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An Onofri-type Inequality on the Sphere with Two Conical Singularities

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Abstract. In this paper, we give a new proof of the Onofri-type inequality

$$\int_S e^{2u}\,ds^2 \le 4\pi(\beta+1)\exp\Big\{\frac{1}{4\pi(\beta+1)}\int_S |\nabla u|^2\,ds^2 + \frac{1}{2\pi(\beta+1)}\int_S u\,ds^2\Big\}$$

on the sphere S with Gaussian curvature 1 and with conical singularities divisor $\mathcal{A} = \beta\cdot p_1 + \beta\cdot p_2$ for $\beta \in (-1,0)$; here p_1 and p_2 are antipodal.

1 Introduction

On a smooth compact Riemannian surface Σ, the Moser–Trudinger inequality says that any function $u \in H^1(\Sigma)$

$$\int_\Sigma e^u\,dA \le C\exp\Big\{\frac{1}{16\pi}\int_\Sigma |\nabla u|^2\,dA + \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma u\,dA\Big\} \tag{1}$$

where C is a positive constant.

It is quite important to know which constant C is optimal for this inequality. It was shown firstly by Onofri [On] and Hong [H] that on the standard sphere S^2, using the Trudinger inequality, the best constant C is 4π. Consequently they got an inequality

$$\int_{S^2} e^u\,dA \le 4\pi\exp\Big\{\frac{1}{16\pi}\int_{S^2} |\nabla u|^2\,dA + \frac{1}{4\pi}\int_{S^2} u\,dA\Big\}$$

which is called Onofri inequality. Recently Li Suyu and Zhu Meijun [LZ1] gave a new proof of this inequality by using an inequality named the sharp local inequality which was shown in their recent paper [LZ1] and [LZ2]. In particular, their proof is independent of the Trudinger inequality.

We would like to remind the reader that there is another well-known best constant in the inequality, $\frac{1}{16\pi}$, which was obtained by Moser in [M].

A natural question is: Can one generalize these results to surfaces with singularities?

In this paper, we will discuss a similar inequality on a sphere with two conical singularities, called an Onofri-type inequality. Let us first recall the definition, which

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was first given in [T1]. A conformal metric ds^2 on a Riemannian surface Σ (possibly with boundary) has a *conical singularity* of order β (a real number with $\beta > -1$) at a point $p \in \Sigma \cup \partial\Sigma$ if in some neighborhood of p

$$ds^2 = e^{2u}|z - z(p)|^{2\beta}\,|dz|^2,$$

where z is a coordinate of Σ defined in this neighborhood and u is smooth away from p and continuous at p. The point p is then said to be a conical singularity of *angle* $\theta = 2\pi(\beta + 1)$ if $p \notin \partial\Sigma$ and a *corner* of angle $\theta = \pi(\beta + 1)$ if $p \in \partial\Sigma$. For example, a football has two singularities with equal angles, while a teardrop has only one singularity. Both these examples correspond to the case $-1 < \beta < 0$; in case $\beta > 0$, the angle is larger than 2π, leading to a different geometric picture. Such singularities appear in orbifolds and branched coverings. They can also describe the ends of complete Riemannian surfaces with finite total curvature. If (Σ, ds^2) has conical singularities of order $\beta_1, \beta_2, \ldots, \beta_n$ at $p_1, p_2, \ldots, p_n$, then ds^2 is said to represent the divisor $\mathbf{A} := \Sigma_{i=1}^{n}\beta_i p_i$.

Associated to ds^2 one can define gradient ∇ and Laplacian $\triangle$ operator in the usual way. One can also define the Hilbert space $H^1(\Sigma)$ with norm $\|\nabla u\|_2 + \|u\|_2$, where $\|u\|_p = (\int_\Sigma |u|^p\, dA)^{\frac{1}{p}}$ is the L^p-norm.

There are not many results about the Sobolev inequality on singular surfaces. Troyanov [T2] was the first author to consider Trudinger inequality on singular surfaces. He has shown that

$$\int_\Sigma e^{bu^2}\, dA \leq C \tag{2}$$

holds for all $u \in H^1(\Sigma)$ satisfying $\|\nabla u\|_2 \leq 1$ and $\int_\Sigma u\, dA = 0$, where Σ is a compact Riemannian surface with conical singularities of divisor $\beta = \sum_{j=1}^{k} \beta_i p_i$, and $b < 4\pi \min_i\{1, 1+\beta_i\}$. Furthermore, W. Chen [Ch] showed that this inequality holds for $b_0 = 4\pi \min_i\{1, 1 + \beta_i\}$, and the constant b_0 is sharp. As a direct consequence, we have the Moser–Trudinger inequality on this singular surface

$$\int_\Sigma e^u\, dA \leq C \exp\left\{ \frac{1}{4b_0} \int_\Sigma |\nabla u|^2\, dA + \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma u\, dA \right\}.$$

Lately, W. Chen and C. Li [CL] obtained an Onofri-type inequality on a sphere S with two singularities of equal angle, $0 < \theta_1 = \theta_2 < 2\pi$. Letting $\alpha = \frac{\theta_1}{2\pi}$ and taking the singularities as the poles and the metric as $ds^2 = L^2\big(dr^2 + (\frac{\alpha}{\pi}\sin\pi r)^2 d\theta^2\big)$, they showed that

$$\int_S e^u\, dA \leq \frac{4L^2\alpha}{\pi} \exp\left\{ \frac{1}{16\pi\alpha} \int_S |\nabla u|^2\, dA + \frac{\pi}{4L^2\alpha} \int_S u\, dA \right\} \tag{3}$$

holds for all $u \in H^1(S)$, where $\frac{4L^2\alpha}{\pi}$ is the smallest possible constant.

It is clear that (3) will be reduced to the standard Onofri inequality on the smooth sphere when $L = \pi$ and $\alpha = 1$. From the Gauss–Bonnet formula

$$\frac{1}{2\pi} \int_S K\, dA + \frac{1}{2\pi} \int_{\partial S} k\, ds = \chi(S, \beta)$$

where $\chi(S,\beta)=\chi(S)+\sum_{i=1}^{k}(\frac{\theta_i}{2\pi}-1)$, χ is the topological Euler characteristic of S, we know that the inequality (3) holds only in the critical case $\chi(S,\beta)=\min_i\{2,\frac{2\theta_i}{\pi}\}$, which is defined by Troyanov when he studied the prescribing Gaussian curvatures problem on singular surfaces. Their method to establish inequality (3) was based on the Trudinger inequality (2) and the distribution of mass analysis.

It is well known that the Onofri inequality plays a very important role in conformal geometry. Based on the Onofri inequality, many results on prescribing Gaussian curvatures on the standard sphere S^2 were obtained by using various techniques, see [CD1], [CD2], [CY1], [CY2]. Furthermore, in virtue of the uniformization theorem and the Onofri inequality, one can show that the Liouville energy on a topological two-dimensional sphere will be bounded from below. In the singular case, with the help of this inequality (3), Chen Wenxiong and Li Congming successfully generalized some of their previous results on prescribing Gaussian curvatures from the standard sphere S^2 to such a singular surface.

In this paper, we want to give a new proof of the Onofri-type inequality on a singular sphere S for $\beta\in(-1,0)$. Fortunately, Troyanov [T1] has shown that there is an explicit expression of metric on a sphere with constant curvature and with two conical singularities. More precisely, if we take $S=C\cup\infty$, the only metric (up to a change of coordinate $z\to pz$, $p\in\mathbb{C}$ is a constant) on sphere S with Gaussian curvature 1 and with conical singularities at $z=0$ and $z=\infty$ is

$$ds^2=\frac{(2+2\beta)^2|z|^{2\beta}|dz|^2}{(|1+\mu z^{\beta+1}|^2+|z|^{2\beta+2})^2},$$

where $\beta\in(-1,+\infty)$ such that either β is an integer or $\mu=0$. In virtue of the expression of this metric, we can state our main theorem without using the Trudinger inequality (2) and the distribution of mass analysis:

Theorem 1.1 *Let the sphere S with Gaussian curvature* 1 *and with conical singularities divisor $\mathcal{A}=\beta\cdot p_1+\beta\cdot p_2$ for $\beta\in(-1,0)$, here p_1 and p_2 are antipodal. Then we have for all $u\in H^1(S)$*

$$\int_S e^{2u}\,ds^2\le 4\pi(\beta+1)\exp\Big\{\frac{1}{4\pi(\beta+1)}\int_S|\nabla u|^2\,ds^2+\frac{1}{2\pi(\beta+1)}\int_S u\,ds^2\Big\}. \tag{4}$$

To prove this theorem, we will follow closely the method used in [LZ1]. We will first establish an inequality which is essential to our main theorem, *i.e.*,

$$\int_{B_r}|\nabla w|^2\,dx\ge(4+4\beta)\pi\Big(\ln\frac{(2+2\beta)ae^{-2b}}{2\pi r^{2+2\beta}}+\frac{2\pi r^{2+2\beta}}{(2+2\beta)ae^{-2b}}-1\Big)$$

holds with any function $w(x)$ that satisfies $w(x)-b\in W_0^{1,2}\big(B_r(0)\big)$ and

$$\int_{B_r(0)}|x|^{2\beta}e^{2w}\,dx=a$$

(see Section 2). In the third section, we then use the explicit expression of conformal metric of S to demonstrate our theorem.

2 An Inequality

In this section, we will establish an inequality that is essential to the main theorem. Let $B_r(0) \in \mathbb{R}^2$ be a ball in $\mathbb{R}^2$ with radius r centered at the origin, and

$$D_a^b\big(B_r(0)\big) = \Big\{ f(y) : f(y) - b \in W_0^{1,2}\big(B_r(0)\big), \int_{B_r(0)} |y|^{2\beta} e^{2f}\, dy = a \Big\}.$$

We have the following proposition.

Proposition 2.1 *For any $w \in D_a^b\big(B_r(0)\big)$, we have the following inequality*

$$\int_{B_r} |\nabla w|^2\, dx \geq (4+4\beta)\pi\Big(\ln \frac{(2+2\beta)ae^{-2b}}{2\pi r^{2+2\beta}} + \frac{2\pi r^{2+2\beta}}{(2+2\beta)ae^{-2b}} - 1\Big)$$

for any $w \in D_a^b(B_r)$ with $\beta \in (-1,0)$.

In order to prove Proposition 2.1, we need the following lemma.

Lemma 2.2 *For any nonnegative function $u \in C^1[0,+\infty)$ with $u(0) = 0$ and $\int_0^\infty e^{2u-2s-\beta s}\, ds = a$ with $a > \frac{1}{2+\beta}$, we have*

$$\int_0^\infty |u_r|^2\, dr \geq (2+\beta)\Big[\ln(2+\beta)a + \frac{1}{(2+\beta)a} - 1\Big]. \tag{5}$$

Proof When $\beta = 0$, Li and Zhou have shown it in [LZ1]. Their method is also valid for $\beta > 0$. For convenience of the reader, we give the proof in detail. We divide the proof into three steps.

Step 1 For any ε_0 with $\varepsilon_0 > \frac{1}{\sqrt{2+\beta}}$ we have

$$\int_R^{+\infty} e^{2u-(2+\beta)r}\, dr = o_R(1) \exp\Big\{ \varepsilon_0^2 \int_0^{+\infty} |u_r|^2\, dr \Big\} \tag{6}$$

where $o_R(1) \to 0$ as $R \to +\infty$.

We now show Step 1. Since $u(r)$ is any function in $C^1[0,+\infty)$ with $u(0) = 0$, we have for any $\varepsilon > 0$:

$$\begin{aligned} u(r) &= \int_0^r u_s\, ds \leq \int_0^r |u_s|\, ds \\ &\leq \Big(\int_0^r |u_s|^2\, ds\Big)^{\frac{1}{2}} r^{\frac{1}{2}} \\ &\leq \frac{1}{2}\Big(\varepsilon^2 \int_0^r |u_s|^2\, ds + \frac{r}{\varepsilon^2}\Big). \end{aligned}$$

Thus

$$\int_0^{+\infty} e^{2u-(2+\beta)s}\,ds \le \int_0^{+\infty} \exp\Big\{\varepsilon^2 \int_0^s |u_r|^2\,dr + \Big(\frac{1}{\varepsilon^2} - 2 - \beta\Big)s\Big\}\,ds$$
$$\le \exp\Big\{\varepsilon^2 \int_0^{+\infty} |u_r|^2\,dr\Big\} \cdot \int_0^{+\infty} e^{(\frac{1}{\varepsilon^2}-2-\beta)s}\,ds.$$

If we choose $\varepsilon = \varepsilon_0 > \frac{1}{\sqrt{2+\beta}}$, then the infinite integral

$$\int_0^{+\infty} e^{(\frac{1}{\varepsilon_0{}^2}-2-\beta)s}\,ds$$

is convergent. It follows that

$$\int_R^{+\infty} e^{2u-(2+\beta)r}\,dr \le \exp\Big\{\varepsilon_0^2 \int_0^{+\infty} |u_r|^2\,dr\Big\} \cdot \int_R^{+\infty} e^{(\frac{1}{\varepsilon_0^2}-2-\beta)s}\,ds$$
$$= o_R(1)\exp\Big\{\varepsilon_0^2 \int_0^{+\infty} |u_r|^2\,dr\Big\}.$$

Step 2 Define $D_a = \{u(r) \in W^{1,2}(\mathbb{R}^+) : u(0) = 0, \int_0^{+\infty} e^{2u-r(2+\beta)}\,dr = a\}$. There exists a $v \in D_a$ such that

$$\int_0^{+\infty} |v_r|^2\,dr = \inf_{u\in D_a} \int_0^{+\infty} |u_r|^2\,dr.$$

In fact, if we assume that v^i is a minimizing sequence of $\inf_{u\in D_a} \int_0^{+\infty} |u_r|^2\,dr$, then there is a $v \in W^{1,2}(\mathbb{R}^+)$ such that

$$v^i \rightharpoonup v \quad \text{in } W^{1,2}(\mathbb{R}^+)$$

and

$$\int_0^{+\infty} |v_r|^2\,dr \le \lim_{i\to\infty} \int_0^{+\infty} |v_r^i|^2\,dr \le \inf_{u\in D_a} \int_0^{+\infty} |u_r|^2\,dr.$$

Then by Step 1, the Sobolev embedding theorem and the Arzela–Ascoli Lemma we can show that $\int_0^{+\infty} e^{2v-(2+\beta)s}\,ds = a$, that is $v \in D_a$.

Step 3 In this step we will show that the minimizer v in Step 2 satisfies

$$\int_0^{+\infty} |v_r|^2\,dr = (2+\beta)\Big(\ln(2+\beta)a + \frac{1}{(2+\beta)a} - 1\Big),$$

and consequently we obtain the inequality (5).

In fact, the minimizer v satisfies the following Euler–Lagrange equation is

$$v_{rr} = -\tau e^{2v-(2+\beta)r} \tag{7}$$

for some $\tau > 0$ with $v(0) = 0$. The general solution to the ordinary differential equation (7) is given by

$$v(r) = \ln \frac{1}{\lambda_0 + e^{-(2+\beta)r}} - \frac{1}{n} \ln \frac{\tau}{(2+\beta)^n \lambda_0},$$

where λ_0 is a positive constant. Since $v(r)$ is a solution of (7), we can obtain that $\tau = \frac{(2+\beta)^n \lambda_0}{(1+\lambda_0)^n}$. Therefore

$$v(r) = \ln \frac{1}{\lambda_0 + e^{-(2+\beta)r}} - \ln \frac{1}{\lambda_0 + 1}.$$

Since $\int_0^{+\infty} e^{2v-(2+\beta)r}\, dr = a$, then by making change of variables $r = -\ln s$ we have

$$\begin{aligned} a &= \int_0^{+\infty} \Big(\frac{\lambda_0 + 1}{\lambda + e^{-(2+\beta)r}} \Big)^2 e^{-(2+\beta)r}\, dr \\ &= \int_0^1 \Big(\frac{\lambda_0 + 1}{\lambda_0 + s^{(2+\beta)}} \Big)^2 s^{1+\beta}\, ds \\ &= \frac{\lambda_0 + 1}{(2+\beta)\lambda_0}. \end{aligned}$$

Hence

$$\lambda_0 = \frac{1}{(2+\beta)a - 1}.$$

Now we compute the norm of v:

$$\begin{aligned} \int_0^{+\infty} |v_r|^2\, dr &= \int_0^{+\infty} \Big| \frac{(2+\beta)e^{-(2+\beta)r}}{\lambda_0 + e^{-(2+\beta)r}} \Big|^2 dr \\ &= (2+\beta) \int_1^{1+\frac{1}{\lambda_0}} \frac{s-1}{s^2}\, ds \\ &= (2+\beta)\Big(\ln \frac{\lambda_0 + 1}{\lambda_0} + \frac{\lambda_0}{\lambda_0 + 1} - 1 \Big) \\ &= (2+\beta)\Big(\ln(2+\beta)a + \frac{1}{(2+\beta)a} - 1 \Big). \end{aligned}$$

Thus we have finished the proof of the lemma. ∎

Now we can prove Proposition 2.1 by using the above lemma.

Proof of Proposition 2.1 Without loss of the generality, we show the proposition when $b = 0$ and $r = 1$. We assume $w \in D_a^0(B_1)$. Let $\bar{w} = \bar{w}(r)$ be the symmetric rearrangement of w, *i.e.*, $\bar{w}(r)$ is non-increasing and

$$\operatorname{meas}\{(r,\theta) \mid \bar{w}(r) \geq t\} = \operatorname{meas}\{(r,\theta) \mid w(r,\theta) \geq t\}$$

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for all $-\infty < t < +\infty$. By taking a translate of variable $r = -\ln s$, obviously we have

$$\int_{B_1} |\nabla w|^2\,dx \geq 2\pi \int_0^1 |\bar{w}_s|^2 s\,ds = 2\pi \int_0^\infty \bar{w}_r^2\,dr,$$

$$\int_{B_1} |x|^{2\beta} e^{2w}\,dx = 2\pi \int_0^1 e^{2\bar{w}} s^{2\beta+1}\,ds = 2\pi \int_0^\infty e^{2\bar{w}-2r-\beta r}\,dr.$$

Since $\bar{w}(e^{-r})$ is increasing for $r \in [0,\infty\}$ and $\bar{w}(1) = 0$, from Lemma 2.2, we have

$$\int_{B_1} |\nabla w|^2\,dx \geq (4+4\beta)\pi\Big(\ln \frac{(2+2\beta)\int_{B_1}|x|^{2\beta}e^{2w}\,dx}{2\pi} + \frac{2\pi}{(2+2\beta)\int_{B_1}|x|^{2\beta}e^{2w}\,dx} - 1\Big).$$

3 Proof of the Main Theorem

Proof of Theorem 1.1 Due to the rearrangement, we only need to prove Onofri-type inequality (4) for $u \in C^1(S \setminus \{p_1, p_2\}) \cap C(S)$ that depends only on x_3 and is monotonically decreasing in x_3, where p_1 and p_2 are the north pole and south pole of S respectively. Also, we can assume that $u(x_3)|_{x_3=1} = 0$ (otherwise, we replace $u(x)$ by $u(x) - u(1)$). We can approximate $u(x)$ by a sequence of functions $u_i \in C^1(S)$ such that $u_i(x) = u_i(x_3)$ is monotonically decreasing in x_3, and $u_i(x) = 0$ in the geodesic ball $B_{1/i}(p_1)$ of the north pole p_1 for $i \in \mathbb{N}$, and $u_i(x) = u(p_2)$ in the geodesic ball $B_{1/i}(p_2)$ of the south pole p_2. Denote $S_i = S \backslash \big(B_{1/i}(p_1) \cup B_{1/i}(p_2)\big)$.

By Troyanov's Theorem, on the singular surface S, there exists a unique conformal metric

$$ds^2 = \frac{(2+2\beta)^2|z|^{2\beta}|dz|^2}{(1+|z|^{2\beta+2})^2} = e^{2\varphi(z)}|dz|^2$$

such that its Gaussian curvature is 1 and its conical singularities are $z = 0$ and $z = \infty$. We set

$$\varphi(z) = \ln \frac{(2+2\beta)|z|^\beta}{1+|z|^{2\beta+2}}.$$

Then

$$-\triangle\varphi = e^{2\varphi} \quad \text{in } \mathbb{R}^2\backslash\{0\}.$$

Now we set

$$\widetilde{\varphi}(z) = \ln \frac{(2+2\beta)}{1+|z|^{2\beta+2}}.$$

It is clear that $\varphi(z) = \widetilde{\varphi}(z) + \beta \ln|z|$ and $\widetilde{\varphi}(z)$ satisfies that

$$-\triangle\widetilde{\varphi} = |z|^{2\beta} e^{2\widetilde{\varphi}} \quad \text{in } \mathbb{R}^2\backslash\{0\}.$$

Let Φ be the conformal map from S to $\mathbb{R}^2 \cup \infty$ such that its conformal factor is $e^{2\varphi}$. Then we have $\Phi(S_i) = B_{R_i}\backslash B_{r_i}$. It is obvious that $R_i \to +\infty$ and $r_i \to 0$ as $i \to +\infty$. Set

$$w_i(z) = u_i(x) + \varphi(z) = u_i\big(\Phi^{-1}(z)\big) + \varphi(z),$$

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and

$$\widetilde{w}_i(z) = u_i\big(\Phi^{-1}(z)\big) + \widetilde{\varphi}(z).$$

It is clear that

$$w_i(z) = \widetilde{w}_i(z) + \beta \ln z.$$

Since $\int_S e^{2u}\,dv$ is conformally invariant, we have

$$\int_{S_i} e^{2u_i(z)}\,dv = \int_{B_{R_i}\setminus B_{r_i}} e^{2w_i(z)}\,dz = \int_{B_{R_i}\setminus B_{r_i}} e^{2\widetilde{w}_i(z)}|z|^{2\beta}\,dz. \tag{8}$$

Now we define $a_i = \int_{B_{R_i}} e^{2\widetilde{w}_i(z)}|z|^{2\beta}\,dz$. By a direct computation, we obtain

$$\begin{aligned}
\int_{B_{R_i}\setminus B_{r_i}} |\nabla \widetilde{w}_i|^2\,dz &= \int_{B_{R_i}\setminus B_{r_i}} |\nabla(u_i\circ\Phi^{-1})|^2\,dz \\
&\quad + 2\int_{B_{R_i}\setminus B_{r_i}} \nabla(u_i\circ\Phi^{-1})\nabla\widetilde{\varphi}\,dz + \int_{B_{R_i}\setminus B_{r_i}} |\nabla\widetilde{\varphi}|^2\,dz \\
&= \int_{S_i} |\nabla u_i|^2\,dv - 2\int_{B_{R_i}\setminus B_{r_i}} u_i\circ\Phi^{-1}\triangle\widetilde{\varphi}\,dz \\
&\quad + \int_{B_{R_i}\setminus B_{r_i}} |\nabla\widetilde{\varphi}|^2\,dz - 2\int_{\partial B_{r_i}} u_i\circ\Phi^{-1}\frac{\partial\widetilde{\varphi}}{\partial n}\,d\sigma \\
&= \int_{S_i} |\nabla u_i|^2\,dv + 2\int_{S_i} u_i\,dv \\
&\quad + \int_{B_{R_i}\setminus B_{r_i}} |\nabla\widetilde{\varphi}|^2\,dz - 2u(p_2)\int_{\partial B_{r_i}} \frac{\partial\widetilde{\varphi}}{\partial n}\,d\sigma.
\end{aligned} \tag{9}$$

Furthermore, we note that

$$\int_{B_{R_i}} |\nabla\widetilde{\varphi}|^2\,dz = 4\pi(\beta+1)\Big(\ln(1+R_i^{2\beta+2}) + \frac{1}{1+R_i^{2\beta+1}} - 1\Big),$$

$$\int_{B_{r_i}} |\nabla\widetilde{\varphi}|^2\,dz = 4\pi(\beta+1)\Big(\ln(1+r_i^{2\beta+2}) + \frac{1}{1+r_i^{2\beta+1}} - 1\Big) \to 0 \text{ as } r_i \to 0,$$

$$\int_{\partial B_{r_i}} \frac{\partial\widetilde{\varphi}}{\partial n}\,d\sigma = -\frac{2\pi(2\beta+2)r_i^{2\beta+2}}{1+r_i^{2\beta+2}} \to 0 \text{ as } r_i \to 0,$$

$$\int_{B_{r_i}} |\nabla\widetilde{w}_i|^2\,dz = \int_{B_{r_i}} |\nabla\widetilde{\varphi}|^2\,dz \to 0 \text{ as } r_i \to 0.$$

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Since $\widetilde{w}_i(z)|_{\partial B_{R_i}} = \ln \frac{2+2\beta}{1+R_i^{2\beta+2}}$, by Proposition 2.1, we have

(10)
$$\int_{B_{R_i}} |\nabla \widetilde{w}_i|^2 \, dz$$
$$\geq (4+4\beta)\pi \Big\{ \ln \Big[\frac{(2+2\beta)a_i}{2\pi R_i^{2\beta+2}} \Big(\frac{1+R_i^{2\beta+2}}{2+2\beta} \Big)^2 \Big] + \frac{2\pi R_i^{2\beta+2}}{(2+2\beta)a_i(\frac{1+R_i^{2\beta+2}}{2+2\beta})^2} - 1 \Big\}.$$

We conclude from (8), (9) and (10) that

$$\begin{aligned}
\int_{S_i} |\nabla u_i|^2 \, ds + 2\int_{S_i} u_i \, ds
&= \int_{B_{R_i}} |\nabla \widetilde{w}_i|^2 \, dz - \int_{B_{R_i}} |\nabla \widetilde{\varphi}|^2 \, dz + o(r_i) \\
&\geq (4+4\beta)\pi \Big\{ \ln \frac{a_i(1+R_i^{2\beta+2})^2}{4\pi(\beta+1)R_i^{2\beta+1}} + \frac{4\pi(\beta+1)R_i^{2\beta+1}}{a_i(1+R_i^{2\beta+2})^2} - \frac{1}{1+R_i^{2\beta+2}} \Big\} + o(r_i),
\end{aligned}$$

Where $o(r_i) \to 0$ as $r_i \to 0$. Let $i \to +\infty$, we obtain that

$$\begin{aligned}
\int_S |\nabla u|^2 \, ds + 2\int_S u \, ds &\geq 4\pi(\beta+1) \ln \frac{a}{4\pi(\beta+1)} \\
&= 4\pi(\beta+1) \ln \frac{\int_S e^{2u} \, ds}{4\pi(\beta+1)}.
\end{aligned}$$

Thus we complete the proof of the theorem. ∎

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