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MULTIPLE SOLUTIONS FOR A DIRICHLET PROBLEM WITH p-LAPLACIAN AND SET-VALUED NONLINEARITY

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Abstract

The existence of a negative solution, of a positive solution, and of a sign-changing solution to a Dirichlet eigenvalue problem with p-Laplacian and multi-valued nonlinearity is investigated via sub- and supersolution methods as well as variational techniques for nonsmooth functions.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, let $1 < p < +\infty$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in each variable separately. Given a real parameter λ, consider the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u - g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. If $p = 2$ then the existence of multiple solutions to (1.1) has been widely investigated; see [1, 2, 18, 19] and the references therein. All these papers treat the case where $(x, t) \mapsto g(x, t)$ does not depend on x and is suitably regular, for example, continuously differentiable [1] or Lipschitz continuous [18, 19]. Roughly speaking, the results obtained are as follows. Let the function g exhibit a superlinear behaviour at both zero and infinity. Under a further technical condition, which may vary from one situation to another, problem (1.1) possesses at least three nontrivial solutions provided $\lambda > \lambda_2$, the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$. Combining the method of sub- and supersolutions with variational techniques chiefly based on the second deformation lemma, two very recent papers [4, 17] examine a much more general situation, that is, $1 < p < +\infty$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ of Carathéodory's type only.

We next point out that Struwe's result [19, Theorem 10.5] was extended to a wider class of problems, the so-called elliptic hemivariational inequalities, in [11].

The same nonsmooth framework of [11] is adopted here, but the technical approach exploited is based on that of [4]. More precisely, setting, for g merely bounded on bounded sets,

$$G(x, \xi) := \int_0^\xi g(x, t)\, dt, \quad (x, \xi) \in \Omega \times \mathbb{R},$$

we shall be concerned with the problem

$$\begin{cases} -\Delta_p u \in \lambda |u|^{p-2} u - \partial G(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\partial G(x, u(x))$ indicates the generalized gradient of $\xi \mapsto G(x, \xi)$ at the point $u(x)$. Obviously, (1.2) reduces to (1.1) if g satisfies Carathéodory's conditions. We say that $u \in W_0^{1,p}(\Omega)$ is a solution of (1.2) if there exists an $\eta \in L^{p/(p-1)}(\Omega)$ such that

$$\eta(x) \in \partial G(x, u(x)) \quad \text{almost everywhere in } \Omega, \tag{1.3}$$

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, dx + \int_\Omega (\eta - \lambda |u|^{p-2} u) \varphi\, dx = 0 \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \tag{1.4}$$

The main result of this paper, Theorem 4.1 below, establishes the existence of at least three nontrivial solutions $u_-, u_+, u_0 \in C_0^1(\overline{\Omega})$ to (1.2) such that $u_- < 0 < u_+$, while u_0 changes sign, in Ω provided $\lambda > \lambda_2$, the second eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$. It represents a nonsmooth version of [4, Theorem 4.1] and includes both [17, Theorem 3.9] and [11, Corollary 3.2] as special cases. Accordingly, Theorem 4.1 also extends the results of [1, 2, 18, 19] to problem (1.2). We subsequently note that it exhibits significant qualitative properties of the solutions obtained. For other multiplicity results under different assumptions, see [9, 12, 15] and the references therein.

Problems like (1.2) are sometimes called elliptic hemivariational inequalities. They arise in the mathematical formulation of several complicated mechanical and engineering questions, where the relevant energy functionals turn out to be neither convex nor smooth (the so-called superpotentials). The monographs [9, 10, 14, 16] are general works on this subject.

2. Basic assumptions and preliminary results

Let $(X, \|\cdot\|)$ be a real Banach space. Given a set $V \subseteq X$, write ∂V for the boundary of V, $\operatorname{int}(V)$ for the interior of V, and $\overline{V}$ for the closure of V. If $x, z \in X$ and $\delta > 0$ then

$$B_\delta(x) := \{w \in X : \|w - x\| < \delta\}, \quad [x, z] := \{(1-t)x + tz : t \in [0, 1]\}.$$

The symbol X^* denotes the dual space of X, while $\langle \cdot, \cdot \rangle$ indicates the duality pairing between X and X^*. A function $\Phi : X \to \mathbb{R}$ is called coercive if

$$\lim_{\|x\| \to +\infty} \Phi(x) = +\infty.$$

If to every $x \in X$ there correspond a neighbourhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \text{for all } z, w \in V_x,$$

then we say that Φ is locally Lipschitz continuous. In this case, $\Phi^0(x; z)$, $x, z \in X$, denotes the generalized directional derivative of Φ at the point x along the direction z, that is,

$$\Phi^0(x; z) := \limsup_{w \to x,\, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.$$

The generalized gradient of the function Φ in x is the set

$$\partial \Phi(x) := \{x^* \in X^* : \langle x^*, z \rangle \leq \Phi^0(x; z)\ \forall\, z \in X\}.$$

Then [6, Proposition 2.1.2] ensures that $\partial \Phi(x)$ turns out to be nonempty, convex, in addition to weak* compact, and that

$$\Phi^0(x; z) = \max\{\langle x^*, z \rangle : x^* \in \partial \Phi(x)\}, \qquad z \in X.$$

Hence, it makes sense to write

$$m_\Phi(x) := \min\{\|x^*\|_{X^*} : x^* \in \Phi(x)\}.$$

The classical Palais–Smale condition for C^1 functions here takes the following form (see [5, Definition 2]).

(PS) *Every sequence* $\{x_n\} \subseteq X$ *such that* $\{\Phi(x_n)\}$ *is bounded and* $\lim_{n \to +\infty} m_\Phi(x_n) = 0$ *possesses a convergent subsequence.*

We say that $x \in X$ is a critical point of Φ if $0 \in \partial \Phi(x)$, that is, $\Phi^0(x; z) \geq 0$ for all $z \in X$. Obviously, each local minimizer or maximizer of Φ turns out to be a critical point of Φ; see [5, Proposition 10]. Put

$$K(\Phi) := \{x \in X : 0 \in \partial \Phi(x)\}.$$

The following nonsmooth version of the Ambrosetti–Rabinowitz mountain pass theorem is essentially due to Chang [5, Theorem 3.4] and will be exploited in Section 4.

THEOREM 2.1. *Let* X *be reflexive and let* Φ *satisfy* (PS). *If there exist* $x_0, x_1 \in X$, $r > 0$ *such that* $\|x_1 - x_0\| > r$ *and* $\max\{\Phi(x_0), \Phi(x_1)\} < \inf_{x \in \partial B_r(x_0)} \Phi(x)$ *then* Φ *has a critical point* $\widehat{x} \in X$ *such that*

$$\inf_{x \in \partial B_r(x_0)} \Phi(x) \leq \Phi(\widehat{x}) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)),$$

where $\Gamma := \{\gamma \in C^0([0, 1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}$.

An operator $A : X \to X^*$ is called coercive if

$$\lim_{\|x\|\to+\infty} \frac{\langle A(x), x\rangle}{\|x\|} = +\infty.$$

We say that A is of type $(\mathrm{S})_+$ if $x_n \rightharpoonup x$ in X and $\limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle \le 0$ imply $x_n \to x$.

Throughout this paper, Ω denotes a bounded domain of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \ge 3$, with a smooth boundary $\partial\Omega$, $p \in\,]1, +\infty[$, and $p' := p/(p-1)$. The symbol $W_0^{1,p}(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On $W_0^{1,p}(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p}.$$

Denote by p^* the critical exponent for the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$ and $p^* = +\infty$ if $p \ge N$. As usual, we write

$$L^q(\Omega)_+ := \{u \in L^q(\Omega) : u(x) \ge 0 \text{ almost everywhere in } \Omega\},$$
$$C_0^1(\overline{\Omega})_+ := \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \ \forall\, x \in \Omega\}.$$

It is known (see, for example, [10, Remark 6.2.10]) that

$$\mathrm{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0,\ \frac{\partial u}{\partial n}(x) < 0 \ \forall\, x \in \Omega\right\},$$

with $n(x)$ being the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.

Let $-\Delta_p : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ be defined by

$$\langle -\Delta_p u, v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\, dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega), \tag{2.1}$$

and let λ_1 (λ_2) its first (second) eigenvalue. One usually refers to Δ_p as the p-Laplacian operator. The following properties of λ_1, λ_2, and $-\Delta_p$ can be found in [10, Section 6.2]; see also [8].

(p_1) $0 < \lambda_1 < \lambda_2$.

(p_2) *There exists an eigenfunction* φ_1 *corresponding to* λ_1 *such that* $\varphi_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ *as well as* $\|\varphi_1\|_{L^p(\Omega)} = 1$.

(p_3) *If* $S := \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ *and*

$$\Gamma_0 := \{\gamma \in C^0([-1, 1], S) : \gamma(-1) = -\varphi_1,\ \gamma(1) = \varphi_1\},$$

then

$$\lambda_2 = \inf_{\gamma\in\Gamma_0}\ \max_{u\in\gamma([-1,1])} \|u\|^p.$$

(p_4) *The operator* $-\Delta_p$ *is maximal monotone, coercive, and of type* $(\mathrm{S})_+$.

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Finally, for notational convenience, we define, for $u, v : \Omega \to \mathbb{R}$,

$$\Omega(u \le v) := \{x \in \Omega : u(x) \le v(x)\}, \quad u^+ := \max\{u, 0\}, \quad u^- := \min\{u, 0\},$$

and henceforth 'measurable' will always mean Lebesgue measurable. Suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

(g_1) *g is measurable in each variable separately.*

(g_2) *There exist $a_1 > 0$, $q \in]p, p^*[$ such that*

$$|g(x, t)| \le a_1(1 + |t|^{q-1}),$$

for almost every $x \in \Omega$ and every $t \in \mathbb{R}$.

Then the functions $G(x, \cdot) : \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : L^q(\Omega) \to \mathbb{R}$ given by

$$G(x, \xi) := \int_0^{\xi} g(x, t)\, dt \quad \text{for all } \xi \in \mathbb{R},$$
$$\mathcal{G}(u) := \int_{\Omega} G(x, u(x))\, dx \quad \text{for all } u \in L^q(\Omega), \tag{2.2}$$

respectively, are well defined and locally Lipschitz continuous. So, it makes sense to consider their generalized gradients $\partial G(x, \cdot)$ and $\partial \mathcal{G}$. For every $(x, t) \in \Omega \times \mathbb{R}$, set

$$g_1(x, t) := \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|\tau - t| < \delta} g(x, \tau), \quad g_2(x, t) := \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|\tau - t| < \delta} g(x, \tau).$$

Then [14, Proposition 1.7] ensures that

$$\partial G(x, \xi) = [g_1(x, \xi), g_2(x, \xi)], \tag{2.3}$$

while [10, Theorem 4.5.19] leads to

$$\partial \mathcal{G}(u) \subseteq \{w \in L^{q'}(\Omega) : g_1(x, u(x)) \le w(x) \le g_2(x, u(x)) \text{ almost everywhere in } \Omega\}, \tag{2.4}$$

with $q' := q/(q - 1)$. The next result is an immediate consequence of [6, Proposition 2.1.5], apart from the choice of q.

LEMMA 2.2. *Suppose $u_n \to u$ in $W_0^{1,p}(\Omega)$, $w_n \rightharpoonup w$ in $L^{p'}(\Omega)$, and $w_n \in \partial \mathcal{G}(u_n)$ for all $n \in \mathbb{N}$. Then $w \in \partial \mathcal{G}(u)$.*

We shall further make the following assumptions.

(g_3) $\lim_{t \to 0}(g(x, t)/|t|^{p-2}t) = 0$ *uniformly for almost all $x \in \Omega$.*

(g_4) $\lim_{|t| \to +\infty}(g(x, t)/|t|^{p-2}t) = +\infty$ *uniformly for almost all $x \in \Omega$.*

REMARK 2.3. For $p = 2$ and if $(x, t) \mapsto g(x, t)$ does not depend on x and is continuous, hypotheses (g_3)–(g_4) have previously been introduced in [1, 18]. The very recent paper [11] deals with possibly discontinuous nonlinearities.

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REMARK 2.4. Assumption (g_3) forces $g_1(x, 0) \le 0 \le g_2(x, 0)$ for almost all $x \in \Omega$. Hence, in view of (2.3), problem (1.2) always possesses the trivial solution.

A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution to (1.2) if $\underline{u}|_{\partial\Omega} \le 0$ and there exists an $\underline{\eta} \in L^{p'}(\Omega)$ such that

$$\underline{\eta}(x) \in \partial G(x, \underline{u}(x)) \quad \text{for almost every } x \in \Omega, \tag{2.5}$$

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \varphi \, dx + \int_\Omega (\underline{\eta} - \lambda |\underline{u}|^{p-2} \underline{u}) \varphi \, dx \le 0$$
$$\text{for all } \varphi \in W^{1,p}(\Omega) \cap L^p(\Omega)_+. \tag{2.6}$$

Likewise, we say that $\overline{u} \in W^{1,p}(\Omega)$ is a supersolution of problem (1.2) if $\overline{u}|_{\partial\Omega} \ge 0$ and there exists an $\overline{\eta} \in L^{p'}(\Omega)$ fulfilling

$$\overline{\eta}(x) \in \partial G(x, \overline{u}(x)) \quad \text{for almost every } x \in \Omega, \tag{2.7}$$

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \varphi \, dx + \int_\Omega (\overline{\eta} - \lambda |\overline{u}|^{p-2} \overline{u}) \varphi \, dx \ge 0$$
$$\text{for all } \varphi \in W_0^{1,p}(\Omega) \cap L^p(\Omega)_+. \tag{2.8}$$

As a result of (p_4) the operator $-\Delta_p$ turns out to be surjective; see, for instance, [10, Corollary 3.2.21]. Thus, we can find a function $e \in W_0^{1,p}(\Omega)$ such that $-\Delta_p \, e = 1$. Gathering [9, Theorems 1.5.6 and 1.5.7] together yields $e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

We are now in a position to establish the existence of sub- and supersolutions to problem (1.2).

THEOREM 2.5. *Let* (g_1)–(g_4) *be satisfied. Then, for every* $\lambda > \lambda_1$, *there exists a constant* $a_\lambda > 0$ *such that* $-a_\lambda e$ ($a_\lambda e$) *is a subsolution (supersolution) of (1.2). Moreover,* $\varepsilon\varphi_1$ ($-\varepsilon\varphi_1$) *is a subsolution (supersolution) to (1.2) and* $\varepsilon\varphi_1 < a_\lambda e$ *in* Ω *for any sufficiently small* $\varepsilon > 0$.

PROOF. Pick $\lambda > \lambda_1$. Hypothesis (g_4) produces a $t_\lambda > 0$ such that

$$\frac{g(x, t)}{|t|^{p-2}t} > \lambda \quad \text{provided } |t| > t_\lambda. \tag{2.9}$$

Through (g_2) we can find a $c_\lambda > 0$ fulfilling

$$|g(x, t) - \lambda |t|^{p-2} t| \le c_\lambda \quad \text{for all } |t| \le t_\lambda. \tag{2.10}$$

Both inequalities above hold almost everywhere in Ω. Moreover, combining (2.9) with (2.10), we achieve

$$-\Delta_p(-c_\lambda^{1/(p-1)} e) + \lambda c_\lambda e^{p-1} + g_2(x, -c_\lambda^{1/(p-1)} e) \le 0 \tag{2.11}$$

as well as

$$-\Delta_p(c_\lambda^{1/(p-1)} e) - \lambda c_\lambda e^{p-1} + g_1(x, c_\lambda^{1/(p-1)} e) \ge 0. \tag{2.12}$$

Therefore, on account of (2.3), the first conclusion is true once we put $a_\lambda := c_\lambda^{1/(p-1)}$.

Next, since $\lambda > \lambda_1$, assumption (g_3) yields a $\delta_\lambda > 0$ such that

$$\frac{|g(x,t)|}{|t|^{p-1}} < \lambda - \lambda_1 \quad \text{provided } 0 < |t| \le \delta_\lambda. \tag{2.13}$$

Fix a positive number $\varepsilon \le \delta_\lambda / \|\varphi_1\|_{L^\infty(\Omega)}$. From ($p_2$) and (2.13), which holds almost everywhere in Ω, it easily follows that

$$-\Delta_p(\varepsilon\varphi_1) - \lambda\varepsilon^{p-1}\varphi_1^{p-1} + g_2(x, \varepsilon\varphi_1) < 0.$$

Likewise,

$$-\Delta_p(-\varepsilon\varphi_1) + \lambda\varepsilon^{p-1}\varphi_1^{p-1} + g_1(x, -\varepsilon\varphi_1) > 0.$$

Hence, the function $\varepsilon\varphi_1$ ($-\varepsilon\varphi_1$) turns out to be a subsolution (supersolution) of (1.2). Finally, as $e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, for any sufficiently small $\varepsilon > 0$,

$$e - \frac{\varepsilon}{a_\lambda}\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+),$$

that is, $\varepsilon\varphi_1 < a_\lambda e$ in Ω. This completes the proof. □

3. Constant-sign solutions

Two nonzero, constant-sign, extremal solutions to problem (1.2) can be achieved when $\lambda > \lambda_1$, the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$. The next result represents a preliminary step in this direction.

THEOREM 3.1. *If* (g_1)–(g_4) *hold true, while* $\lambda > \lambda_1$, *then for every* $\varepsilon > 0$ *small enough (1.2) has a minimal positive solution* $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \cap [\varepsilon\varphi_1, a_\lambda e]$ *and a maximal negative solution* $u_- \in -\operatorname{int}(C_0^1(\overline{\Omega})_+) \cap [-a_\lambda e, -\varepsilon\varphi_1]$, *with* $a_\lambda > 0$ *given by Theorem 2.5.*

PROOF. Since similar reasoning is used for u_- and u_+, we shall confine ourselves to the case of u_+. Let

$$\underline{u} := \varepsilon\varphi_1, \qquad \overline{u} := a_\lambda e.$$

Theorem 2.5 ensures that $\underline{u}$ ($\overline{u}$) turns out to be a subsolution (supersolution) of (1.2) lying in $\operatorname{int}(C_0^1(\overline{\Omega})_+)$ and that $\underline{u} < \overline{u}$ provided $\varepsilon > 0$ is sufficiently small. Put

$$U := \{u \in W_0^{1,p}(\Omega) : \underline{u}(x) \le u(x) \le \overline{u}(x) \text{ for almost every } x \in \Omega\}.$$

Thanks to (g_1), (g_2) the functional $E_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$E_\lambda(u) := \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\|u\|^p_{L^p(\Omega)} + \mathcal{G}(u) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

with $\mathcal{G}$ as in (2.2), is well defined, locally Lipschitz continuous, weakly sequentially lower semicontinuous, and coercive in U. Hence, there exists a $u_\lambda \in U$ fulfilling

$$E_\lambda(u_\lambda) = \inf_{u \in U} E_\lambda(u). \tag{3.1}$$

We claim that u_λ solves problem (1.2). Indeed, pick $v \in W_0^{1,p}(\Omega)$, $\alpha > 0$, and set

$$w(x) := \begin{cases} \underline{u}(x) & \text{if } x \in \Omega(u_\lambda + \alpha v \le \underline{u}), \\ u_\lambda(x) + \alpha v(x) & \text{if } x \in \Omega(\underline{u} \le u_\lambda + \alpha v) \cap \Omega(u_\lambda + \alpha v \le \overline{u}), \\ \overline{u}(x) & \text{if } x \in \Omega(\overline{u} \le u_\lambda + \alpha v). \end{cases}$$

Obviously, $w \in U$. Consequently, $tw + (1-t)u_\lambda \in U$ for all $t \in [0, 1]$. If $I_{[0,1]}$ denotes the indicator function of $[0, 1] \subseteq \mathbb{R}$,

$$f(t) := E_\lambda(tw + (1-t)u_\lambda) \quad \text{and} \quad \widetilde{f}(t) := f(t) + I_{[0,1]}(t), \qquad t \in \mathbb{R},$$

then, due to (3.1), the function $\widetilde{f}$ attains its minimum at $t = 0$. Accordingly, by [16, Proposition 2.1],

$$f^0(0; t-0) + I_{[0,1]}(t) - I_{[0,1]}(0) \ge 0 \quad \text{for all } t \in \mathbb{R},$$

which means, in particular, that $f^0(0; 1) \ge 0$. Since $f^0(0; 1) = \max\{z : z \in \partial f(0)\}$, we can find a $z \in \partial f(0) \cap [0, +\infty[$. Using the chain rule [6, Theorem 2.3.10] yields

$$\partial f(0) \subseteq \partial E_\lambda(u_\lambda) \cdot (w - u_\lambda). \tag{3.2}$$

On account of (3.2) and (2.4) there thus exists a $w_\lambda \in L^{q'}(\Omega)$ such that

$$w_\lambda(x) \in \partial G(x, u_\lambda(x)) \quad \text{almost everywhere in } \Omega, \tag{3.3}$$

$$\langle -\Delta_p u_\lambda, w - u_\lambda \rangle - \lambda \int_\Omega u_\lambda^{p-1}(w - u_\lambda)\, dx + \int_\Omega w_\lambda(w - u_\lambda)\, dx = z \ge 0. \tag{3.4}$$

We explicitly note that $w - u_\lambda \in L^q(\Omega)$ because $q \le p^*$. If $\underline{\eta}$ ($\overline{\eta}$) belongs to $L^{p'}(\Omega)$ and satisfies (2.6) ((2.8)) then, by the choice of w, inequality (3.4) becomes

$$\begin{aligned} 0 \le \alpha & \int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \cdot \nabla v\, dx - \alpha \int_\Omega (\lambda u_\lambda^{p-1} - w_\lambda) v\, dx \\ & - \int_{\Omega(\overline{u} \le u_\lambda + \alpha v)} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla(u_\lambda + \alpha v - \overline{u})\, dx \\ & + \int_{\Omega(\overline{u} \le u_\lambda + \alpha v)} (\lambda \overline{u}^{p-1} - \overline{\eta})\, (u_\lambda + \alpha v - \overline{u})\, dx \\ & + \int_{\Omega(u_\lambda + \alpha v \le \underline{u})} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla(\underline{u} - u_\lambda - \alpha v)\, dx \\ & - \int_{\Omega(u_\lambda + \alpha v \le \underline{u})} (\lambda \underline{u}^{p-1} - \underline{\eta})\, (\underline{u} - u_\lambda - \alpha v)\, dx \\ & + \int_{\Omega(\overline{u} \le u_\lambda + \alpha v)} (\lambda \overline{u}^{p-1} - \overline{\eta} - \lambda u_\lambda^{p-1} + w_\lambda)\, (\overline{u} - u_\lambda - \alpha v)\, dx \\ & + \int_{\Omega(u_\lambda + \alpha v \le \underline{u})} (\lambda \underline{u}^{p-1} - \underline{\eta} - \lambda u_\lambda^{p-1} + w_\lambda)\, (\underline{u} - u_\lambda - \alpha v)\, dx \end{aligned}$$

$$
\begin{aligned}
&-\int_{\Omega(u_\lambda+\alpha v\le \underline{u})} (|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla \underline{u}|^{p-2}\nabla \underline{u})\cdot\nabla(u_\lambda-\underline{u})\,dx\\
&-\alpha\int_{\Omega(u_\lambda+\alpha v\le \underline{u})} (|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla \underline{u}|^{p-2}\nabla \underline{u})\cdot\nabla v\,dx\\
&+\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)} (|\nabla \overline{u}|^{p-2}\nabla \overline{u} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\cdot\nabla(u_\lambda-\overline{u})\,dx\\
&+\alpha\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)} (|\nabla \overline{u}|^{p-2}\nabla \overline{u} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\cdot\nabla v\,dx. \qquad (3.5)
\end{aligned}
$$

Now, putting $\varphi := (u_\lambda + \alpha v - \overline{u})^+$ in (2.8) leads to

$$
\begin{aligned}
-&\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)} |\nabla \overline{u}|^{p-2}\nabla \overline{u}\cdot\nabla(u_\lambda+\alpha v-\overline{u})\,dx\\
&+\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)} (\lambda\overline{u}^{p-1}-\overline{\eta})\,(u_\lambda+\alpha v-\overline{u})\,dx \le 0, \qquad (3.6)
\end{aligned}
$$

while (2.6) written for $\varphi := (\underline{u} - u_\lambda - \alpha v)^+$ gives

$$
\begin{aligned}
&\int_{\Omega(u_\lambda+\alpha v\le \underline{u})} |\nabla \underline{u}|^{p-2}\nabla \underline{u}\cdot\nabla(\underline{u}-u_\lambda-\alpha v)\,dx\\
&\quad+\int_{\Omega(u_\lambda+\alpha v\le \underline{u})} (\underline{\eta}-\lambda\underline{u}^{p-1})\,(\underline{u}-u_\lambda-\alpha v)\,dx \le 0. \qquad (3.7)
\end{aligned}
$$

Since $\underline{u} \le u_\lambda \le \overline{u}$ in Ω, the result is

$$
\int_{\Omega(u_\lambda+\alpha v\le \underline{u})} (\underline{u}^{p-1}-u_\lambda^{p-1})\,(\underline{u}-u_\lambda-\alpha v)\,dx \le 0, \qquad (3.8)
$$

$$
\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)} (\overline{u}^{p-1}-u_\lambda^{p-1})\,(\overline{u}-u_\lambda-\alpha v)\,dx \le 0. \qquad (3.9)
$$

Next, assumption (g_2), (2.3), and the continuity of $\underline{u}$, $\overline{u}$ on $\overline{\Omega}$ ensure that both $\partial G(x, \underline{u}(x))$ and $\partial G(x, \overline{u}(x))$ are uniformly bounded with respect to $x \in \Omega$. So, in view of (2.5), (2.7), and (3.3), there exists a constant $a_2 > 0$ fulfilling

$$
\begin{aligned}
&\int_{\Omega(u_\lambda+\alpha v\le \underline{u})} (-\underline{\eta}+w_\lambda)\,(\underline{u}-u_\lambda-\alpha v)\,dx\\
&\quad\le a_2\int_{\Omega(u_\lambda+\alpha v\le \underline{u}<u_\lambda)} (\underline{u}-u_\lambda)\,dx - \alpha\int_{\Omega(u_\lambda+\alpha v\le \underline{u})(-\underline{\eta}+w_\lambda)} v\,dx \qquad (3.10)
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)} (-\overline{\eta}+w_\lambda)\,(\overline{u}-u_\lambda-\alpha v)\,dx\\
&\quad\le a_2\int_{\Omega(u_\lambda<\overline{u}\le u_\lambda+\alpha v)} (u_\lambda-\overline{u})\,dx + \alpha\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)(\overline{\eta}-w_\lambda)} v\,dx. \qquad (3.11)
\end{aligned}
$$

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Finally, by virtue of (p_4) we obtain

$$-\int_{\Omega(u_\lambda+\alpha v\le \underline{u})} (|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla \underline{u}|^{p-2}\nabla \underline{u})\cdot\nabla(u_\lambda-\underline{u})\,dx \le 0 \tag{3.12}$$

and

$$\int_{\Omega(\overline{u}\le u_\lambda+\alpha v)} (|\nabla \overline{u}|^{p-2}\nabla \overline{u} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\cdot\nabla(u_\lambda-\overline{u})\,dx \le 0. \tag{3.13}$$

At this point, gathering (3.5)–(3.13) together and dividing by $\alpha > 0$ yields

$$\begin{aligned}
0 \le{} & \int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla v\,dx - \int_\Omega(\lambda u_\lambda^{p-1}-w_\lambda)v\,dx \\
& - a_2\int_{\Omega(u_\lambda+\alpha v\le \underline{u}<u_\lambda)} v\,dx + a_2\int_{\Omega(u_\lambda<\overline{u}\le u_\lambda+\alpha v)} v\,dx \\
& + \int_{\Omega(u_\lambda+\alpha v\le \underline{u})}(\underline{\eta}-w_\lambda)v\,dx + \int_{\Omega(\overline{u}\le u_\lambda+\alpha v)}(\overline{\eta}-w_\lambda)v\,dx \\
& - \int_{\Omega(u_\lambda+\alpha v\le \underline{u})}(|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla \underline{u}|^{p-2}\nabla \underline{u})\cdot\nabla v\,dx \\
& + \int_{\Omega(\overline{u}\le u_\lambda+\alpha v)}(|\nabla \overline{u}|^{p-2}\nabla \overline{u} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\cdot\nabla v\,dx.
\end{aligned} \tag{3.14}$$

For $\alpha \to 0^+$, inequality (3.14) becomes

$$0 \le \int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla v\,dx - \int_\Omega(\lambda u_\lambda^{p-1}-w_\lambda)v\,dx,$$

because u_λ lies in U and we have $\underline{\eta}\ge w_\lambda$ on $\Omega(\underline{u}=u_\lambda)$, $\overline{\eta}\le w_\lambda$ on $\Omega(\overline{u}=u_\lambda)$ (see (2.12)). As $v\in W_0^{1,p}(\Omega)$ was arbitrary, it results in

$$-\Delta_p u_\lambda = \lambda u_\lambda^{p-1} - w_\lambda, \tag{3.15}$$

that is, u_λ is a positive solution of (1.2). From $u_\lambda, w_\lambda\in L^\infty(\Omega)$ it follows that $\Delta_p u\in L^\infty(\Omega)$. Then [10, Theorem 6.2.7] forces $u_\lambda\in C_0^1(\overline{\Omega})$. Due to (2.13) and ($g_2$) we can find a constant $\widetilde{c}_\lambda > 0$ satisfying

$$|g(x,t)| \le \widetilde{c}_\lambda t^{p-1}, \quad \text{for all } t\in[0, \|\overline{u}\|_{L^\infty(\Omega)}] \tag{3.16}$$

Hence, by (3.15), (3.3), and (3.16),

$$\Delta_p u_\lambda \le (\lambda+\widetilde{c}_\lambda)u_\lambda^{p-1}. \tag{3.17}$$

The Vázquez maximum principle [9, Theorem 1.5.7] thus provides $u_\lambda\in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Denote by $\mathcal{U}$ the set of all solutions $u\in\operatorname{int}(C_0^1(\overline{\Omega})_+)$ to problem (1.2) such that $\underline{u}\le u\le\overline{u}$ in Ω. $\mathcal{U}$ is nonempty since $u_\lambda\in\mathcal{U}$. Arguing as in the proofs of [3, Lemma 4.23] and [3, Corollary 4.24], and using [9, Theorem 1.5.7] once more, we then see that $\mathcal{U}$ possesses a minimal element, say u_+, with respect to the pointwise usual order. □

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Two extremal solutions of (1.2) having opposite constant sign can now be obtained via Theorem 3.1.

THEOREM 3.2. *Let* (g_1)–(g_4) *be fulfilled. Then for every* $\lambda > \lambda_1$ *there exist a minimal positive solution* $u_+ \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+) \cap [0, a_\lambda e]$ *and a maximal negative solution* $u_- \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+) \cap [-a_\lambda e, 0]$ *to problem (1.2), where* $a_\lambda > 0$ *is given by Theorem 2.5.*

PROOF. Fix $\lambda > \lambda_1$. Since similar reasoning is used for u_- and u_+, we shall confine ourselves to the case of u_+. Retain the notation introduced in the proof of Theorem 3.1. By that result, for every positive integer n sufficiently large there is a minimal solution $u_n \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+) \cap [\frac{1}{n}\varphi_1, \overline{u}]$ to (1.2). The minimality property of u_n gives

$$u_n \downarrow u_+ \quad \text{pointwise in } \Omega \tag{3.18}$$

for some $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \le u_+ \le \overline{u}$. We claim that

$$\text{the function } u_+ \text{ turns out to be a solution of problem (1.2).} \tag{3.19}$$

In fact, from (1.4), with $u := u_n$, it follows that

$$\langle -\Delta_p u_n, \varphi \rangle = \int_\Omega (\lambda |u_n|^{p-2} u_n - \eta_n)\varphi \, dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega), \tag{3.20}$$

where $\eta_n \in L^{p'}(\Omega)$ and $\eta_n(x) \in \partial G(x, u_n(x))$ for almost all $x \in \Omega$. If $\varphi := u_n$ then

$$\|u_n\|^p = \int_\Omega (\lambda u_n^p - \eta_n) u_n \, dx, \qquad n \in \mathbb{N}. \tag{3.21}$$

Due to (g_2), besides the inequality $0 \le u_n \le \overline{u}$, the sets $\partial G(x, u_n(x))$, $x \in \Omega$, $n \in \mathbb{N}$, are uniformly bounded. Hence, there exists an $a_3 > 0$ such that

$$|\eta_n(x)| \le a_3, \qquad \text{almost everywhere in } \Omega, \text{ for all } n \in \mathbb{N}. \tag{3.22}$$

Thus, by (3.21), the sequence $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ is bounded too. Taking a subsequence when necessary, we may suppose that

$$u_n \rightharpoonup u_+ \quad \text{in } W_0^{1,p}(\Omega), \qquad u_n \to u_+ \quad \text{in } L^p(\Omega). \tag{3.23}$$

On account of (3.20) with $\varphi := u_n - u_+$,

$$\begin{aligned} \langle -\Delta_p u_n, u_n - u_+ \rangle = \lambda & \int_\Omega (|u_n|^p - |u_n|^{p-2} u_n u_+) \, dx \\ & - \int_\Omega \eta_n (u_n - u_+) \, dx \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Now, by virtue of (3.23), (3.18), and the Lebesgue dominated convergence theorem, this forces $\lim_{n\to\infty} \langle -\Delta_p u_n, u_n - u_+ \rangle = 0$. Thanks to (p_4), we then obtain

$$u_n \to u_+ \quad \text{in } W_0^{1,p}(\Omega). \tag{3.24}$$

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Using (3.22) yields a function $\eta_+ \in L^{p'}(\Omega)$ such that $\eta_n \rightharpoonup \eta_+$ in $L^{p'}(\Omega)$. By (3.24), Lemma 2.2 can be applied to obtain $\eta_+(x) \in \partial G(x, u_+(x))$ for almost every $x \in \Omega$. From (3.20) it finally follows that

$$\langle -\Delta_p u_+, \varphi\rangle = \int_\Omega (\lambda |u_+|^{p-2} u_+ - \eta_+)\varphi\, dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega),$$

that is to say,

$$-\Delta_p u_+ = \lambda |u_+|^{p-2} u_+ - \eta_+,$$

and (3.19) is proved.

Next, since $u_+ \in L^\infty(\Omega)$, assumption (g$_2$) produces $\Delta_p u_+ \in L^\infty(\Omega)$. Owing to (3.16) we achieve, as before,

$$\Delta_p u_+ \le (\lambda + \widetilde{c}_\lambda) u_+^{p-1}.$$

The Vázquez maximum principle [9, Theorem 1.5.7] ensures that either $u_+ \equiv 0$ or $u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. If the assertion

$$u_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \tag{3.25}$$

were false then $u_+ \equiv 0$. Accordingly, in view of (3.18),

$$u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega\,. \tag{3.26}$$

Setting

$$\widetilde{u}_n = \frac{u_n}{\|u_n\|}, \qquad n \in \mathbb{N},$$

we may suppose that (along a relabelled subsequence, when necessary)

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{in } W_0^{1,p}(\Omega), \qquad \widetilde{u}_n \to \widetilde{u} \quad \text{in } L^p(\Omega), \tag{3.27}$$

as well as

$$|\widetilde{u}_n(x)| \le w(x) \quad \text{for all } n \in \mathbb{N}, \qquad \widetilde{u}_n(x) \to \widetilde{u}(x) \quad \text{for almost all } x \in \Omega, \tag{3.28}$$

with $w \in L^p(\Omega)_+$. By virtue of (3.20) this leads to

$$\langle -\Delta_p \widetilde{u}_n, \varphi\rangle = \lambda \int_\Omega \widetilde{u}_n^{p-1}\varphi\, dx - \int_\Omega \frac{\eta_n}{u_n^{p-1}} \widetilde{u}_n^{p-1}\varphi\, dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \tag{3.29}$$

If $\varphi := \widetilde{u}_n - \widetilde{u}$ then

$$\langle -\Delta_p \widetilde{u}_n, \widetilde{u}_n - \widetilde{u}\rangle = \lambda \int_\Omega \widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u})\, dx - \int_\Omega \frac{\eta_n}{u_n^{p-1}} \widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u})\, dx. \tag{3.30}$$

By (3.16), (3.28) there exists a constant $\overline{c}_\lambda > 0$ fulfilling

$$\frac{|\eta_n(x)|}{u_n(x)^{p-1}} \widetilde{u}_n(x)^{p-1} |\widetilde{u}_n(x) - \widetilde{u}(x)| \le \overline{c}_\lambda w(x)^{p-1} |\widetilde{u}_n(x) - \widetilde{u}(x)| \le 2\overline{c}_\lambda w(x)^p$$

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almost everywhere in Ω. Due to (3.28), besides the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \frac{\eta_n}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})\,dx=0.$$

Hence, from (3.30) and (3.28) it again follows that $\lim_{n\to\infty}\langle -\Delta_p\widetilde{u}_n, \widetilde{u}_n-\widetilde{u}\rangle=0$, which, on account of ($p_4$), forces

$$\widetilde{u}_n\to\widetilde{u}\quad\text{in } W_0^{1,p}(\Omega). \tag{3.31}$$

So, in particular, $\|\widetilde{u}\|=1$. Gathering (3.29), (3.31), (3.26), and (g_3) together gives

$$\langle -\Delta_p\widetilde{u}, \varphi\rangle=\lambda\int_\Omega \widetilde{u}^{p-1}\varphi\,dx\quad\text{for all } \varphi\in W_0^{1,p}(\Omega),$$

that is, $\widetilde{u}$ is an eigenfunction of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ corresponding to the eigenvalue $\lambda>\lambda_1$. By [10, Proposition 6.2.15], the function $\widetilde{u}$ must change sign in Ω, whereas (3.28) and (3.26) imply that $\widetilde{u}(x)\geq 0$ for almost all $x\in\Omega$. Therefore, (3.25) holds.

Let us finally verify that

$$u_+ \text{ is a minimal positive solution of (1.2) within } [0,\overline{u}]. \tag{3.32}$$

If $u\in W_0^{1,p}(\Omega)\cap[0,\overline{u}]$, $0<u\leq u_+$ in Ω, and u satisfies (1.2) then, by [10, Theorem 6.2.7], $u\in C_0^1(\overline{\Omega})$. The same argument as employed before regarding u_λ and u_+ now yields $u\in\operatorname{int}(C_0^1(\overline{\Omega})_+)$. Consequently, $u\in[(1/n)\varphi_1,\overline{u}]$ for any sufficiently large n. Since u_n is a minimal solution of (1.2) in $[(1/n)\varphi_1,\overline{u}]$, it turns out that $u_n\leq u$. As n was arbitrary, (3.18) leads to $u_+\leq u$, and the conclusion follows. □

4. Sign-changing solutions

A third nonzero, sign-changing solution to (1.2) can be obtained when $\lambda>\lambda_2$, the second eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, as the next result shows.

THEOREM 4.1. *Under assumptions* (g_1)–(g_4), *for every $\lambda>\lambda_2$, problem (1.2) possesses a positive solution $u_+\in\operatorname{int}(C_0^1(\overline{\Omega})_+)$, a negative solution $u_-\in-\operatorname{int}(C_0^1(\overline{\Omega})_+)$, and a nontrivial sign-changing solution $u_0\in C_0^1(\overline{\Omega})$.*

PROOF. Fix $\lambda>\lambda_2$. If u_+ and u_- are given by Theorem 3.2 then there exist $\eta_+,\eta_-\in L^{p'}(\Omega)$ such that

$$-\Delta_p u_+(x)=\lambda u_+(x)^{p-1}-\eta_+(x),\qquad -\Delta_p u_-(x)=\lambda|u_-(x)|^{p-2}u_-(x)-\eta_-(x),$$

as well as

$$\eta_+(x)\in\partial G(x,u_+(x)),\qquad \eta_-(x)\in\partial G(x,u_-(x)),$$

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for almost every $x \in \Omega$. Define, whenever $(x, t) \in \Omega \times \mathbb{R}$,

$$\tau_+(x, t) := \begin{cases} 0 & \text{if } t < 0, \\ t & \text{if } 0 \le t \le u_+(x), \\ u_+(x) & \text{if } t > u_+(x), \end{cases}$$
$$\tau_-(x, t) := \begin{cases} u_-(x) & \text{if } t < u_-(x), \\ t & \text{if } u_-(x) \le t \le 0, \\ 0 & \text{if } t > 0, \end{cases}$$
$$\tau_0(x, t) := \begin{cases} u_-(x) & \text{if } t < u_-(x), \\ t & \text{if } u_-(x) \le t \le u_+(x), \\ u_+(x) & \text{if } t > u_+(x), \end{cases}$$

and

$$g_+(x, t) := \begin{cases} 0 & \text{if } t < 0, \\ g(x, t) & \text{if } 0 \le t \le u_+(x), \\ \eta_+(x) & \text{if } t > u_+(x), \end{cases}$$
$$g_-(x, t) := \begin{cases} \eta_-(x) & \text{if } t < u_-(x), \\ g(x, t) & \text{if } u_-(x) \le t \le 0, \\ 0 & \text{if } t > 0, \end{cases}$$
$$g_0(x, t) := \begin{cases} \eta_-(x) & \text{if } t < u_-(x), \\ g(x, t) & \text{if } u_-(x) \le t \le u_+(x), \\ \eta_+(x) & \text{if } t > u_+(x). \end{cases}$$

Moreover, provided that $u \in W_0^{1,p}(\Omega)$, set

$$E_+(u) := \frac{1}{p}\,\|u\|^p - \int_\Omega \left(\int_0^{u(x)} (\lambda \tau_+(x, t)^{p-1} - g_+(x, t))\, dt \right) dx,$$
$$E_-(u) := \frac{1}{p}\,\|u\|^p - \int_\Omega \left(\int_0^{u(x)} (\lambda |\tau_-(x, t)|^{p-2} \tau_-(x, t) - g_-(x, t))\, dt \right) dx,$$
$$E_0(u) := \frac{1}{p}\,\|u\|^p - \int_\Omega \left(\int_0^{u(x)} (\lambda |\tau_0(x, t)|^{p-2} \tau_0(x, t) - g_0(x, t))\, dt \right) dx.$$

Due to (g_2), besides the regularity properties of u_+ and u_-, the functionals $E_+, E_-, E_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ are locally Lipschitz continuous. We claim that

$$\text{each critical point of } E_+ \text{ belongs to } [0, u_+]. \tag{4.1}$$

In fact, if $u \in W_0^{1,p}(\Omega)$ fulfils $0 \in \partial E_+(u)$ then $-\Delta_p u = \lambda \tau_+(x, u)^{p-1} - w$ for some $w \in L^{p'}(\Omega)$ such that $w(x) \in \partial G_+(x, u(x))$ almost everywhere in Ω, with

$$G_+(x, \xi) := \int_0^\xi g_+(x, t)\, dt, \qquad (x, \xi) \in \Omega \times \mathbb{R}.$$

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Choosing the test function $\varphi := (u - u_+)^+$ gives

$$\begin{aligned}&\langle -\Delta_p u + \Delta_p u_+, (u - u_+)^+\rangle\\ &\qquad = \int_\Omega [\lambda\tau_+(x, u)^{p-1} - w - \lambda u_+^{p-1} + \eta_+]\,(u - u_+)^+\,dx = 0.\end{aligned}$$

On account of (p_4), this implies that $u \le u_+$. Similarly, from

$$\langle -\Delta_p u, -u^-\rangle = -\int_\Omega [\lambda\tau_+(x, u)^{p-1} - w]u^-\,dx = 0,$$

it follows that $u \ge 0$. Hence, assertion (4.1) holds.

An easy verification ensures that the functional E_+ is bounded below, weakly sequentially lower semicontinuous, and coercive. So there exists a $v_+ \in W_0^{1,p}(\Omega)$ satisfying

$$E_+(v_+) = \inf_{u\in W_0^{1,p}(\Omega)} E_+(u), \tag{4.2}$$

which forces both $v_+ \in [0, u_+]$, on account of (4.1), and $-\Delta_p v_+ \in \lambda v_+^{p-1} - \partial G_+(x, v_+)$. Since $\partial G_+(x, v_+(x)) \subseteq \partial G(x, v_+(x))$, $x \in \Omega$, the function v_+ turns out to be a solution of (1.2). Moreover, $v_+ \ne 0$. Indeed, by virtue of (3.25) we obtain

$$t\varphi_1 \le u_+, \qquad t\|\varphi_1\|_{L^\infty(\Omega)} \le \delta_\lambda,$$

with δ_λ as in (2.13), provided that $t > 0$ is sufficiently small. Consequently, by (4.2), (p_2), and (2.13),

$$E_+(v_+) \le E_+(t\varphi_1) = \frac{\lambda_1}{p}\,t^p - \int_\Omega\left(\int_0^{t\varphi_1(x)} (\lambda s^{p-1} - g(x, s))\,ds\right)dx < 0, \tag{4.3}$$

that is to say, $v_+ \ne 0$. At this point, the same argument as exploited in the proof of Theorem 3.2 to achieve (3.25) shows here that

$$v_+ \in \operatorname{int}(C_0^1(\overline{\Omega})_+). \tag{4.4}$$

Gathering (4.4), the inequality $v_+ \le u_+$, and Theorem 3.2 together gives $v_+ = u_+$. Thus, due to (4.2), besides (4.4), the function u_+ is a local minimizer of E_0 in $C_0^1(\overline{\Omega})$. Then [9, Proposition 4.6.10] guarantees that u_+ enjoys the same property in the space $W_0^{1,p}(\Omega)$. Likewise, replacing the functional E_+ with E_- one realizes that u_- is a local minimizer of E_0 in $W_0^{1,p}(\Omega)$.

Since E_0 is bounded below, weakly sequentially lower semicontinuous, and coercive, there exists a $v_0 \in W_0^{1,p}(\Omega)$ fulfilling $E_0(v_0) = \inf_{u\in W_0^{1,p}(\Omega)} E_0(u)$. Moreover, as before, it results in $v_0 \ne 0$ as well as

$$\text{each critical point of } E_0 \text{ belongs to } [u_-, u_+]. \tag{4.5}$$

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Therefore, $v_0 \in [u_-, u_+]$ and v_0 is a nontrivial solution of (1.2). Without loss of generality we may suppose that $v_0 = u_+$ or $v_0 = u_-$, because otherwise the extremality of u_+ and u_- established in Theorem 3.2 would force a change of sign for v_0, which completes the proof. So, let $v_0 = u_+$ (similar reasoning applies when $v_0 = u_-$). We may assume also that u_- is a strict local minimizer of E_0. In fact, if this were false then infinitely many sign-changing solutions to (1.2) might be found via (4.5), besides the extremality of u_+, u_-, and the conclusion follows. Pick a $\rho \in]0, \|u_+ - u_-\|[$ such that

$$E_0(u_+) \le E_0(u_-) < \inf_{u \in \partial B_\rho(u_-)} E_0(u). \tag{4.6}$$

The functional E_0 satisfies condition (PS) because it is bounded below, locally Lipschitz continuous, and coercive; see, for example, [13, Corollary 2.4]. Bearing in mind (4.6), Theorem 2.1 can be applied. Hence, there is a $u_0 \in W_0^{1,p}(\Omega)$ complying with $0 \in \partial E_0(u_0)$ and

$$\inf_{u \in \partial B_\rho(u_-)} E_0(u) \le E_0(u_0) = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} E_0(\gamma(t)), \tag{4.7}$$

where

$$\Gamma := \{\gamma \in C^0([-1, 1], W_0^{1,p}(\Omega)) : \gamma(-1) = u_-, \ \gamma(1) = u_+\}.$$

By (4.5), $\partial G_0(x, u_0(x)) \subseteq \partial G(x, u_0(x))$, $x \in \Omega$, that is, u_0 solves (1.2). Moreover, thanks to (4.6) and (4.7), $u_0 \ne u_-$ and $u_0 \ne u_+$. The proof is thus complete once we show that $u_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$. Let us start with $u_0 \ne 0$. This immediately comes out from the inequality

$$E_0(u_0) < 0, \tag{4.8}$$

which, in view of (4.7), holds if we construct a $\widehat{\gamma} \in \Gamma$ such that

$$E_0(\widehat{\gamma}(t)) < 0 \quad \text{for all } \ t \in [-1, 1]. \tag{4.9}$$

Set $S := \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ and fix $\mu \in]0, \lambda - \lambda_2[$. Assumption ($g_3$) yields a $\delta_\mu > 0$ such that

$$\frac{|g(x, t)|}{|t|^{p-1}} \le \mu \quad \text{provided } 0 < |t| \le \delta_\mu. \tag{4.10}$$

If $\rho_0 \in]0, \lambda - \lambda_2 - \mu[$ then, due to (p_3), there exists a $\gamma \in \Gamma_0$ satisfying

$$\max_{t \in [-1,1]} \|\gamma(t)\|^p < \lambda_2 + \frac{\rho_0}{2}.$$

Now, define $S_C := S \cap C_0^1(\overline{\Omega})$ and consider on S_C the topology induced by that of $C_0^1(\overline{\Omega})$. Clearly, S_C is a dense subset of S. So, given $r > 0$, with

$$r \le (\lambda_2 + \rho_0)^{1/p} - \left(\lambda_2 + \frac{\rho_0}{2}\right)^{1/p},$$

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we can find a $\gamma_0 \in C^0([-1, 1], S_C)$ such that $\gamma_0(-1) = -\varphi_1$, $\gamma_0(1) = \varphi_1$, and

$$\max_{t\in[-1,1]} \|\gamma(t) - \gamma_0(t)\| < r.$$

This obviously forces

$$\max_{t\in[-1,1]} \|\gamma_0(t)\|^p < \lambda_2 + \rho_0. \tag{4.11}$$

Let $\varepsilon_1 > 0$ fulfil

$$\varepsilon_1 \max_{x\in\overline{\Omega}} |u(x)| \le \delta_\mu, \quad \text{for all } u \in \gamma_0([-1, 1]). \tag{4.12}$$

Since $u_+, -u_- \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, to every $u \in \gamma_0([-1, 1])$ and every bounded neighbourhood V_u of u in $C_0^1(\overline{\Omega})$ there corresponds a $\nu > 0$ such that

$$u_+ - \frac{1}{m} v \in \operatorname{int}(C_0^1(\overline{\Omega})_+), \quad -u_- + \frac{1}{n} v \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$$
$$\text{whenever } m, n \ge \nu,\ v \in V_u.$$

Through the compactness of $\gamma_0([-1, 1])$ in $C_0^1(\overline{\Omega})$ we thus obtain an $\varepsilon_0 > 0$ satisfying

$$u_-(x) \le \varepsilon u(x) \le u_+(x) \quad \text{for all } x \in \Omega, \quad u \in \gamma_0([-1, 1]), \varepsilon \in]0, \varepsilon_0[. \tag{4.13}$$

The function $t \mapsto \varepsilon\gamma_0(t)$, $t \in [0, 1]$, is a continuous path in S_C joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. Moreover, if $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ then (4.11), (4.13), (4.12), and (4.10) yield

$$\begin{aligned} E_0(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p} \|\gamma_0(t)\|^p - \frac{\varepsilon^p}{p}\lambda + \int_\Omega \left(\int_0^{\varepsilon\gamma_0(t)\,(x)} g(x, \tau_0(x, s))\, ds \right) dx \\ &\le \frac{\varepsilon^p}{p}(\lambda_2 + \rho_0 - \lambda) + \int_\Omega \left(\int_0^{\varepsilon\gamma_0(t)\,(x)} g(x, s)\, ds \right) dx \\ &\le \frac{\varepsilon^p}{p}(\lambda_2 + \rho_0 - \lambda + \mu) < 0 \quad \text{for all } t \in [-1, 1]. \end{aligned} \tag{4.14}$$

Next, write

$$a_4 := E_+(u_+), \quad U_+ := \{u \in W_0^{1,p}(\Omega) : E_+(u) < 0\}.$$

Then clearly $a_4 < 0$, because $u_+ = v_+$ and $E_+(v_+) < 0$ by (4.3). Hence, U_+ turns out to be nonempty. Moreover, $a_4 = \inf_{u\in W_0^{1,p}(\Omega)} E_+(u)$ on account of (4.2). Gathering (4.1) and Theorem 3.2 together ensures that E_+ has no critical point u with $a_4 < E_+(u) < 0$ and that $K(E_+) \cap E_+^{-1}(a_4) = \{u_+\}$. Finally, since E_+ is bounded below, locally Lipschitz continuous, and coercive, it satisfies condition (PS). So, thanks to [7, Theorem 2.10], there exists a continuous function $h : [0, 1] \times U_+ \to U_+$ fulfilling

$$h(0, \cdot) = \mathrm{id}|_{U_+}, \quad h(1, U_+) = \{u_+\},$$
$$E_+(h(t, u)) \le E_+(u) \quad \text{for all } (t, u) \in [0, 1] \times U_+.$$

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Let $\gamma_+ : [0, 1] \to U_+$ defined by $\gamma_+(t) := h(t, \varepsilon\varphi_1)^+$ for every $t \in [0, 1]$. Then $\gamma_+(0) = \varepsilon\varphi_1$, $\gamma_+(1) = u_+$, as well as

$$E_0(\gamma_+(t)) = E_+(\gamma_+(t)) \le E_+(h(t, \varepsilon\varphi_1)) \le E_+(\varepsilon\varphi_1) < 0, \quad t \in [0, 1]. \tag{4.15}$$

In a similar way, but with E_- in place of E_+, we can construct a continuous function $\gamma_- : [0, 1] \to W_0^{1,p}(\Omega)$ such that $\gamma_-(0) = -\varepsilon\varphi_1$, $\gamma_-(1) = u_-$ and

$$E_0(\gamma_-(t)) < 0 \quad \text{for all } t \in [0, 1]. \tag{4.16}$$

Concatenating γ_-, γ_0, and γ_+ produces a path $\widehat{\gamma} \in \Gamma$ which, in view of (4.14)–(4.16), satisfies (4.9). This shows (4.8) and, consequently, $u_0 \neq 0$. To complete the proof we simply note that the same argument, based on [10, Theorem 6.2.7], as exploited before leads to $u_0 \in C_0^1(\overline{\Omega})$. □

REMARK 4.2. The preceding proof is patterned after that of [4, Theorem 4.1]; see also [17, Theorem 3.9]. If the function $t \mapsto g(x, t)$ is continuous on $\mathbb{R}$ then $\partial G(x, \xi) = \{g(x, \xi)\}$, and problem (1.2) reduces to (1.1). However, even in this setting the result above is more general than [17, Theorem 3.9], because we do not assume that $g(x, t)t \ge 0$ for all $t \in \mathbb{R}$.

REMARK 4.3. Theorem 4.1 improves [11, Corollary 3.2]. In fact, let $p = 2$, let $u \in W_0^{1,2}(\Omega)$ be a solution of (1.2) with $g(x, t) \equiv g(t)$, $(x, t) \in \Omega \times \mathbb{R}$, and let $\eta \in L^{p'}(\Omega)$ fulfil (1.3)–(1.4). By the definition of $\partial G(u(x))$, for any $\varphi \in W_0^{1,2}(\Omega)$,

$$\eta(x)\varphi(x) \le G^0(u(x); \varphi(x)) \quad \text{almost everywhere in } \Omega.$$

Hence, due to (1.4),

$$-\int_\Omega \nabla u \cdot \nabla\varphi \, dx + \lambda \int_\Omega u\varphi \, dx = \int_\Omega \eta\varphi \, dx \le \int_\Omega G^0(u(x); \varphi(x)) \, dx \quad \text{for all } \varphi \in W_0^{1,2}(\Omega),$$

that is, u turns out to be a solution of the hemivariational inequality studied in [11]. Since the hypotheses of [11, Corollary 3.2] imply (g_1)–(g_4), the assertion follows.

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