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# TWO FURTHER RAMANUJAN PAIRS

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**Abstract**

In a recent article, George E. Andrews considers a generalization of the Rogers–Ramanujan identities involving a pair of infinite sequences of positive integers, which he calls a 'Ramanujan pair'. He lists the known Ramanujan pairs and conjectures that there are no more. The object of this note is to establish the existence of two further Ramanujan pairs.

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## 1

George E. Andrews (1979), p. 88, makes the following definition: A 'Ramanujan pair' is a pair of infinite sequences of positive integers $\{a_n\}, \{b_n\}$ with the property that

$$\prod_{n \geq 1} (1-q^{a_n})^{-1} = 1 + \sum_{n \geq 1} \frac{q^{b_1+\dots+b_n}}{(q)_n} \tag{1.1}$$

(where, as usual, $(q)_n$ denotes $(1-q)(1-q^2)\dots(1-q^n)$). He conjectures that the only Ramanujan pairs are those he lists, namely

$$\begin{aligned} \{a_n\} &= \{m, m+1, m+2, m+3, \dots\}, \\ \{b_n\} &= \{m, m, m, m, \dots\} \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \{a_n\} &= \{m, m+1, m+2, \dots, 2m-1, 2m+1, 2m+3, \dots\}, \\ \{b_n\} &= \{m, m+1, m+2, m+3, \dots\} \end{aligned} \tag{1.3}$$





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which arise from results of Euler, as well as

$$\text{(1.4)} \qquad \begin{aligned} \{a_n\} &= \{1, 4, 6, 9, \ldots\} \\ &= \{m > 0 : m \equiv \pm 1 \bmod 5\}, \\ \{b_n\} &= \{1, 3, 5, 7, \ldots\} \end{aligned}$$

and

$$\text{(1.5)} \qquad \begin{aligned} \{a_n\} &= \{2, 3, 7, 8, \ldots\} \\ &= \{m > 0 : m \equiv \pm 2 \bmod 5\}, \\ \{b_n\} &= \{2, 4, 6, 8, \ldots\}, \end{aligned}$$

which correspond to the Rogers–Ramanujan identities.

Ramanujan had conjectured that $a_n = b_n = p_n$, the $n$'th prime, provided a Ramanujan pair, but this is false (loc. cit.).

However, Andrews has overlooked two further Ramanujan pairs, and they are

$$\text{(1.6)} \qquad \begin{aligned} \{a_n\} &= \{1, 4, 6, 7, 9, 10, 12, 15, \ldots\} \\ &= \{m > 0\colon m \equiv \pm 1, \pm 4, \pm 6, \pm 7 \bmod 16\}, \\ \{b_n\} &= \{1, 3, 3, 5, 5, 7, 7, \ldots.\} \end{aligned}$$

and

$$\text{(1.7)} \qquad \begin{aligned} \{a_n\} &= \{2, 3, 4, 5, 11, 12, 13, 14, \ldots\} \\ &= \{m > 0\colon m \equiv \pm 2, \pm 3, \pm 4, \pm 5 \bmod 16\}, \\ \{b_n\} &= \{2, 2, 4, 4, 6, 6, \ldots\}. \end{aligned}$$

These correspond respectively to the identities

$$\text{(1.8)} \qquad \prod_{\substack{m>0 \\ m \equiv \pm 1, \pm 4, \pm 6, \pm 7 \bmod 16}} (1-q^m)^{-1} = \\ = 1 + \frac{q^1}{(q)_1} + \frac{q^{1+3}}{(q)_2} + \frac{q^{1+3+3}}{(q)_3} + \frac{q^{1+3+3+5}}{(q)_4} + \ldots$$

and

$$\text{(1.9)} \qquad \prod_{\substack{m>0 \\ m \equiv \pm 2, \pm 3, \pm 4, \pm 5 \bmod 16}} (1-q^m)^{-1} = \\ = 1 + \frac{q^2}{(q)_1} + \frac{q^{2+2}}{(q)_2} + \frac{q^{2+2+4}}{(q)_3} + \frac{q^{2+2+4+4}}{(q)_4} + \ldots.$$

Identities (1.8) and (1.9) have not, as far as I am aware, previously been stated explicitly, but they are implicit in Hirschhorn (1979), Theorems 3 and 4. (Theorems 1 and 2 of that paper do not, however, give rise to Ramanujan pairs.)

Section 2 is devoted to a proof of identity (1.8). The proof of (1.9) is similar, so is omitted.

Whether the list of Ramanujan pairs is yet complete remains an open question.

## 2

We require only the classical identities of Euler and Jacobi, namely

$$(2.1)\qquad (-a;q)_\infty = \sum_0^\infty \frac{q^{\frac{1}{2}(r^2-r)}a^r}{(q)_r}$$

and

$$(2.2)\qquad (-a^{-1}q;q^2)_\infty(-aq;q^2)_\infty(q^2;q^2)_\infty = \sum_{-\infty}^{\infty} a^r q^{r^2}.$$

Both (2.1) and (2.2) follow easily from

$$(2.3)\qquad (-a^{-1}q;q^2)_i(-aq;q^2)_j = \sum_{-i}^{j} a^r q^{r^2}\begin{bmatrix} i+j \\ i+r \end{bmatrix}_{(q^2)},$$

proved in Hirschhorn (1976). (2.3) is ascribed there to P. A. MacMahon, but I have since been informed by Richard Askey that it is much older.

Consider the sum

$$1+\frac{q^1}{(q)_1}+\frac{q^{1+3}}{(q)_2}+\frac{q^{1+3+3}}{(q)_3}+\frac{q^{1+3+3+5}}{(q)_4}+\dots$$
$$=\sum_{r\geqslant 0}\left(\frac{q^{2r^2+2r}}{(q)_{2r}}+\frac{q^{2r^2+4r+1}}{(q)_{2r+1}}\right).$$

This collapses to

$$\sum_{r\geqslant 0}\frac{q^{2r^2+2r}(1-q^{2r+1})+q^{2r^2+4r+1}}{(q)_{2r+1}} = \sum_{r\geqslant 0}\frac{q^{2r^2+2r}}{(q)_{2r+1}},$$

which can be written as, essentially, the difference between two series summable via (2.1):

$$\sum_{r\geqslant 0}\frac{q^{2r^2+2r}}{(q)_{2r+1}} = \sum_{\substack{r\geqslant 0\\ r \text{ odd}}}\frac{q^{\frac{1}{2}(r^2-1)}}{(q)_r}$$
$$=\tfrac{1}{2}q^{-\frac{1}{2}}\left\{\sum_{r\geqslant 0}\frac{q^{\frac{1}{2}r^2}}{(q)_r}-\sum_{r\geqslant 0}(-1)^r\frac{q^{\frac{1}{2}r^2}}{(q)_r}\right\}$$
$$=\tfrac{1}{2}q^{-\frac{1}{2}}\{(-q^{\frac{1}{2}};q)_\infty-(q^{\frac{1}{2}};q)_\infty\}.$$

If we now extract and re-insert the product $(q^2;q^2)_\infty$, we obtain two products which can be expanded via (2.2):

$$\begin{aligned}
&\tfrac{1}{2}q^{-\frac{1}{2}}\{(-q^{\frac{1}{2}};q)_\infty-(q^{\frac{1}{2}};q)_\infty\}\\
&\quad=\tfrac{1}{2}q^{-\frac{1}{2}}(q^2;q^2)_\infty^{-1}\{(-q^{\frac{1}{2}};q^2)_\infty(-q^{3/2};q^2)_\infty(q^2;q^2)_\infty\\
&\qquad\qquad-(q^{\frac{1}{2}};q^2)_\infty(q^{3/2};q^2)_\infty(q^2;q^2)_\infty\}\\
&\quad=\tfrac{1}{2}q^{-\frac{1}{2}}(q^2;q^2)_\infty^{-1}\left\{\sum_{-\infty}^{\infty}q^{r^2-\frac{1}{2}r}-\sum_{-\infty}^{\infty}(-1)^r q^{r^2-\frac{1}{2}r}\right\}.
\end{aligned}$$

The two sums coalesce to give, again via (2.2),

$$\begin{aligned}
&(q^2;q^2)_\infty^{-1}\cdot\sum_{r\text{ odd}}q^{r^2-\frac{1}{2}r-\frac{1}{2}}\\
&\quad=(q^2;q^2)_\infty^{-1}\sum_{-\infty}^{\infty}q^{4r^2+3r}\\
&\quad=(q^2;q^2)_\infty^{-1}(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty.
\end{aligned}$$

Using the device $(-q^a;q^b)_\infty=(q^a;q^b)_\infty^{-1}(q^{2a};q^{2b})_\infty$, we find easily that the product

$$(q^2;q^2)_\infty^{-1}(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty=\prod_{\substack{m>0\\ m\equiv\pm1,\pm4,\pm6,\pm7\bmod 16}}(1-q^m)^{-1}$$

which completes the proof of (1.8).

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