundary $M$:
Let $K(x, \xi)$ be the Martin kernel and $\mu$ be the harmonic measure on $M$. Define new measures $\tilde{\mu}$ and $\bar{\mu}$ on $M$ by $d\tilde{\mu}(\xi) = k(\xi)d\mu(\xi)$ and $d\bar{\mu}(\xi) = \frac{1}{k(\xi)}d\mu(\xi)$, where $k(\xi) = \int K(x, \xi)dx$.

Then we shall show that for any $\varphi \in L^1(\bar{\mu})$ with $\int \varphi(\xi)d\mu(\xi) = 0$, there exists a square integrable harmonic function $h$ on $\Omega$ with $D(h) < \infty$ such that $\gamma(\varphi) = \varphi$ if and only if $\varphi$ is a Nikodym domain (Lemma 8). As an application of this fact we shall solve the above boundary value problems as follows.

Assume that $\Omega$ is a Nikodym domain, then

(a) for any $\varphi$ and $\psi$ in $L^1(\bar{\mu})$ with $\int \varphi(\xi)d\mu(\xi) = 0$ there exists a biharmonic function $u$ such that $\gamma(\varphi) = \varphi$ and $\gamma(\Delta u) = \psi$;

(b) for any $f \in L^1(\bar{\mu})$ and $\varphi \in L^1(\bar{\mu})$ with $\int \varphi(\xi)d\mu(\xi) = -\int H_f(x)dx$ there exists a biharmonic function $u$ such that $\gamma(\Delta u) = f$ and $\gamma(u) = \varphi$;

(c) for any $f \in L^1(\mu)$ and $\varphi \in L^1(\bar{\mu})$ with $\int \varphi(\xi)d\mu(\xi) = 0$ there exists a biharmonic function $u$ such that $\gamma(\varphi) = f$ and $\gamma(\Delta u) = \varphi$.

Moreover the uniqueness of the above solutions will be shown.

2. Preliminaries

Let $\Omega$ be an arbitrary bounded domain of the $n$-dimensional Euclidean space $\mathbb{R}^n (n \geq 2)$ and $G(x, y)$ be it's Green function with respect to the equation $\Delta u = 0$, that is $(-\Delta)G(x, y) = \delta_x$ in $\Omega$.

We shall mention the definition of the Martin boundary of $\Omega$.

We put

$$K(x, y) = \frac{G(x, y)}{G(x_0, y)}$$

on $\Omega \times \Omega$ if $y \neq x_0$ and $K(x, x_0) = 0$ if $x \neq x_0$ and $K(x_0, x_0) = 1$, where $x_0$ is a fixed reference point in $\Omega$.

We take a fixed exhaustion $\{\Omega_n\}$ of $\Omega$ such that $x_0 \in \Omega_1$, and put
\[ d(x_1, x_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x \in B_n} \left| \frac{K(x, x_1)}{1 + K(x, x_1)} - \frac{K(x, x_2)}{1 + K(x, x_2)} \right|. \]

Then \( d \) defines a metric on \( \Omega \). We denote by \( \Omega^* \) the completion of \( \Omega \) by this metric. For a point \( \xi \in \Omega^* - \Omega \), we can find a sequence \( \{y_n\} \) in \( \Omega \) such that \( d(\xi, y_n) \to 0 \) and so we can define

\[ K(x, \xi) = \lim_{n \to \infty} K(x, y_n). \]

We say that \( \Omega^* \) is the Martin compactification of \( \Omega \) and the set \( M = \Omega^* - \Omega \) is called the Martin boundary of \( \Omega \). The function \( K(x, \xi) \) on \( \Omega \times \Omega^* \) is called the Martin kernel. We denote by \( \mu \) the harmonic measure on \( M \) with respect to the fixed reference point \( x_0 \).

Now let \( G_\lambda(x, y) \) be the Green function of \( \Omega \) with respect to the equation \((\Delta - 1)u = 0\), that is \((-\Delta y + 1)G_\lambda(x, y) = \varepsilon_\lambda\) in \( \Omega \). For \( x \in \Omega \) and \( \xi \in M \), we put

\[ K_\lambda(x, \xi) = K(x, \xi) - \int G_\lambda(x, y)K(y, \xi)dy. \]

We set for \( f \in L^1(\mu) \),

\[ H_f(x) = \int K(x, \xi)f(\xi)d\mu(\xi) \]

and

\[ H_\lambda f(x) = \int K_\lambda(x, \xi)f(\xi)d\mu(\xi). \]

Denote by \( D(u) \) the Dirichlet integral of \( u \) on \( \Omega \). For measurable functions \( f \) and \( g \) on \( M \), we put

\[ D(f, g) = \frac{1}{2} \int M \int M (f(\xi) - f(\eta))(g(\xi) - g(\eta))\theta(\xi, \eta)d\mu(\xi)d\mu(\eta) \]

and \( D(f) = D(f, f) \), where \( \theta(\xi, \eta) \) is the Naim kernel (c.f. [7]).

The following lemma is obtained by Doob [3].

**Lemma 1.** If \( u \) is a harmonic function with \( D(u) < \infty \), then \( u \) has a fine boundary function \( u' \) and \( D(u') = D(u) \). Conversely if \( f \) is an arbitrary measurable function on \( M \) with \( D(f) < \infty \), then \( f \in L^1(\mu) \) and \( D(H_f) = D(f) \).
Put \( k(\xi) = \int K(x, \xi)dx \), and \( k(\xi) \) is a strictly positive lower semicontinuous function on \( M \) and so \( \inf_{\xi \in M} k(\xi) = c > 0 \). Since

\[
\int k(\xi)d\mu(\xi) = \int \left( \int K(x, \xi)d\mu(\xi) \right)dx = |\Omega| \quad \text{(area of } \Omega) ,
\]

we see that \( k(\xi) \in L^1(\mu) \).

Define new measures \( \bar{\mu} \) and \( \bar{\bar{\mu}} \) on \( M \) by \( d\bar{\mu}(\xi) = k(\xi)d\mu(\xi) \) and \( d\bar{\bar{\mu}}(\xi) = \frac{1}{k(\xi)}d\mu(\xi) \) respectively, and we have the following relations

(6) \( B(M) \subset L^1(\bar{\mu}) \subset L^1(\mu) \subset \mathbb{L}(\bar{\bar{\mu}}) \subset L^1(\mu) \),

where \( B(M) \) is the space of all bounded measurable functions on \( M \). We also see that

(7) \( \| f \|_{L^1(\bar{\mu})} \leq \frac{1}{\sqrt{c}} \| f \|_{L^1(\mu)} \leq \frac{1}{c} \| f \|_{L^1(\bar{\mu})} \)

for any \( f \in L^1(\bar{\mu}) \).

By the Fubini theorem, \( \int H_{\mu}(x)dx < \infty \) for any \( f \in L^1(\bar{\mu}) \). Hence we know

\[
\int H_{\|f\|}(x)H_{\|g\|}(x)dx \leq \int H_{\|f\|}(x)H_{\|g\|}(x)dx \\
\leq \left( \int (H_{\|f\|}(x))^2dx \cdot \int (H_{\|g\|}(x))^2dx \right)^{1/2} \\
\leq \left( \int H_{\|f\|}(x)dx \cdot \int H_{\|g\|}(x)dx \right)^{1/2} < \infty
\]

for any \( f \) and \( g \) in \( L^1(\bar{\mu}) \).

**Lemma 2.** Let \( f \) and \( g \) be in \( L^1(\bar{\mu}) \). Then

(8) \( \int H_f(x)H_g^*(x)dx = \int H_f(x)H_g^*(x)dx \)

and

(9) \( \int H_f(x)H_g^*(x)dx \leq \int (H_f(x))^2dx \leq c' \int H_f(x)H_g^*(x)dx \)

for some constant \( c' \geq 1 \).

**Proof.** By the definition of \( K_i(x, \xi) \) and the resolvent equation,
(10) \[ H'_f(x) = H_f(x) - \int G_1(x, y)H_f(y)dy \]
and
(11) \[ H_f(x) = H'_f(x) + \int G(x, y)H'_f(y)dy . \]

Hence
\[
\int H_\phi(x)H'_f(x)dx = \int H_\phi(x)\left( H_f(x) - \int G_1(x, y)H_f(y)dy \right)dx \\
= \int H_\phi(x)H_f(x)dx - \int H_f(y)\left( \int G_1(x, y)H_\phi(x)dx \right)dy \\
= \int H_\phi(x)H_f(x)dx - \int H_f(y)(H_\phi(y) - H'_f(y))dy \\
= \int H_f(x)H'_f(x)dx
\]

and
(12) \[
\int (H_f(x))^2dx - \int H_f(x)H'_f(x)dx = \int H_f(x)(H_f(x) - H'_f(x))dx \\
= \int H_f(x)\left( \int G_1(x, y)H_f(y)dy \right)dx \\
= \int \int G_1(x, y)H_f(x)H_f(y)dxdy \geq 0 .
\]

By (11)
\[
\int (H_f(x))^2dx - \int H_f(x)H'_f(x)dx = \int H_f(x)\left( \int G(x, y)H'_f(y)dy \right)dx
\]

and hence
\[
\left( \int (H_f(x))^2dx - \int H_f(x)H'_f(x)dx \right)^2 \\
\leq \int (H_f(x))^2dx \cdot \left( \int \int G(x, y)dy \cdot \int G(x, y)(H'_f(y))^2dy \right)dx \\
\leq c_0^2 \cdot \int (H_f(x))^2dx \cdot \int (H'_f(x))^2dx
\]

where \( c_0 = \sup_{x \in \bar{a}} \int G(x, y)dy \). Similarly to (12), we know
\[
\int H_f(x)H'_f(x)dx - \int (H'_f(x))^2dx \geq 0 ,
\]
and so we have an inequality
\[
\int (H_f(x))^2 dx - \int H_f(x)H'_f(x) dx \\
\leq c_0 \left( \int (H_f(x))^2 dx \right)^{1/2} \left( \int H_f(x)H'_f(x) dx \right)^{1/2}.
\]
Hence
\[
\int (H_f(x))^2 dx \leq c' \int H_f(x)H'_f(x) dx
\]
for some constant \( c' \geq 1 \). This completes the proof.

Now we set
\[
\tilde{H}(M) = \{ f ; f \in L^2(\mu) \text{ and } D(f) < \infty \},
\]
and define two inner products on \( \tilde{H}(M) \) by
\[
(f, g)_1 = D(f, g) + \int H_f(x)H_g(x) dx
\]
and
\[
(f, g)_2 = D(f, g) + \int H_f(x)H'_g(x) dx
\]
for functions \( f \) and \( g \) in \( \tilde{H}(M) \). By the above lemma, we know that \((\cdot, \cdot)_2\) is an inner product on \( \tilde{H}(M) \). We put \( \| f \|_2^2 = (f, f)_1 \) and \( \| f \|_2 = (f, f)_2 \) for \( f \in \tilde{H}(M) \). Then we have

**Lemma 3.** Norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent and \( \tilde{H}(M) \) is a Hilbert space with respect to these norms.

**Proof.** By the above lemma,
\[
\| f \|_2 \leq \| f \|_1 \leq \left( \max (1, c') \right)^{1/2} \| f \|_2,
\]
and so these norms are equivalent. Let \( f \) be in \( \tilde{H}(M) \). Then by the Riesz decomposition of \( -(H_f)^2 \) we have
\[
(H_f)^2 = H_{1\mathcal{A}} - \int G(\cdot, y)d\nu_f(y).
\]
Since \( D(H_f) = \frac{1}{2} \int d\nu_f \), we have
\( \| f \|_{L^2(\Omega)} = \int H_f(x) \, dx \)
\[ \begin{align*}
&= \int ((H_f(x))^2 + \int G(x, y) \, dv_f(y)) \, dx \\
&\leq \int (H_f(x))^2 \, dx + c_\Phi \int dv_f \\
&\leq \max (1, 2c_\Phi) \left( \int (H_f(x))^2 \, dx + D(H_f) \right) \\
&= \max (1, 2c_\Phi) \left( \int (H_f(x))^2 \, dx + D(f) \right) \\
&= \max (1, 2c_\Phi) \| f \|_1^2.
\end{align*} \]

Hence we see that \( \tilde{H}(M) \) is a Hilbert space.

3. **Definitions of \( \gamma_o(u) \) and \( \gamma_1(u) \) for a function \( u \) on \( \Omega \)**

   We shall define \( \gamma_o(u) \) and \( \gamma_1(u) \) for a function \( u \) on \( \Omega \) as follows.

   **DEFINITION 1.** If a function \( u \) on \( \Omega \) has a fine boundary function \( f \) on \( M \), we denote \( f \) by \( \gamma_o(u) \).

   The definition of \( \gamma_1(u) \) is a slight modification of the definition of the generalized normal derivative of \( u \) (c.f. Doob [3]).

   **DEFINITION 2.** Consider the function \( u(x) = H_f(x) + u_p(x) \), where \( f \) is a measurable function on \( M \) with \( D(f) < \infty \) and \( u_p \) is a potential of a measure \( \nu \) on \( \Omega \). We assume that for any \( g \in H(M), H_\Phi \) is integrable on \( \Omega \) with respect to the absolute variation of \( \nu \). If there exists a function \( \varphi \) on \( M \) such that \( \int \varphi(\xi) g(\xi) d\mu(\xi) < +\infty \) and

   \[ D(f, g) = -\int \varphi(\xi) g(\xi) d\mu(\xi) + \int H_\Phi(x) dv(x) \]

   for any \( g \in \tilde{H}(M) \), we denote \( \varphi \) by \( \gamma_1(u) \).

   We shall show the following

   **LEMMA 4.** Let \( \varphi \) be in \( L(\tilde{\Omega}) \). Then there exists a unique function \( f \in \tilde{H}(M) \) such that \( \gamma_1(u) = \varphi \), where

   \[ u(x) = H_f(x) - \int G(x, y) H_f(y) \, dy. \]

   **Proof.** In the Hilbert space \( \tilde{H}(M) \) with the norm \( \| \cdot \|_1 \), the mapping
g \rightarrow -\int g(\xi)\varphi(\xi)d\mu(\xi) is a linear functional. By the Schwarz inequality and (17), we have
\[\left| -\int g(\xi)\varphi(\xi)d\mu(\xi) \right|^2 \leq \left( \int |g(\xi)|^2 \frac{1}{k(\xi)^{1/2}} |\varphi(\xi)| d\mu(\xi) \right)^2 \]
\[\leq \|\varphi\|_{L^2(\mathfrak{F})} \cdot \|g\|_{L^2(\mathfrak{F})} \]
\[\leq \max (1, 2c_\varphi) \|\varphi\|_{L^2(\mathfrak{F})} \cdot \|g\|_{L^2}^2 .\]

Hence the above mapping is bounded on \( \hat{H}(M) \). Therefore there exists a unique function \( f \in \hat{H}(M) \) such that \( (f, g)_1 = -\int \varphi(\xi)g(\xi)d\mu(\xi) \), namely
\[\mathcal{D}(f, g) = -\int \varphi(\xi)g(\xi)d\mu(\xi) + \int H_f(x)(-H_f(x))dx\]
for any \( g \in \hat{H}(M) \). If we put \( u(x) = H_f(x) - \int G(x, y)H_f(y)dy \), then from the definition we have \( \gamma(\mu)(u) = \varphi \).

Similarly we have

**Lemma 5.** Let \( \varphi \) be in \( L^2(\mathfrak{F}) \). Then there exists a unique function \( f \in \hat{H}(M) \) such that \( \gamma_1(H_f) = \varphi \).

**Proof.** By Lemma 3, the mapping \( g \rightarrow -\int g(\xi)\varphi(\xi)d\mu(\xi) \) is a bounded linear functional on the Hilbert space \( \hat{H}(M) \) with the norm \( \| \cdot \|_2 \).
Hence there exists a unique function \( f \in \hat{H}(M) \) such that
\[\mathcal{D}(f, g) = -\int \varphi(\xi)g(\xi)d\mu(\xi) + \int H_f(x)(-H_f(x))dx\]
for any \( g \in \hat{H}(M) \). Since \( H_f(x) = H_f(x) - \int G(x, y)H_f(y)dy \), we have \( \gamma_1(H_f) \)
\[= \varphi .\]

We set
\[\hat{H}(M) = \{ f \in \hat{H}(M) \; ; \; \text{there exists} \; \gamma_1(H_f) \in L^2(\mathfrak{F}) \} .\]

Then we have similarly to Folgesatz 17.27 in [1] and Theorem 6 in [6] the following

**Lemma 6.** \( \hat{H}(M) \) is dense in \( \hat{H}(M) \).
Proof. Let \( f_0 \) be in \( \tilde{H}(M) \) and \((f_0, g)_i = 0 \) for any \( g \in \tilde{H}(M) \). Then we have

\[
D(f_0, g) + \int H_{f_0}(x) H_g(x) dx = 0.
\]

Since \( f_0 \) is in \( L^2(\mathbb{R}) \), by Lemma 4 there exists \( f_0' \in \tilde{H}(M) \) such that

\[
\gamma_i \left( H_{f_0} - \int G(\cdot, y) H_{f_0}(y) dy \right) = f_0.
\]

On the other hand

\[
\gamma_i \left( \int G(\cdot, y) H_{f_0}(y) dy \right) = \int K(x, \cdot) H_{f_0}(x) dx
\]

and

\[
\left\| \int K(x, \cdot) H_{f_0}(x) dx \right\|_{L^2(\mathbb{R})} \leq \| f_0' \|_{L^2(\mathbb{R})} < \infty.
\]

Hence \( \gamma_i \left( H_{f_0} \right) \in L^2(\mathbb{R}) \) and \( f_0' \) is in \( \tilde{H}(M) \). By (19), we have

\[
D(f_0, f_0') + \int H_{f_0}(x) H_{f_0}(x) dx = 0
\]

and by (20),

\[
D(f_0, f_0') = -\int f_0'(\xi) d\mu(\xi) - \int H_{f_0}(x) H_{f_0}(x) dx
\]

therefore we know that \( f_0 = 0 \). This completes the proof.

4. Nikodym domain

In this section we shall treat the problem whether we are able to find \( f \in \tilde{H}(M) \) such that \( \gamma_i (H_f) = \varphi \) for any \( \varphi \in L^2(\mathbb{R}) \) with \( \int \varphi(\xi) d\mu(\xi) = 0 \).

DEFINITION 3. (Deny-Lions [2]) We shall say that \( \Omega \) is a Nikodym domain if every distribution \( T \) with \( \frac{\partial}{\partial x_i} T \in L^2(\Omega) \) \( (1 \leq i \leq n) \) is in \( L^2(\Omega) \).

We set \( \mathcal{S}_{L^2}(\Omega) = \left\{ u; \ u \in L^2(\Omega) \text{ and } \frac{\partial}{\partial x_i} u \in L^2(\Omega) \ (1 \leq i \leq n) \right\} \).

A necessary and sufficient condition for \( \Omega \) to be a Nikodym domain is given by the following inequality of Poincaré: there exists a constant \( P(\Omega) \) such that
\[ \int (u(x))^2 dx - \frac{1}{|\Omega|} \left| \int u(x) dx \right|^2 \leq P(\Omega) D(u) \]

for any \( u \in \mathcal{E}_1^L(\Omega) \) (c.f. [2]).

Deny-Lions [2] gives another characterization of a Nikodym domain by setting
\[
\mathcal{N} = \left\{ u \in \mathcal{E}_1^L(\Omega); \, \Delta u \in L^1(\Omega) \text{ and } (-\Delta u, v)_{L^2(\Omega)} = D(u, v) \right\}
\]
for any \( v \in \mathcal{E}_1^L(\Omega) \).

**Lemma 7.** (Deny-Lions) For any \( F \in L^1(\Omega) \) with \( \int F(x) dx = 0 \) we can find \( u \) in \( \mathcal{N} \) (unique up to an additive constant) such that \( -\Delta u = F \) if and only if \( \Omega \) is a Nikodym domain.

The following lemma gives an answer to our above problem and it gives a characterization of a Nikodym domain.

**Lemma 8.** For any \( \phi \in L^1(\tilde{\mu}) \) with \( \int \phi(\xi)d\tilde{\mu}(\xi) = 0 \) we can find \( f \) in \( \tilde{H}(M) \) (unique up to an additive constant) such that \( \gamma(I_1) = \phi \) if and only if \( \Omega \) is a Nikodym domain.

**Proof.** Assume that \( \Omega \) is a Nikodym domain. Let \( \phi \) be in \( L^1(\tilde{\mu}) \) with \( \int \phi(\xi)d\tilde{\mu}(\xi) = 0 \). Then by Lemma 4 there exists a unique function \( f_\phi \in \tilde{H}(M) \) such that
\[
\gamma(I_1) \left( H_{f_\phi} - \int G(\cdot, y)H_{f_\phi}(y)dy \right) = \phi .
\]

Hence
\[
D(f_\phi, g) = -\int \phi(\xi)g(\xi)d\tilde{\mu}(\xi) + \int H_\phi(x)(-H_{f_\phi}(x))dx
\]
for any \( g \in \tilde{H}(M) \). We put \( g = 1 \) in (21), then \( \int H_{f_\phi}(x)dx = 0 \) from the condition \( \int \phi(\xi)d\tilde{\mu}(\xi) = 0 \).

Since \( f_\phi \) is in \( \tilde{H}(M) \), \( H_{f_\phi} \in L^1(\Omega) \) and \( D(H_{f_\phi}) = D(f_\phi) < \infty \). Therefore by Lemma 7, we can find \( u \) in \( \mathcal{N} \) (unique up to an additive constant) such that \( -\Delta u = H_{f_\phi} \). Hence we know that \( \Delta u = 0, \, u \in L^1(\Omega) \) and \( D(u) < \infty \) and so by the uniqueness of the Royden decomposition of \( u \), we have
\[
\begin{align*}
   u(x) &= h(x) - \int G(x,y)\Delta u(y)dy \\
   &= h(x) + \int G(x,y)H_f(y)dy
\end{align*}
\]

for some harmonic function \( h \in L^1(\Omega) \) with \( D(h) < \infty \). From (17), \( h \) has a fine boundary function \( h' \) in \( L^1(\partial) \) and so \( h = H_{h'} \) with \( h' \in \tilde{\mathcal{H}}(M) \).

Since \( u \) is in \( N \) and \( \{H_g : g \in \tilde{\mathcal{H}}(M)\} \subset \mathcal{E}_L(\Omega) \), we have

\[
\int H_g(x)(-\Delta u(x))dx = D(u,H_g)
\]

for any \( g \in \tilde{\mathcal{H}}(M) \). Hence we have

\[
D(h', g) - \int H_g(x)H_f(x)dx
\]

\[
= D(h, H_g) - \int H_g(x)(-\Delta u(x))dx
\]

\[
= D(h, H_g) - D(u, H_g)
\]

\[
= D(h - u, H_g)
\]

\[
= D\left(\int G(\cdot , y)\Delta u(y)dy, H_g\right) = 0
\]

for any \( g \in \tilde{\mathcal{H}}(M) \) and so \( \gamma_1(u) = 0 \).

Now we put \( f = f_0 + h' \), then \( f \) is determined (uniquely up to an additive constant) in \( \tilde{\mathcal{H}}(M) \) and we have

\[
\gamma_1(H_f) = \gamma_1(H_{f_0} + h)
\]

\[
= \gamma_1\left(H_{f_0} - \int G(\cdot, y)H_f(y)dy + u\right)
\]

\[
= \varphi.
\]

Conversely assume that for any \( \varphi \in L^1(\partial) \) with \( \int \varphi(\xi)d\mu(\xi) = 0 \) we can find \( f \) in \( \tilde{\mathcal{H}}(M) \) such that \( \gamma_1(H_f) = \varphi \). We shall show that for any \( v \in L^1(\Omega) \) with \( \int v(x)dx = 0 \), we can find \( u \) in \( N \) (unique up to an additive constant) such that \( -\Delta u = v \). Then by Lemma 7 we conclude that \( \Omega \) is a Nikodym domain. Let \( v \) be in \( L^1(\Omega) \) with \( \int v(x)dx = 0 \). Since

\[
\int |v(x)|\cdot|H_g(x)| dx < \infty
\]

for any \( g \in \tilde{\mathcal{H}}(M) \), we know
\[ \gamma_1 \left( -\int G(\cdot, y)v(y)dy \right) = -\int K(x, \cdot)v(x)dx . \]

Put \( \varphi_v = \gamma_1 \left( -\int G(\cdot, y)v(y)dy \right) \), and we know
\[
\int \varphi_v(\xi)d\mu(\xi) = \int \left( -\int K(x, \xi)v(x)dx \right)d\mu(\xi) = -\int v(x)dx = 0 .
\]

Hence we can find \( f \) in \( \tilde{H}(M) \) (unique up to an additive constant) such that \( \gamma_1(H_f) = \varphi_v \). We put
\[ u(x) = H_f(x) + \int G(x, y)v(y)dy \]
thus \( u \) is determined (uniquely up to an additive constant) in \( \mathcal{E}^1_{L^1}(\Omega) \), \(-\Delta u = v\) and \( \Delta u \in L^1(\Omega) \).

Now we shall show that \( u \) is in \( N \), that is \( D(u, w) = (-\Delta u, w)_{L^2(\Omega)} \) for any \( w \) in \( \mathcal{E}^1_{L^1}(\Omega) \).

We have the following decomposition of \( \mathcal{E}^1_{L^1}(\Omega) \):
\[ \mathcal{E}^1_{L^1}(\Omega) = \{H_v; g \in \tilde{H}(M)\} \oplus L^2D_0(\Omega) , \]
where \( L^2D_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( D(\cdot) + ||\cdot||_{L^2(\Omega)} \). In case \( w = H_v \) for some \( g \in \tilde{H}(M) \), we have
\[
D(u, w) = D(u, H_v) \\
= D(H_f, H_v) - D \left( \int G(\cdot, y)v(y)dy, \int G(\cdot, y)H_v(y)dy \right) \\
= D(f, g) - \int v(x) \left( \int G(x, y)H_v(y)dy \right)dx .
\]

Since \( \gamma_1(u) = \gamma_1(H_f) + \int K(x, \cdot)v(x)dx = \varphi_v - \varphi_v = 0 \), we know
\[ D(f, g) = \int v(x)H_v(x)dx \]
for any \( g \in \mathcal{H}(M) \). Hence we have

\[
D(u, H_\partial) = \int v(x) \left( H_\partial(x) - \int G(x, y)H_\partial(y)dy \right) dx \\
= -\int \Delta u(x)H_\partial(x)dx.
\]

In case \( w \) is in \( C_0^\infty(\Omega) \) we know that

\[
w(x) = \int G(x, y)(-\Delta w(y))dy.
\]

Hence

\[
D(u, w) = D\left( \int G(\cdot, y)v(y)dy, \int G(\cdot, y)(-\Delta w(y))dy \right) \\
= \int v(x) \left( \int G(x, y)(-\Delta w(y))dy \right) dx \\
= -\int \Delta u(x)w(x)dx.
\]

For any \( w \) in \( L^2D_0(\Omega) \), we can find a sequence \( \{w_n\} \) in \( C_0^\infty(\Omega) \) such that \( w_n \rightarrow w \) in \( L^2D_0(\Omega) \). Since \( D(u, w_n) = -\int \Delta u(x)w_n(x)dx \), letting \( n \rightarrow \infty \), we have \( D(u, w) = -\int \Delta u(x)w(x)dx \). Therefore we know

\[
D(u, w) = (-\Delta u, w)_{L^2(\Omega)},
\]

for any \( w \in \mathcal{D}'_{L^2}(\Omega) \) and so \( u \) is in \( N \). This completes the proof.

5. Boundary value problems

In this section we shall solve the boundary value problems described in section 1 as an application of Lemma 8. We put

\[
\mathcal{S}_1 = \{ u \in C^4(\Omega); \ u \text{ and } \Delta u \text{ are in } \mathcal{D}'_{L^2}(\Omega) \},
\]

\[
\mathcal{S}_2 = \{ u \in C^4(\Omega); \ u \text{ is in } \mathcal{D}'_{L^2}(\Omega) \text{ and } \Delta u \text{ is in } L^2(\Omega) \}
\]

and

\[
\mathcal{S}_3 = \{ u \in C^4(\Omega); \ \Delta u \text{ is in } \mathcal{D}'_{L^2}(\Omega) \}.
\]

Then we shall show

THEOREM. Assume that \( \Omega \) is a Nikodym domain, then

(a) for any \( \varphi \) and \( \psi \) in \( L^2(\Omega) \) with \( \int \psi(\xi)d\mu(\xi) = 0 \), there exists \( u \) in
$\mathcal{P}_1$ unique up to an additive constant such that $\Delta u = 0$, $\gamma_1(u) = \varphi$ and $\gamma_1(\Delta u) = \psi$;

(b) for any $f$ in $L^2(\mu)$ and $\varphi$ in $L^2(\mu)$ with

\begin{equation}
\int \varphi(\xi)d\mu(\xi) = -\int H_f(x)dx,
\end{equation}

there exists $u$ in $\mathcal{P}_2$ unique up to an additive constant such that $\Delta u = 0$, $\gamma_1(\Delta u) = f$ and $\gamma_1(u) = \varphi$;

(c) for any $f$ in $L^2(\mu)$ and $\varphi$ in $L^2(\mu)$ with $\int \varphi(\xi)d\mu(\xi) = 0$, there exists $u$ in $\mathcal{P}_2$ such that $\Delta u = 0$, $\gamma_1(\Delta u) = \varphi$.

Proof. (a) For any $\varphi$ and $\psi$ in $L^2(\mu)$ with $\int \varphi(\xi)d\mu(\xi) = 0$, by Lemma 8 there exists $f$ in $\tilde{H}(M)$ such that $\gamma_1(H_f) = \psi$ and

\begin{equation}
\int \left( \varphi(\xi) + \int K(x, \xi)H_f(x)d\mu(\xi) \right)d\mu(\xi) = 0.
\end{equation}

Since $\varphi + \int K(x, \cdot)H_f(x)d\mu$ is in $L^1(\mu)$ and (23), there exists $f_0$ in $\tilde{H}(M)$ such that $\gamma_1(H_{f_0}) = \varphi + \int K(x, \cdot)H_f(x)d\mu$.

We put

$$u(x) = H_{f_0}(x) - \int G(x, y)H_f(y)dy.$$ 

Then we know that $u$ is in $\mathcal{P}_1$, $\Delta u = 0$, $\gamma_1(u) = \varphi$ and $\gamma_1(\Delta u) = \psi$.

Next we shall show the uniqueness of the solution. Let $w$ be in $\mathcal{P}_1$ such that $\Delta w = 0$, $\gamma_1(w) = 0$ and $\gamma_1(\Delta w) = 0$. By the uniqueness of the Royden decomposition of $w$, there exists $f_w$ and $g_w$ in $\tilde{H}(M)$ such that

$$w = H_{f_w} - \int G(\cdot, y)\Delta w(y)dy$$

and $\Delta w = H_{g_w}$. Since $\gamma_1(w) = 0$, we have

\begin{equation}
D(H_{f_w}, H_\varphi) + \int \Delta w(x)H_\varphi(x)dx = 0
\end{equation}

for any $g$ in $\tilde{H}(M)$. Hence
(25) \[ D(w, w) = D(H_{f_w}, H_{f_w}) + \iint G(x, y) \Delta w(x) \Delta w(x) dx dy \]
\[ = -\int \Delta w(x) H_{f_w}(x) dx + \int \Delta w(x) \left( \int G(x, y) \Delta w(y) dy \right) dx \]
\[ = -\int \Delta w(x) w(x) dx . \]

Since \( \gamma_1(\Delta w) = 0 \), we have

(26) \[ D(\Delta w, H_g) = 0 \]
for any \( g \) in \( \tilde{H}(M) \). We put \( g = g_w \) in (24) and \( g = f_w \) in (26), then we know that \( \Delta w = 0 \) and so \( w = \text{constant} \) by (25).

(b) First we shall remark that the condition (22) is necessary for the existence of the solution. Let \( u \) be a solution, then

\[ u(x) = H_{f_u}(x) - \int G(x, y) \Delta u(y) dy \]

for some \( f_u \in \tilde{H}(M) \). Since \( \gamma_0(\Delta u) = f \) and \( \gamma_1(u) = \varphi \), we know \( \Delta u = H_f \) and

(27) \[ D(H_{f_u}, H_g) = -\int \varphi(\xi) g(\xi) d\mu(\xi) + \int H_g(x)(-\Delta u(x)) dx \]
for any \( g \in \tilde{H}(M) \). Put \( g = 1 \) in (27) and we have (22).

For any \( f \) in \( L^2(\tilde{\mu}) \) and \( \varphi \) in \( L^2(\tilde{\mu}) \) we know that \( \int K(x, \cdot) H_f(x) dx \) is in \( L^2(\tilde{\mu}) \) and by (22)

\[ \int \left( \varphi(\xi) + \int K(x, \xi) H_f(x) dx \right) d\mu(\xi) = 0. \]

Hence there exists \( f_0 \) in \( \tilde{H}(M) \) such that

\[ \gamma_1(H_{f_0}) = \varphi + \int K(x, \cdot) H_f(x) dx . \]

We put

\[ u(x) = H_{f_0}(x) - \int G(x, y) H_f(y) dy . \]

Then \( u \) is in \( \mathcal{F}_z, D u = 0, \gamma_0(\Delta u) = f \) and \( \gamma_1(u) = \varphi \).

The uniqueness of the solution is shown in a similar manner to (a).

Let \( w \) be in \( \mathcal{F}_z \) such that \( D w = 0, \gamma_0(\Delta w) = 0 \) and \( \gamma_1(w) = 0 \), then we have
\[ D(w, w) + \int \Delta w(x) w(x) dx = 0. \]

Since \( \Delta w \) is harmonic and \( \gamma_{\phi}(\Delta w) = 0 \), we know \( \Delta w = 0 \) and so \( w = \) constant.

(c) Put
\[ u(x) = H_{\phi}(x) - \int G(x, y) H_{\phi}(y) dy, \]
where \( \phi \) is in \( \mathcal{H}(M) \) such that \( \gamma_{\phi}(H_{\phi}) = \varphi \), and \( u \) is the desired solution. This completes the proof.

Remark 1. In the case of (c) the uniqueness of the solution is interpreted as follows. If \( u_0 \) is a solution of (c), then every solution is given by \( u_0 + a \int G(\cdot, y) dy \), where \( a \) is some constant.

In fact if \( w \) is in \( \mathcal{S} \), \( \Delta w = 0 \), \( \gamma_{\phi}(\Delta w) = 0 \) and \( \gamma_{\phi}(\Delta w) = 0 \), then \( h(x) = w(x) + \int G(x, y) \Delta w(y) dy \) is harmonic and \( \gamma_{\phi}(h) = 0 \). Hence we have
\[ w(x) = -\int G(x, y) \Delta w(y) dy. \]
Since \( \gamma_{\phi}(\Delta w) = 0 \), we know \( w(x) = a \int G(x, y) dy \) for some constant \( a \).

Remark 2. Lemma 8 asserts that if one of the above boundary value problems has always a solution, then \( \Omega \) is necessarily a Nikodym domain. Hence the above problems are solved if and only if \( \Omega \) is a Nikodym domain.

REFERENCES


Department of Mathematics
Saitama University