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An Estimate For a Restricted X-Ray Transform

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Abstract. This paper contains a geometric proof of an estimate for a restricted x-ray transform. The result complements one of A. Greenleaf and A. Seeger.

Fix $d \geq 2$ and, for $s \in \mathbb{R}$, let $\gamma(s) = (1, s, s^2, \ldots, s^{d-1}) \in \mathbb{R}^d$. Let $\Pi(s)$ be the space of vectors in $\mathbb{R}^d$ which are orthogonal to $\gamma(s)$. We consider a restricted x-ray transform R on $\mathbb{R}^d$ defined by

$$Rf(s, p) = \int_{-\infty}^{\infty} f\big(p + t\gamma(s)\big)\, dt, \quad s \in \mathbb{R},\ p \in \Pi(s).$$

Our interest is in estimates

$$\Big(\int_{\{|s|\leq 1\}} \int_{\{p\in\Pi(s):|p|\leq 1\}} |Rf(s, p)|^q \, dp\, ds\Big)^{\frac{1}{q}} \leq C(p, q)\|f\|_p \tag{1}$$

related to the mapping properties of R as an operator on $L^p(\mathbb{R}^d)$. It is observed in [GS] that if (1) holds then $(\frac{1}{p}, \frac{1}{q})$ must belong to the triangle $\mathcal{T} = \text{hull}\,(A, B, C)$, where $A = (0, 0)$, $B = (1, 1)$, $C = (\frac{d^2-d+2}{d^2+d}, \frac{d^2-d}{d^2+d})$. In case $d = 4$, the converse is nearly true. This follows by interpolation from trivial estimates and a result in [GSW] which shows that R maps $L^{10/7,1}(\mathbb{R}^4)$ into $L^{5/3,\infty}(\mathbb{R}^4)$. In general, let $D = (\frac{d^2-d+2}{2d^2-2d}, \frac{1}{2})$, so that D is on the lower edge AC of $\mathcal{T}$. Proposition 5.2 in [GS] shows that R is bounded if $(\frac{1}{p}, \frac{1}{q})$ belongs to hull (A, B, D). Our purpose here is to prove another partial result of this type: let $E = (\frac{d-1}{d}, \frac{d-2}{d-1})$, so that E lies on the upper edge CB of $\mathcal{T}$.

Theorem *R is of restricted weak type $(\frac{d}{d-1}, \frac{d-1}{d-2})$.*

It follows by interpolation that R is bounded if $(\frac{1}{p}, \frac{1}{q}) \neq E$ belongs to hull (A, B, E). The method of proof is geometrical. It is an adaptation of the proof in [O1]. One motivation for studying the mapping properties of R is that R can be regarded as a model for the operator $Tf(x) = \int_0^1 f\big(x - \Gamma(s)\big)\, ds$, where $\Gamma(s) = (s, s^2/2, \ldots s^d/d)$—see [O2], [M], [O3], [O4], [C2].

Proof of Theorem In addition to its usual usages as norm or absolute value, the notation $|\cdot|$ will stand for the Lebesgue measure of a set in $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$ or $\{(s, p) : s \in \mathbb{R}, p \in$

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$\Pi(s)\} \sim \mathbb{R}^d$, the choice being clear from the context. Fix $\lambda > 0$ and a measurable set $E \subseteq \mathbb{R}^d$. Let $F = \{|s|, |p| \leq 1, R\chi_E(s,p) > \lambda\}$.

We need to show that

$$\lambda |F|^{\frac{d-2}{d-1}} \leq C|E|^{\frac{d-1}{d}} \tag{2}$$

for some C independent of E and λ. The trivial estimate

$$\lambda |F| \leq 2|E|$$

gives (2) if $\lambda |F|^{-1/(d-1)}$ is small, so we consider the other case. Let k be a positive number so small that $k/2 - (2k)^d c_1 > 0$—here c_1 is a positive constant that will appear later in the proof. Assuming that $k\lambda^{1/(d-1)}|F|^{-1/(d-2)^2} \geq 1$, *i.e.*, that $\lambda|F|^{-1/(d-1)}$ is not small, choose an integer N such that

$$k\lambda^{\frac{1}{d-1}}|F|^{\frac{-1}{(d-1)^2}} \leq N < k\lambda^{\frac{1}{d-1}}|F|^{\frac{-1}{(d-1)^2}} + 1 \leq 2k\lambda^{\frac{1}{d-1}}|F|^{\frac{-1}{(d-1)^2}}. \tag{3}$$

For $|s| \leq 1$, let $\tilde{E}(s) = \{p \in \Pi(s) : |p| \leq 1, R\chi_E(s,p) > \lambda\}$. Then $\int_{-1}^{1} |\tilde{E}(s)|\, ds = |F|$. Assume without loss of generality that $\int_0^1 |\tilde{E}(s)|\, ds \geq |F|/2$. Choose $s_1, \dots, s_N$ with

$$0 < s_1 < s_2 < \cdots < s_N, \quad s_{j+1} - s_j = \frac{1}{N},$$
$$\frac{1}{N}\sum_{j=1}^{N} |\tilde{E}(s_j)| \geq \int_0^1 |\tilde{E}(s)|\, ds \geq \frac{|F|}{2}.$$

By shrinking some of the $\tilde{E}(s_j)$ if necessary, we can assume that

$$\frac{|F|}{2} \leq \frac{1}{N}\sum_{j=1}^{N} |\tilde{E}(s_j)| \leq |F|.$$

Let P_j denote the orthogonal projection of $\mathbb{R}^d$ onto the plane $\Pi(s_j)$ and set

$$E(s_j) = \{x \in E : P_j x \in \tilde{E}(s_j)\}.$$

Then

$$|E(s_j)| \geq \int_{\Pi(s_j)} R\chi_E(s_j, p)\chi_{\tilde{E}(s_j)}(p)\, dp \geq \lambda |\tilde{E}(s_j)|, \tag{4}$$

where the first inequality follows because $\gamma(s_j)$ has norm at least one and the second inequality follows from the definition of $\tilde{E}(s)$. To simplify notation we will write E_j for $E(s_j)$ and $\tilde{E}_j$ for $\tilde{E}(s_j)$. We will establish (2) by using a counting lemma to estimate $|E|$ from below.

Lemma 1 *The following inequality holds:*

$$\Big|\bigcup_{j=1}^{N} E_j\Big| \geq \frac{1}{d}\Big(\sum_{j=1}^{N} |E_j| - \sum_{1 \leq j_1 < \cdots < j_d \leq N} |E_{j_1} \cap \cdots \cap E_{j_d}|\Big).$$

Proof of Lemma 1 Let $G_k = \{x \in E_k : \sum_{1\le j<k} \chi_{E_j}(x) < d-1\}$. Then

$$\chi_{E_k}\Big(1 - \sum_{1\le j_1<\cdots<j_{d-1}<k} \chi_{E_{j_1}}\cdots\chi_{E_{j_{d-1}}}\Big) \le \chi_{G_k}, \quad \sum_1^N \chi_{G_k} \le d\chi_{\bigcup_1^N E_j},$$

and so

$$\sum_1^N \Big(\chi_{E_k} - \sum_{1\le j_1<\cdots<j_{d-1}<k} \chi_{E_{j_1}}\cdots\chi_{E_{j_{d-1}}}\chi_{E_k}\Big) \le d\chi_{\bigcup_1^N E_j}.$$

Integrating this inequality yields the desired result. ■

Lemma 2 $|E_{j_1} \cap \cdots \cap E_{j_d}| \le C \prod_{1\le k<\ell\le d} |s_{j_k} - s_{j_\ell}|^{\frac{-1}{d-1}} \prod_{p=1}^d |\tilde{E}_{j_p}|^{\frac{1}{d-1}}$.

Proof of Lemma 2 The change of variables $(u_1, \dots, u_d) \mapsto x \doteq \sum_{\ell=1}^d u_\ell \gamma(s_{j_\ell})$ has Jacobian J given by the Vandermonde determinant

$$\prod_{1\le k<\ell\le d} (s_{j_k} - s_{j_\ell}).$$

Thus

$$\begin{aligned}
|E_{j_1} \cap \cdots \cap E_{j_d}| &= |J| \int \prod_{k=1}^d \chi_{E_{j_k}}\big(\sum_{\ell=1}^d u_\ell\gamma(s_{j_\ell})\big)\, du_1\cdots u_d \\
&\le |J| \int \prod_{k=1}^d \chi_{\tilde{E}_{j_k}}\Big(P_{j_k}\Big(\sum_{\ell=1}^d u_\ell\gamma(s_{j_\ell})\Big)\Big)\, du_1\cdots u_d \\
&= |J| \int \prod_{k=1}^d \chi_{\tilde{E}_{j_k}}\Big(P_{j_k}\Big(\sum_{\substack{\ell=1\\ \ell\ne k}}^d u_\ell\gamma(s_{j_\ell})\Big)\Big)\, du_1\cdots u_d.
\end{aligned}$$

The following result of Blei (see Lemma 2.4 of [B]) applies to the last integral: suppose $f_1, \dots, f_d$ are nonnegative functions of $u_1, \dots, u_d$ with f_k independent of u_k for $k = 1, \dots, d$, so that f_k is a function on $\mathbb{R}^{d-1}$. Then

$$\int \prod_{k=1}^d f_k(u_1, \dots, u_d)\, du_1\cdots u_d \le \prod_{k=1}^d \|f_k\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

This leads to

$$|E_{j_1} \cap \cdots \cap E_{d_j})| \le |J| \prod_{k=1}^d \Big(\int \chi_{\tilde{E}_{j_k}}\Big(P_{j_k}\Big(\sum_{\substack{\ell=1\\ \ell\ne k}}^d u_\ell\gamma(s_{j_\ell})\Big)\Big)\, du_1\cdots u_{k-1}u_{k+1}\cdots u_d\Big)^{\frac{1}{d-1}}.$$

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Now

$$\int \chi_{\tilde{E}_{j_k}}\Big(P_{j_k}\Big(\sum_{\substack{\ell=1\\ \ell\neq k}}^{d} u_\ell \gamma(s_{j_\ell})\Big)\Big)\, du_1 \cdots u_{k-1}u_{k+1}\cdots u_d = \frac{|\gamma(s_{j_k})|}{|J|}|\tilde{E}_{j_k}|,$$

which one can see, for example, by considering an orthogonal transformation of $\mathbb{R}^d$ that takes $\gamma(s_{j_k})$ onto $(|\gamma(s_{j_k})|, 0, 0, \dots, 0)$. Lemma 2 follows since $|s| \le 1$. ■

Lemma 2 will be applied in conjunction with a multilinear inequality for nonnegative sequences:

$$\sum_{j_1,\dots,j_d>0} \prod_{\ell=1}^{d} a_{j_\ell} \prod_{1\le k<\ell\le d} \frac{1}{1+|j_k - j_\ell|^{\frac{1}{d-1}}} \le C\Big(\sum a_j^2\Big)^{\frac{d}{2}}. \tag{5}$$

This inequality is a discrete version of Proposition 2.2 in [C1] and follows from that result or from its proof. Using Lemma 2, (5), and the equality $|s_{j_k} - s_{j_\ell}| = N^{-1}|j_k - j_\ell|$, we get

$$\begin{aligned}
\sum_{1\le j_1<\cdots<j_d\le N} |E_{j_1}\cap\cdots\cap E_{j_d}| &\le c_1 N^{\frac{d}{2}}\Big(\sum_1^N |\tilde{E}_j|^{\frac{2}{d-1}}\Big)^{\frac{d}{2}}\\
&\le c_1 N^{\frac{d}{2}+\frac{(d-3)d}{2(d-1)}}\Big(\sum_1^N |\tilde{E}_j|\Big)^{\frac{d}{d-1}}\\
&= c_1 N^{\frac{d}{2}+\frac{(d-3)d}{2(d-1)}+\frac{d}{d-1}}\Big(\frac{1}{N}\sum_1^N |\tilde{E}_j|\Big)^{\frac{d}{d-1}}\\
&\le c_1 N^d (|F|)^{\frac{d}{d-1}}.
\end{aligned}$$

The last inequality here follows from the fact that the $\tilde{E}_j$ were chosen so that

$$\frac{1}{N}\sum_{j=1}^{N} |\tilde{E}_j| \le |F|.$$

Since also

$$\frac{|F|}{2} \le \frac{1}{N}\sum_{j=1}^{N} |\tilde{E}_j|,$$

Lemma 1 and (4) yield

$$d\Big|\bigcup_{j=1}^{N} E_j\Big| \ge \lambda N \frac{|F|}{2} - c_1 N^d |F|^{\frac{d}{d-1}}.$$

With (3) this gives

$$d\Big|\bigcup_{j=1}^{N} E_j\Big| \ge \lambda^{\frac{d}{d-1}} |F|^{\frac{d(d-2)}{(d-1)^2}}\Big(\frac{k}{2} - (2k)^d c_1\Big).$$

Since $\bigcup E_j \subseteq E$ and since k was chosen so that $k/2 - (2k)^d c_1$ is positive, (2) follows and the proof is complete. ■

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